**ANTI-WINDUP STRATEGY FOR LINEAR SYSTEMS WITH AMPLITUDE AND DYNAMICS RESTRICTED ACTUATOR**

S. Tarbouriech, G. Garcia*, P. Langouët

LAAS-CNRS,
7 Avenue du Colonel Roche, 31077 Toulouse cedex 4, France.
E-mail: tarbour@laas.fr, garcia@laas.fr, plangoue@laas.fr

* This author is also with INSA, Toulouse, France.

**Keywords:** Anti-windup scheme, bounded controlled output, amplitude and dynamics restricted actuator, stability regions.

**Abstract**

This paper addresses the problem of the determination of regions of stability for linear systems with amplitude and successive dynamics restricted actuator through anti-windup strategies. The objective by designing anti-windup gains is to guarantee the stability of the closed-loop system and the respect of the controlled output constraints for a region of admissible initial states as large as possible. Based on the modeling of the closed-loop system resulting from the controller plus the anti-windup loop as a linear system with a dead-zone and dynamics restricted nonlinearities, constructive stability conditions are formulated by using quadratic and Lure approach associated with the Finsler’s lemma. Numerical procedures are discussed.

**1 Introduction**

Physical, safety or technological constraints generally induce that the control actuators cannot provide unlimited amplitude signals neither unlimited speed of reaction. That means that the control systems are generally subject to amplitude and dynamics actuator saturations. The control problems of combat aircraft prototypes and launchers offer interesting examples of the difficulties due to these major constraints. Neglecting both amplitude and dynamics actuator limitations can be source of undesirable even catastrophic behaviors for the closed-loop system (as the lost of the closed-loop stability) [4]. For these reasons, the study of the control problem or analysis stability problem with respect to systems subject to both amplitude and rate actuator saturations has received the attention of many researchers in the last years (see, for example, [20], [9], [11]).

The anti-windup fits the approach consisting in taking into account the effect of saturations in a second step after a previous design performed disregarding the saturation terms. The objective then consists in introducing control modifications in order to recover, as much as possible, the performance induced by a previous design carried out on the basis of the unsaturated system. In particular, anti-windup schemes have been successfully applied in order to avoid or minimize the windup of the integral action in PID controllers, largely applied in the industry. In this case, most of the related literature focuses on the performance improvement in the sense of avoiding large and oscillatory transient responses (see, among others, [2], [1], [7]).

More recently, a special attention has been paid to the influence of the anti-windup schemes in the stability and the performances of the closed-loop system (see, for example, [3], [5], [12], [14], [15], [17], [19]). Several results on the anti-windup problem are concerned with achieving global stability properties. Since global results cannot be achieved for open-loop unstable linear systems in the presence of actuator saturation, local results have to be developed. In this context, a key issue concerns the determination of domains of stability for the closed-loop system. Most of the local results available in the anti-windup literature do not provide explicit characterization of the domain of stability. It is worth to notice that the basin of attraction is modified by the anti-windup loop. If the resulting basin of attraction is not sufficiently large, the system can present a divergent behavior depending on its initialization and the action of disturbances.

In this paper, we consider the structure of the observer-based anti-windup [2, 1]. With respect to this structure, we can cite [12] in which passivity arguments are invoked or still [18] in which a local \( H_\infty \) design is provided in terms of matrix inequalities. More recently, in [6], some constructive conditions are proposed both to determine suitable anti-windup gains and to quantify the closed-loop region of stability in the case of amplitude saturation actuator. Differently from the papers cited above, in this paper we focus our attention on linear systems with amplitude and successive dynamics restricted actuator and bounded controlled outputs. Our aim is the characterization of stability regions for this class of systems through anti-windup strategies. Especially, we are interested in the anti-windup gains design in order to ensure the closed-loop stability for regions of admissible initial states as large as possible. Based on the modeling of the closed-loop system resulting from the controller plus the anti-windup loop as a linear system with a dead-zone and dynamics restricted nonlinearities, constructive stability conditions are formulated by using quadratic and Lure approaches associated with the Finsler’s lemma. Numerical procedures based on the solution of some iterative convex optimization problems with LMI constraints are proposed for computing the anti-windup gains that lead to the maximization of the size of the associated region of stability. At our knowledge, the current paper constitutes the first study with respect to the considered class of systems.
Matrices

2 Problem Statement

In this paper, we consider a class of nonlinear systems which are obtained by cascading linear systems with actuators containing some nonlinearities of saturation type. The actuator under consideration is a dynamic system containing amplitude and dynamics restrictions, that is, it is described via successive time-derivatives of the input of the plant. By setting

\[
x_\alpha(t) = A_\alpha x_\alpha(t) + B_\alpha \sigma x_\alpha(t)
\]

and

\[
\sigma x_\alpha(t) = sat_{\alpha}(u(t)) \in \mathbb{R}^m
\]

where \(u(t)\) denotes the \(q\)-order time-derivative of \(u\), the model of the actuator reads as follows:

\[
\begin{aligned}
\dot{x}_\alpha(t) &= A_\alpha x_\alpha(t) + B_\alpha \sigma x_\alpha(t) \\
\sigma x_\alpha(t) &= sat_{\alpha}(u(t))
\end{aligned}
\]

where \(x_\alpha\) is the state of the actuator and \(\sigma x_\alpha\) is the measured output of the actuator and \(y_\alpha \in \mathbb{R}^n\) is the output of the controller. Matrices \(A_\alpha, B_\alpha, j = 0, \ldots, q, \) and \(C_\alpha\) are defined by:

\[
A_\alpha = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 1 \\
0 & \ldots & \ldots & \ldots & 0
\end{bmatrix} \in \mathbb{R}^{mq \times mq}
\]

\[
B_\alpha = \begin{bmatrix}
0 & 0 & \ldots & 0 & T^j
\end{bmatrix} \in \mathbb{R}^{mq \times mq}
\]

\[
C_\alpha = \begin{bmatrix}
1 & 0 & \ldots & 0
\end{bmatrix} \in \mathbb{R}^{q \times mq}
\]

Such a model is the type of actuator encountered in the control of launchers (see [16] in which \(m = 1\) and \(q = 2\)). In (1), the positive vectors \(u_0\) and \(u_j, j = 1, \ldots, q - 1\), may be viewed as bounds on the position and the successive dynamics of the actuator state. Thus, it clearly appears that one cannot have simultaneously position and dynamics saturation.

The plant is a linear continuous-time system defined as:

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) \\
z(t) &= C_2 x(t)
\end{aligned}
\]

where \(x \in \mathbb{R}^n, y(t) \in \mathbb{R}^m\) and \(z(t) \in \mathbb{R}^l\) are the state, the measured output and the controlled output vectors, respectively.

Without saturation terms, that is with \(sat_{\alpha}(C_\alpha x_\alpha) = C_\alpha x_\alpha = u\) and \(sat_{\eta}(sat_{\alpha}(C_\alpha x_\alpha(t)^{(j)})) = C_\eta x_\alpha(t)^{(j)} = u(t)^{(j)}, j = 1, \ldots, q - 1\), system (1)-(3) is linear and reads:

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\dot{y}(t) &= Ax(t) + B_\eta y(t) \\
z(t) &= C_2 x(t)
\end{aligned}
\]

Under the \((A, B)\)-controllability and \((C, A)\)-observability assumptions, we assume that an \(n\)-order dynamic output stabilizer has been determined to stabilize the linear system (4) and is described as follows:

\[
\begin{aligned}
\dot{\eta}(t) &= A_\eta \eta(t) + B_\gamma y(t) \\
y_\alpha(t) &= C_\eta \eta(t) + D_\gamma y(t)
\end{aligned}
\]

where \(\eta(t) \in \mathbb{R}^n\) is the controller state, \(u(t) = y(t)\) is the controller input and \(y_\alpha(t) \in \mathbb{R}^n\) is the controller output.

Furthermore, due to the presence of the saturation terms, in order to mitigate the undesirable effects of windup, caused by input saturation (due to \(y_\alpha\) measurable variable of the actuator), an anti-windup term \((\eta_{\alpha}(C_\alpha x_\alpha(t) - C_\eta x_\alpha(t)))\) can be added to the controller [14] through adequate gain. Thus, considering the dynamic controller and this anti-windup strategy, the closed-loop system reads:

\[
\dot{x}(t) = Ax(t) + Bu(t) + B_\alpha \sigma x_\alpha(t)
\]

\[
\begin{aligned}
\dot{y}(t) &= Ax(t) + B_\alpha \sigma x_\alpha(t) \\
\eta(t) &= A_\eta \eta(t) + B_\gamma y(t) + E_\gamma (sat_{\alpha}(C_\alpha x_\alpha) - C_\alpha x_\alpha(t)) \\
y_\alpha(t) &= C_\eta \eta(t) + D_\gamma y(t) + F_\gamma (sat_{\alpha}(C_\alpha x_\alpha)) - C_\alpha x_\alpha(t) \\
z(t) &= C_2 x(t)
\end{aligned}
\]

where \(E_\gamma\) and \(F_\gamma\) are the two anti-windup gains to be determined. It is worth noticing in system (6) that if the \(j\)th component of the amplitude saturation is effective (i.e., \(C_\alpha x_\alpha^{(j)} > u_\alpha x_\alpha^{(j)}\)) then the corresponding component of the successive dynamics saturation does not affect the system (i.e., \(sat_{\alpha}(C_\alpha x_\alpha(t)^{(j)}) = 0, j = 1, \ldots, q - 1\).)

Problem 1 Determine anti-windup gains \(E_\gamma\) and \(F_\gamma\), and a set \(S_0\) such that:

1. The asymptotic stability of the closed-loop system (6) is ensured for any \([x(0)^T \quad x_\alpha(0)^T \quad \eta(0)^T] \in S_0\), where \(S_0\) is as large as possible.

2. For any \([x(0)^T \quad x_\alpha(0)^T \quad \eta(0)^T] \in S_0\) the measured output \(z\) takes values in the set \(Z_0\) defined by:

\[
Z_0 = \{z \in \mathbb{R}^l; -z_0 \leq z \leq z_0, \sigma_{z_i} > 0, i = 1, \ldots, l\}
\]

The implicit objective in Problem 1 is to compute \(E_\gamma\) and \(F_\gamma\) for enlarging the basin of attraction of the closed-loop system.
3 Preliminaries

Let us first define the \( q \) nonlinearities \( \phi_0 \) and \( \phi_j \), \( j = 1, \ldots, q - 1 \):

\[
\phi_0(C_0 x_a(t)) = y_a(t) - C_0 x_a(t) = sat_{\phi_0}(C_0 x_a(t)) - C_0 x_a(t) \tag{8}
\]

\[
\phi_j(C_0 x_a(t)) = sat_{\phi_j}(y_a^{(j)}(t)) - y_a^{(j)}(t) = sat_{\phi_j}(sat_{\phi_0}(C_0 x_a(t))^{(j)}) - sat_{\phi_0}(C_0 x_a(t))^{(j)} \tag{9}
\]

From the definition of \( \phi_0 \), one gets

\[
sat_{\phi_0}(C_0 x_a(t))^{(j)} = \phi_0^{(j)}(C_0 x_a(t)) + C_0 x_a^{(j)}(t) \tag{10}
\]

Therefore, the system (6) can be written in a compact form. For this, define the extended state vector

\[
\xi(t) = [x(t)' \ x_a(t)' \ \eta(t)']' \in \mathbb{R}^{n+m+n_q} \tag{11}
\]

and the following matrices of appropriate dimensions

\[
\hat{A} = \begin{bmatrix}
A & B C a & 0 \\
B_D & A_a & B a_{q-1} \\
B & B a_b & A_c
\end{bmatrix} \tag{12}
\]

\[
A_0 = \begin{bmatrix}
B a_0 \\
B a_d \end{bmatrix} \quad ; \quad B = \begin{bmatrix}
0 \\
B a_j \end{bmatrix} \quad ; \quad B_q = \begin{bmatrix}
0 \\
B a_q
\end{bmatrix}
\]

\[
0 \quad 0 \quad 1 
\]

\[
A = \begin{bmatrix}
0 & C_a & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad ; \quad A_0 = \begin{bmatrix}
0 & C_a & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

for \( j = 1, \ldots, q - 1 \). Thus, the closed-loop system reads:

\[
\begin{align*}
\dot{\xi}(t) &= \hat{A} \xi(t) + (A_0 + A A_c + A_q F_q) \phi_0(\mathbb{K} \xi(t)) \\
&\quad + \sum_{j=1}^{q-1} B_j \phi_j(\mathbb{K} \xi(t)) + \phi^{(j)}_0(\mathbb{K} \xi(t)) \\
Z(t) &= C_2 \bar{z}(t)
\end{align*}
\tag{13}
\]

In the sequel for simplicity, \( \phi_0(\mathbb{K} \xi(t)) \), \( \phi_j(\mathbb{K} \xi(t)) \) and \( \phi^{(j)}_0(\mathbb{K} \xi(t)) \), \( j = 1, \ldots, q - 1 \) will be denoted \( \phi_0 \), \( \phi_j \) and \( \phi^{(j)}_0 \). Note that in the absence of saturation one gets \( \phi_0 = 0 \), \( \phi_j = 0 \) and \( \phi^{(j)}_0 = 0 \) and by hypothesis the matrix \( \hat{A} \) is assumed to be asymptotically stable.

The nonlinearity \( \phi_0 \) is decentralized, memoryless and satisfies the following sector condition [13] for any diagonal positive definite matrix \( T_0 \):

\[
\phi_j T_0 (\phi_0 + A_0 \mathbb{K} \xi) \leq 0 \tag{14}
\]

provided that \( \xi \) takes values in \( S(\mathbb{K}, u^2_0) \)

\[
S(\mathbb{K}, u^2_0) = \left\{ \xi \in \mathbb{R}^{n+mq+n_q} ; \|\mathbb{K} \xi\|_i \leq \frac{u_{i(0)}}{1-\lambda_{0(i)}}, i = 1, \ldots, m \right\} \tag{15}
\]

where \( A_0 \) is a positive diagonal matrix with \( A_0(i,i) = \lambda_{0(i)} \).

From the definition of each component of \( \phi_j \), \( j = 1, \ldots, q - 1 \), one verifies that for any diagonal positive definite matrix \( T_j \), \( j = 1, \ldots, q - 1 \):

\[
\phi_j T_j (\phi_j + A_j \mathbb{K} \xi(j)) \leq 0 \tag{16}
\]

provided that \( \xi \) takes values in \( S(\mathbb{K}, u^2_j) \)

\[
S(\mathbb{K}, u^2_j) = \left\{ \xi \in \mathbb{R}^{n+mq+n_q} ; \|\mathbb{K} \xi\|_i \leq \frac{u_{j(i)}}{1-\lambda_{j(i)}}, i = 1, \ldots, m \right\} \tag{17}
\]

where \( A_j \) is a positive diagonal matrix with \( A_j(i,i) = \lambda_{j(i)} \).

Furthermore, we can express inherent properties relating \( \phi_0 \), \( \phi_j \) and \( \phi^{(j)}_0 \).

**Lemma 1** The nonlinearities \( \phi_0, \phi_j \) and \( \phi^{(j)}_0 \) satisfy the following properties for \( j = 1, \ldots, q - 1 \):

\[
(\mathbb{K} \xi(j) + \phi_0^{(j)} \phi_0^{(j)}) \phi_0 = 0; (\mathbb{K} \xi(j) + \phi^{(j)}_0 \phi^{(j)}_0) \phi_0 = 0 \tag{18}
\]

\[
\phi'_j \phi_0 = 0; \phi'_j \phi^{(j)}_0 = 0; \phi'_j \phi^{(j+1)}_0 = 0 \tag{19}
\]

4 Stability conditions

Some conditions in terms of matrix inequalities are now presented by using both quadratic Lyapunov and Lur'e Lyapunov functions. For this, define the vectors \( \Phi^d_0 \), \( \Phi_1 \) and \( U \):

\[
\Phi^d_0 = \begin{bmatrix}
\phi_0 \\
\phi_0^{(2)} \\
\vdots \\
\phi_0^{(q-1)} \\
\phi_0^{(q-1)} 
\end{bmatrix} \quad ; \quad \Phi_1 = \begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_{q-1} \\
\phi_{q-1}
\end{bmatrix} \quad ; \quad U_1 = \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
\nu_{q-1}
\end{bmatrix} \tag{20}
\]

and the augmented matrices

\[
B_1 = \begin{bmatrix}
B_1 \\
\vdots \\
B_{q-1}
\end{bmatrix} \quad ; \quad B = \begin{bmatrix}
B_1 \\
\vdots \\
B_{q-1}
\end{bmatrix} \quad ; \quad \mathbb{K} A = \begin{bmatrix}
\mathbb{K} A_1 \\
\vdots \\
\mathbb{K} A_{q-1}
\end{bmatrix} \tag{21}
\]

4.1 Quadratic approach

Consider a quadratic candidate Lyapunov function \( V(\xi) \):

\[
V(\xi) = \xi P \xi \quad \text{with} \quad P = P' > 0 \tag{22}
\]

**Proposition 1** If there exist matrices of appropriate dimensions \( P = P' > 0 \), \( E_1, \ldots, E_q \), \( F_1, \ldots, F_q \), \( G, H, J, L \), diagonal matrices \( N_3, N_4 \) and \( N_5 \), diagonal positive matrices \( T_0, T_1, T_2, A_0, \mathbb{I}, 1, \) positive scalar \( \gamma \) satisfying\(^1\)

\[
\begin{bmatrix}
M_1 & * & * \\
M_2 & M_3 & * \\
M_5 & M_6 & M_8
\end{bmatrix} < 0 \tag{23}
\]

\[
\begin{bmatrix}
(1 - \lambda_{0(i)}) \mathbb{K} \xi(i) \gamma_{u_0(i)}^2 \\
(1 - \lambda_{1(i)}) \mathbb{K} \xi(i) \gamma_{u_1(i)}^2
\end{bmatrix} \geq 0, \quad i = 1, \ldots, m \tag{24}
\]

\[
\begin{bmatrix}
P \\
(1 - \mathbb{I}_{1(j)}) \mathbb{K} \xi(i) \gamma_{u_{1(i)}^2}
\end{bmatrix} \geq 0, \quad i = 1, \ldots, mq \tag{25}
\]

\(^1\)The symbol * stands for symmetric blocks in matrix inequalities.
\[
\begin{bmatrix}
P \\
\mathcal{C}_{2(i)} \\
P_{G(i)}
\end{bmatrix} \geq 0, \quad i = 1, \ldots, l 
\]
(26)

\[
0 \leq \lambda_0 < 1; \quad 0 \leq 1 < 1
\]
(27)

with
\[M_1 = \begin{bmatrix}
\mathcal{C}_{i} & P \mathcal{C}_{i} & \Phi_i \\
P_{G(i)} & P_{G(i)} & \Phi_i \\
0 & 0 & 0
\end{bmatrix}; \quad M_2 = \begin{bmatrix}
\mathcal{C}_{i} & P \mathcal{C}_{i} & \Phi_i \\
P_{G(i)} & P_{G(i)} & \Phi_i \\
0 & 0 & 0
\end{bmatrix}.
\]

In Proposition 1, there appear some nonlinearities in particular to the product between the multipliers \((F, G, H, J\) and \(L)\) and the gains of the anti-windup \(E_c\) and \(F_c\). Moreover, the satisfaction of relation (23) means that \(G + G > 0\) and therefore matrix \(G\) must be nonsingular. From this fact and a suitable choice of multipliers with an adequate change of variables simplify a major part of the inequalities of Proposition 1.

**Corollary 1** If there exist matrices of appropriate dimensions \(W = W^T > 0, S, Z_1, Z_2, V_2\) diagonal positive matrices \(\alpha_i, \beta_i, \gamma, \delta_i\) satisfying

\[
\begin{bmatrix}
\begin{array}{ccc}
\alpha_i + \beta_i S & 0 \\
0 & \gamma \delta_i S
\end{array}
\end{bmatrix} \geq 0, \quad i = 1, \ldots, m 
\]
(31)

then the gains \(E_c, F_c\) and the set \(S_0 = \{\xi \in R^{n+m+\kappa}; \xi^T P \xi \leq 1\}\) solve Problem 1.

**4.2 Lure approach**

Let us now consider a Lure candidate Lyapunov function \(V(\xi)\):

\[
\begin{align*}
V(\xi) &= \xi^T P \xi - 2 \sum_{i=1}^{m} \int_{0}^{\xi(i)} \phi_{0}(\sigma) \mathcal{N}(0, i) d\sigma \\
&\quad - 2 \sum_{j=1}^{q-1} \sum_{i=1}^{m} \int_{0}^{\xi(i)} \phi_{0}(\sigma) \mathcal{N}(0, i) d\sigma
\end{align*}
\]

with \(P = P^T > 0, N_0\) and \(N_j\) diagonal positive matrices. One gets \(\xi > 0\) for all \(\xi \in \mathbb{S}(\mathbb{K}, \mathbb{K}^2) \cap \mathbb{S}(\mathbb{K}, \mathbb{K}^2), \xi \neq 0\).

**Proposition 2** If there exist matrices of appropriate dimensions \(P = P^T > 0, E_c, F_c, F, G, H, J\) diagonal matrices \(N_1, N_2\) diagonal positive matrices \(T_1, T_2, N_0, N_1, N_2, N_0, \text{ positive scalar } \gamma \) satisfying relations (24), (25), (26), (27) and

\[
\begin{bmatrix}
M_1 & M_2 & M_3 & M_4 & M_0 \\
M_2 & M_4 & M_5 & M_0 & M_6 \\
M_3 & M_5 & M_0 & M_6 & M_7
\end{bmatrix} \leq 0
\]
(36)
with $M_1$, $M_3$, $M_4$, $M_5$, $M_6$ defined in Proposition 1 and $u_z -$

\[
\begin{bmatrix}
-\|\beta_n\| & 0 & 0 & \cdots & 0 & 0 \\
-\|\beta_n\| & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
-\|\beta_n\| & 0 & 0 & \cdots & 0 & 0 \\
-\|\beta_n\| & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

then the gains $E_c$, $F_c$ and the set $S_0 = \{\xi \in \mathbb{R}^{m+m+n}; V(\xi) \leq \gamma^{-1}\}$ solve Problem 1.

**Proof.** The proof follows the same lines as that of Proposition 1 by considering the Lure function defined in (35). Hence, note that the time-derivative of this function $V(\xi)$ reads:

\[
V(\xi) = \dot{\xi}^T P \dot{\xi} + \xi^T P \xi - 2\phi_0 N_0 \xi^T - 2\Phi_1 N_1 \xi - 2\Phi_1 \Phi_0^T
\]

where $N_1 = \text{diag}([N_1, \ldots, N_{q-1}])$. Indeed, by definition of different nonlinearities one can prove that $\Phi_1^T \Phi_0^T = 0$. Therefore, one can remark that the new matrix $Q$ is

\[
Q = \begin{bmatrix}
0 & P & 0 & 0 \\
0 & P & 0 & 0 \\
N_0 & 0 & 0 \\
N_1 & 0 & 0
\end{bmatrix}
\]

From the use of this matrix the proof is similar to that of Proposition 1. Moreover, from (35) one has $\xi^T P \xi \leq V(\xi)$ and the following inclusion $\{\xi \in \mathbb{R}^{m+m+n}; V(\xi) \leq \gamma^{-1}\} \subset \{\xi \in \mathbb{R}^{m+m+n} + N; \xi^T P \xi \leq \gamma^{-1}\}$ holds. Hence, conditions (24), (25), (26) allow to verify that $\xi \in S(\mathbb{R}^{m+n}; u_0^{(j)} \cap S(\mathbb{R}^{m+n}; u_1^{(j)}), j = 1, \ldots, q - 1$.

As in Corollary 1, an adequate change of variables and of multipliers exhibits the two gains $E_c$ and $F_c$. 

**Corollary 2** If there exist matrices of appropriate dimensions $W = W^T > 0, S, Z_1, Z_2, V_1, V_2$, diagonal positive matrices $N_0, N_1, R_0, R_1, R_2$, positive scalar $\gamma$ satisfying relations (31), (32), (33), (34) and

\[
\begin{bmatrix}
\Sigma_{0}\xi & \Sigma_{0} \xi & \Sigma_{0} \xi \\
\Sigma_{0} \xi & \Sigma_{0} \xi & \Sigma_{0} \xi \\
\Sigma_{0} \xi & \Sigma_{0} \xi & \Sigma_{0} \xi
\end{bmatrix}
\]

then the gains $E_c = Z_4 R_0^{-1}, F_c = Z_2 R_0^{-1}$ and the set $S_0 = \{\xi \in \mathbb{R}^{m+m+n}; V(\xi) \leq \gamma^{-1}\}$, with $P = S^{-1} W(S)^{-1}, N_0 = R_0^{-1}$ and $N_1 = R_1^{-1}$, solve Problem 1.

Note that if the satisfaction of relation (37) implies that the Lure function $V(\xi)$ defined in (35) verifies $V(\dot{\xi}) < 0$ along the trajectories of the closed-loop system (13), the inclusion $\{\xi \in \mathbb{R}^{m+m+n}; V(\xi) \leq \gamma^{-1}\} \subset \{\xi \in \mathbb{R}^{m+m+n}; \xi^T P \xi \leq \gamma^{-1}\}$ does not imply that the quadratic function $\xi^T P \xi$ is a decreasing function along the trajectories of the closed-loop system (13). Moreover, by setting $N_0 = 0$ and $N_1 = 0$, condition (36) (resp. (37)) is equivalent to (23) (resp. (30)). Moreover, an interesting fact appearing in relation (37) with respect to the equivalent one obtained via classical quadratic approach (see [10]) is that there is no nonlinearities between matrices $N_0 = 0, N_1 = 0$ of the Lure function and the anti-windup gains.

**5 Numerical procedure**

Some relations of Corollaries 1 or 2 are bilinear in decision variables $\Lambda_0$ and $S$, and $\Lambda_1$ and $S$. A way to overcome the computational difficulty of directly solving BMI conditions consists in using relaxation schemes, that is to fix one of the variables and seek for the other ones. In this case, the relations become linear. Moreover, the implicit objective is to maximize the region of stability of the closed-loop system over the choice of the anti-windup gains.

**5.1 Quadratic approach**

From Proposition 1 and Corollary 1, the region of stability associated to the closed-loop system (13) is the ellipsoid $S_0 = \{\xi \in \mathbb{R}^{m+m+n}; \xi^T P \xi \leq \gamma^{-1}\}$. By noting that the volume of $S_0$ is proportional to $\sqrt{\det(\frac{P}{\gamma})}$, it is then possible to maximize its size by minimizing the function $\log(\det(\gamma P))$. By definition of $P$ it follows: $\det(\gamma P) = \det(\gamma P(S)^{-1}) = \gamma^m + m \log(\gamma) + \log(\det(W))$. Hence, an algorithm based on some relaxation schemes can be considered. Let us underline that the maximization of $\log(\det(\gamma P))$ implies to maximize the set $S(\mathbb{R}^{m+n}; u_0^{(j)}) \cap S(\mathbb{R}^{m+n}; u_1^{(j)}), j = 1, \ldots, q - 1$.

As a consequence, one can observe that the size of the set of interest $S_0$ will increase, which contains the set of interest $S_0$. Therefore, this algorithm would add some degrees of freedom to maximize the size of $S_0$. Nevertheless, it is important to note that, in general, the better solution is not obtained for $\Lambda_0 = 0$ and $\Lambda_1 = 0$. Due to the form of the actuator (1), remind us that one cannot have simultaneously $\phi_{(i)}^{(j)} \neq 0$ and $\varphi_{(i)}^{(j)} \neq 0$. Indeed, when $\phi_{(i)}^{(j)} \neq 0$ one gets $\varphi_{(i)}^{(j)} = 0$, and therefore one gets $\lambda_{(i)}^{(j)} \neq 0$ and $\Lambda_{(i)}^{(j)} = \lambda_{(i)}^{(j)} = 0$. This fact means that the numerical tests of relations (30), (31) and (32) will be done by removing some lines and columns in matrix inequality (30).

**5.2 Lure approach**

With respect to the Lure function defined in (35), one gets:

\[
V(\xi) \leq \xi^T (P + \mathbb{R} N_0 \mathbb{E} \xi) + (\Phi_1 + \mathbb{D} \xi)^T N_1 (\Phi_1 + \mathbb{D} \xi)
\]

Recall that by definition of $\Phi_1 + \mathbb{D} \xi$, one gets, $\forall i = 1, \ldots, m$:

\[
\phi_{(i)}^{(j)}(\xi)_{i,j} = \begin{cases}
0 & \text{if } \|\xi\| > u_0(i) \\
\|\xi\|_{i,j} & \text{if } \|\xi\| \leq u_0(i)
\end{cases}
\]

Thus in a certain way, one can consider that the right part of the inequality (38) evolves as $\xi^T (P + \mathbb{R} N_0 \mathbb{E} \xi + \mathbb{D} \xi) = \xi^T \gamma \gamma^{-1} (W + S(S)^{-1})$. From this, in order to maximize the size of the set $S_0 = \{\xi \in \mathbb{R}^{m+m+n}; V(\xi) \leq \gamma^{-1}\}$, we use the same concept as in the previous subsection. Hence, note that $\det(\gamma (P + \mathbb{R} N_0 \mathbb{E} \xi + \mathbb{D} \xi)) = \det(\gamma^{-1} (W + S(S)^{-1})$. Therefore, one can maximize the region of stability of the closed-loop system over the choice of the anti-windup gains.
In this paper, we have addressed the problem of designing anti-windup gains in order to obtain a region of stability, as large as possible, for linear systems with amplitude and dynamics restricted actuator. The anti-windup strategy developed consisted in adding the part due to the saturation of the output actuator (measured part) both in the state evolution and in the output of the controller. Theoretical constructive conditions have been provided in order to associate the synthesized anti-windup gain to a region of stability, while controlled output constraints were satisfied. From these conditions, algorithms based on the solution of LMI-based problems have been proposed in order to optimize the size of the region of stability over the choice of the anti-windup gains $E_c$ and $F_c$. Since the solution $E_c = F_c = 0$ is always admissible, we can conclude that anti-windup gains can always be used in order to obtain larger regions of stability.

**Acknowledgement.** This research was supported in part by PIROLA Project under Grant number F/20062/SA.

**References**


