Quadratic Hamiltonians on non-Euclidean spaces of arbitrary constant curvature

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Abstract—This paper derives explicit solutions for Riemannian and sub-Riemannian curves on non-Euclidean spaces of arbitrary constant cross-sectional curvature. The problem is formulated in the context of an optimal control problem on a 3-D Lie group and an application of Pontryagin’s maximum principle of optimal control leads to the appropriate quadratic Hamiltonian. It is shown that the regular extremals defining the necessary conditions for Riemannian and sub-Riemannian curves can each be expressed as the classical simple pendulum. The regular extremal curves are solved analytically in terms of Jacobi elliptic functions and their projection onto the underlying base space of arbitrary curvature are explicitly derived in terms of Jacobi elliptic functions and an elliptic integral.

Keywords: Riemannian curves, sub-Riemannian curves, non-Euclidean space, optimal control.

I. INTRODUCTION

Let $G$ denote the 3-D isometry group of a simply connected surface $S$ of constant cross-sectional curvature $\varepsilon$, and let $A_1, A_2$ and $A_3$ denote a basis of left-invariant vector fields in the Lie algebra $\mathfrak{g}$ of $G$ with the Lie bracket $[X,Y] = XY - YX$ (with $X,Y \in \mathfrak{g}$ and where $XY$ denotes matrix multiplication of $X$ and $Y$) defined by the commutative relations $[A_1, A_2] = \varepsilon A_1$, $[A_2, A_3] = A_1$ and $[A_1, A_3] = -A_2$. Note that when $\varepsilon = 1, -1, 0$ we obtain the standard 3-D matrix Lie algebras i.e. $G$ is the Special Orthogonal Group $SO(3)$ with Lie algebra $\mathfrak{so}(3)$ when $\varepsilon = 1$, $G$ is the Hyperbolic Group $SO(1,2)$ with Lie algebra $\mathfrak{so}(1,2)$ when $\varepsilon = -1$, and $G$ is the Special Euclidean Group $SE(2)$ with Lie algebra $\mathfrak{se}(2)$ when $\varepsilon = 0$. In each of the standard cases the simply connected surfaces $S$ are the planar forms; the sphere $S^2$, the hyperbola $\mathbb{H}^2$ and the Euclidean plane $\mathbb{R}^2$ with each having constant cross sectional curvature of $\varepsilon = 1, -1$ and 0 respectively. In this paper we generalise to spaces of arbitrary constant cross sectional curvature, with $\varepsilon \in (-\infty,0) \cup (0,\infty)$ ensuring that $G$ is a semi-simple Lie group while the degenerate Euclidean case $\varepsilon = 0$ is considered as a limiting case. This paper considers the problem of minimizing quadratic functions of the form:

$$J = \frac{1}{2} \int_0^T \sum_{i=1}^n c_i v_i^2 \, dt$$

(1)

where $i \leq n \leq 3$ and $c_i > 0$ are constant weights and $v_i$ are functions on the interval $[0,T]$, satisfying the prescribed boundary conditions $g(0) = g_0$ and $g(T) = g_T$ where $g(t) \in G$ satisfies the differential constraint:

$$\frac{dg(t)}{dt} = g(t) \sum_{i=1}^n A_i v_i.$$  

(2)

This class of problem is associated with Riemannian geometry when $n = 3$ where the metric (the integrand of equation (1)) is a positive definite quadratic form defined on the entire Lie algebra. If the metric is defined only partially on the Lie algebra ($n < 3$) the problem is a sub-Riemannian one [1], [2], [3], [4], [5]. The Riemannian problem equates to a statement of the Principle of least action for a free rigid body if $c_1, c_2, c_3$ are equal to the principal moments of inertia, $v_1$ the angular velocities and $\varepsilon = 1$ ($G \in SO(3)$) [6], [7]. In this case the Hamiltonian vector fields defining the necessary conditions for optimality are the Euler equations. In this particular case the Hamiltonian equations of the free rigid body can be reduced to the classical simple pendulum equations under a cylindrical coordinate change of variables [7]. In this paper it is shown that the necessary conditions for optimality can be reduced to the equations of the simple pendulum for a larger class of optimal control problem.

In all other cases, other than the Riemannian problem, this problem statement is associated with sub-Riemannian geometry where the integrand of (1) defines only a partial metric on the Lie algebra ($n < 3$). Sub-Riemannian curves can also be defined equivalently by the Riemannian case but with any single weight $c_j \to \infty$. Note that no more than one constant weight can tend to infinity as for these cases the optimal control problem is not well posed. A particular class of sub-Riemannian curves, called p-curves, were studied in [1] where the partial metric is defined on the vertical vector fields $p$ of the Cartan decomposition. In [1] p-curves are studied for the classic planar forms where their curvature $\varepsilon = -1, 0, 1$. The p-curves, in [1], correspond to the limiting case where $c_3 \to \infty$ or equivalently setting $n = 2$ in equations (1) and (2). In this paper we generalise the analysis of p-curves, in [1], to spaces of arbitrary constant curvature.

Another potentially interesting case is where $A_1$ and $A_3$ in (2) are controlled and $A_2$ is not. In other words $v_2 = 0$ in equations (1) and (2) which corresponds to the limiting Riemannian case as $c_2 \to \infty$. In this case the differential constraint (2) can be viewed analogously to the kinematics of a wheeled robot with a nonholonomic (sliding) constraint where $v_1$ is the velocity in the forward direction and $v_3$ the angular (steering) velocity. It follows that the optimal control problem defines paths of a wheeled robot that minimises a weighted cost function of the forward velocity and the amount of required steering.

This paper solves the extremals for these Riemannian and sub-Riemannian curves in terms of Jacobi elliptic functions and shows that the equations can be reduced to the
classical pendulum through a simple coordinate change. An
integration method is then presented which generalises the
procedure used to project the extremals onto \( g(t) \in SO(3) \)
presented in [8] to spaces of arbitrary cross-sectional cur-
vature. This integration method is then applied to project
the extremals related to Riemannian and sub-Riemannian
curves onto the simply connected surface \( S \). This reveals that
Riemannian and sub-Riemannian curves are described by
Jacobi elliptic functions and an incomplete Elliptic integral
of the third kind.

II. NECESSARY CONDITIONS FOR (SUB-) RIEMANNIAN
CURVES

An application of Pontryagin’s maximum principle of op-
timal control (where the functions \( v_1, v_2, v_3 \) are the assumed
control functions) brings us to the associated (left-invariant)
Hamiltonian formalism. There is a wealth of literature on
the co-ordinate free Maximum principle and in line with the
geometric interpretations of this paper the interested reader
should refer to [8], [5], [9], [3] for details. Each left-invariant
Hamiltonian can be expressed independently of co-ordinates
on \( G \) as a function \( f \) of the extremal curves \( H = f(h_1, h_2, h_3) \)
where \( h_1, h_2, h_3 \in \mathbb{g} \) are the extremal curves and \( h_1 = p(A_i) \)
with \( p(\cdot) \) a scalar function which maps an element of the Lie
algebra to its dual defined through the non-degenerate trace
form (for \( \varepsilon \in (\infty, 0) \cup (0, \infty) \)). Explicitly, minimising the
cost function (1) subject to the constraint on the Lie algebra
from (2) gives the Hamiltonian:

\[
H = \sum_{i=1}^{n} h_i v_i - \rho_0 \sum_{i=1}^{n} c_i v_i^2 \quad (3)
\]

where \( \rho_0 = 0 \) for abnormal extremals and \( \rho_0 = 1 \) for regular
extremals. Proceeding in this paper with an analysis of the
regular extremals and noting that \( H \) is a concave function with
respect to \( v_i \) then the optimal controls are:

\[
v_i^* = \frac{h_i}{c_i} \quad (4)
\]

and substituting (4) into (3) gives the optimal Hamiltonian:

\[
H = \frac{1}{2} \left( \frac{h_1^2}{c_1} + \frac{h_2^2}{c_2} + \frac{h_3^2}{c_3} \right) \quad (5)
\]

where the Hamiltonian corresponds to the Riemannian prob-
lem for arbitrary non-zero constant values of \( c_i \) and to sub-
Riemannian problems whenever any single constant weight
\( c_i \rightarrow \infty \). The Hamiltonian vector fields are then given by
the equation \( X_H[v] = -\partial H / \partial v \) where the Poisson bracket on the
dual of the Lie algebra is defined in terms of the Lie bracket
as \( \{ h_i, h_j \} = -\rho(\{ A_i, A_j \}) \). Then the Hamiltonian vector fields
defining the necessary conditions for optimality are given by:

\[
\begin{align*}
    h_1 &= \{ H, h_1 \} = \frac{1}{c_1} h_2 h_3 - \frac{1}{c_2} h_1 h_3 \\
    h_2 &= \{ H, h_2 \} = \frac{1}{c_2} h_1 h_3 - \frac{1}{c_3} h_1 h_2 \\
    h_3 &= \{ H, h_3 \} = \frac{1}{c_3} h_1 h_2 - \frac{1}{c_1} h_2 h_3 
\end{align*}
\]

It is easily verified that the limiting cases of the Hamiltonian
vector fields as any single \( c_i \rightarrow \infty \), correspond to the limiting
cases of the Hamiltonian function, that is, the equations are
well behaved. For example, as \( c_3 \rightarrow \infty \) the Hamiltonian (5)
yields the Hamiltonian of general p-curves and (6) to the
conventional vector fields defining the necessary conditions
for the existence of p-curves. In addition, it is easily shown that the function:

\[
M = h_1^2 + h_2^2 + \varepsilon h_3^2 \quad (7)
\]
is a Casimir function for (6) i.e. \( \{ H, M \} = 0 \). Furthermore,
the intersection of these functions (that implicitly define surfaces) (7) and (5) geometrically define the extremal curves
[8]. In particular they are the intersection of an Ellip-
soid (Riemannian Case) or elliptic cylinder (sub-Riemannian
case) with an ellipsoid for \( \varepsilon > 0 \) or a hyperbola for \( \varepsilon < 0 \).
It is also well known that the smooth intersection of any
two quadric hypersurfaces in projective three space define
an elliptic curve [10] which can be parameterised by elliptic
functions. This gives us an indication to the form the analytic
solution the extremal solutions will take.

Lemma 1: The real extremal curves associated with Rie-
mannian and sub-Riemannian curves on 2-D simply con-
ected surfaces of constant cross sectional curvature are
described by the equation of the mathematical pendulum of
arbitrary length.

Proof:

define the constants

\[
\lambda_1 = \left( \frac{c_1 - c_3}{c_2 c_3} \right)^2, \quad \lambda_2 = \left( \frac{c_1 - c_2}{c_1 c_2} \right)^2, \quad \lambda_3 = \left( \frac{c_1 - c_2}{c_1 c_2} \right)^2 \quad (8)
\]

then (6) can be expressed as:

\[
(\dot{h}_1)^2 = \lambda_1 h_2^2 h_2^2, \quad (\dot{h}_2)^2 = \lambda_2 h_1^2 h_3^2, \quad (\dot{h}_3)^2 = \lambda_3 h_1^2 h_2^2, \quad (9)
\]

using equations (5) and (7) we can write:

\[
\begin{align*}
    h_2^2 &= \frac{c_3 - c_2}{c_1 - c_2} \left( 2c_2 H_0 + h_1^2 - \frac{(c_1 - c_3) h_2^2}{c_1} - M \right), \\
    h_3^2 &= \frac{c_2 - c_1}{c_1 - c_2} \left( 2c_1 H_0 + h_2^2 - \frac{(c_2 - c_1) h_3^2}{c_2} - M \right), \\
    h_1^2 &= \frac{c_1 - c_2}{c_1 - c_2} \left( 2c_1 H_0 + h_3^2 - \frac{(c_1 - c_2) h_1^2}{c_2} - M \right) \\
\end{align*}
\]

(10)

and again the expressions for the sub-Riemannian case are
the limits of these equations as any single \( c_i \rightarrow \infty \). For
example as \( c_3 \rightarrow \infty \) equation (10) become:

\[
\begin{align*}
    h_2^2 &= 2c_2 H_0 - c_2 h_2^2, \\
    h_3^2 &= \frac{M - 2c_2 H_0}{c_1} + \frac{(c_2 - c_1)}{c_1 c_2} h_3^2, \\
    h_1^2 &= 2c_1 H_0 - c_1 h_1^2 \quad (11)
\end{align*}
\]

then substituting in either (10) or (11) into (9) the
Riemannian and sub-Riemannian curves can be expressed
in the quadratic form:
\[(\dot{h}_i)^2 = \lambda_i(\alpha_i h_i^2 - \beta_i)(k_i h_i^2 - d_i), \quad (12)\]
where \(i = 1, 2, 3\) and \(\lambda_i\) are defined in (8) and for Riemannian curves
\[
\begin{align*}
\alpha_1 &= \frac{c_2(c_1 - c_3)}{c_3(c_1 - c_2)}, & \beta_1 &= -\frac{c_2 c_1}{c_3}, & d_1 &= \frac{2c_1}{c_3}, \\
\alpha_2 &= \frac{c_1(c_2 - c_3)}{c_3(c_1 - c_2)}, & \beta_2 &= \frac{2c_1}{c_3}, & d_2 &= \frac{2c_1}{c_3}, \\
\alpha_3 &= \frac{c_1(c_2 - c_3)}{c_3(c_1 - c_2)}, & \beta_3 &= \frac{2c_1}{c_3}, & d_3 &= \frac{2c_1}{c_3},
\end{align*}
\]
and for example sub-Riemannian curves when \(c_3 \to \infty\) are:
\[
\begin{align*}
\alpha_1 &= -\frac{c_1}{c_2 - c_1}, & \beta_1 &= -2c_1H, & d_1 &= 2c_1H, \\
\alpha_3 &= \frac{c_1(2Hc_2 - M)}{c_1 - c_2}, & k_3 &= \frac{c_1(2Hc_3 - M)}{c_1 - c_2}, & d_3 &= \frac{c_1(2Hc_1 - M)}{c_1 - c_2},
\end{align*}
\]
where \(i = 1\) when \(j = 2\) and \(i = 2\) when \(j = 1\). Then using the change of coordinates \(h_i = \sqrt{\varepsilon} \sin \theta_i\) in (12) yields the equation of the mathematical pendulum:
\[
\dot{\theta} = \pm \sqrt{\Lambda + B \cos \theta} \quad (15)
\]
where the constants \(\Lambda = (4a_i d_i - 2 \kappa_i b_i), B = 2k_i b_i\) where \(a_i = \lambda_i \alpha_i\) and \(b_i = \lambda_i \beta_i\). The recognition of the extremal curves qualitative behaviour as being determined by the mathematical pendulum enables the description of all possible qualitative behaviours of the (sub-) Riemannian curves. Setting \(I = \frac{ad_i}{b_i k_i}\) we define the qualitative behaviours as:
Case A: \(I = 0\) corresponds to the stationary position analogous to the downward position of the mathematical pendulum.
Case B: \(0 < I < 1\) corresponds to oscillatory motion analogous to a pendulum swinging back and forth.
Case C: \(I = 1\) corresponds to the equation of the separatrix connecting the two saddle points of the upward equilibrium position.
Case D: \(I > 1\) corresponds to circulating orbits where the pendulums energy is high enough to carry the pendulum over the top.

**Lemma 2:** The real extremal curves associated with Riemannian and sub-Riemannian curves on 2-D simply connected surfaces of arbitrary curvature for \(b_i k_i < a_i d_i\) are of the analytic form:
\[
h_i = \sqrt{\frac{b_i}{a_i}} \sin(z_i) \quad (16)
\]
where:
\[
z_i = am(\pm \sqrt{a_i d_i t + \beta_i b_i k_i / a_i d_i}) \quad (17)
\]
where \(am(\cdot, \cdot)\) is the Jacobi amplitude function \([12]\) and the constant \(\beta_i = \sin^{-1}(am(\sqrt{\varepsilon} H / \sqrt{K}))\) and for \(b_i k_i > a_i d_i\):
\[
h_i = \sqrt{d_i / k_i} \sin(z_i) \quad (18)
\]
where:
\[
z_i = am(\pm \sqrt{b_i k_i t + \gamma_i a_i d_i / b_i k_i}) \quad (19)
\]
where \(am(\cdot, \cdot)\) is the Jacobi amplitude function \([12]\) and the constant \(\gamma_i = \sin^{-1}(am(\sqrt{K} H / \sqrt{\varepsilon} K A_3))\).

Proof. It is easy to verify by substitution that this solves equation (12). Note that for \(b_i k_i > a_i d_i\) the Jacobi transformation is used \([11]\). □

Here we note that (16) corresponds to Case D of the classical simple pendulum and (18) corresponds to Case A. If \(b_i k_i = a_i d_i\) then each solution degenerates to a hyperbolic tan function defining the heteroclinic connection of Case C.

**Theorem 1:** Riemannian and sub-Riemannian curves on a simply connected surface \(S\) of cross sectional curvature \(\varepsilon \in (-\infty, 0) \cup (0, \infty)\) can be expressed in terms of the extremal curves \(h_1, h_2, h_3\) as:
\[
x = -\frac{h_1}{\sqrt{K^2 - h_3^2}} \cos \phi_1 - \frac{\sqrt{K} h_3}{K} \sin \phi_1
\]
\[
y = -\frac{h_1}{\sqrt{K^2 - h_3^2}} \sin \phi_1 + \frac{\sqrt{K} h_3}{K} \cos \phi_1
\]
\[
z = -\frac{\sqrt{h_3^2}}{K}
\]
where \(K^2 = M\) in (7) and
\[
\phi_1 = \frac{K \sqrt{\varepsilon} \left(\frac{h_1^2}{c_1^2} + \frac{h_3^2}{c_3^2}\right)}{h_1^2 + h_3^2} \quad (21)
\]
Proof: Recall that as the Hamiltonian for sub-Riemannian curves can be viewed as limits of the Riemannian case (5) it suffices to integrate the system down to \(G\) using the expression for the Hamiltonian (5). It is convenient to express the equations describing the extremal curves (6) and their relationship to \(g(t) \in G\) in Lax Pair form defined on the basis of the Lie algebra:
\[
A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\varepsilon & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\varepsilon & 0 \end{pmatrix},
\]
\[
A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
then the (sub-)Riemannian curves are defined by the equations:
\[
\frac{dL(t)}{dt} = [dH, L(t)], \quad \frac{dg(t)}{dt} = g(t) dH
\]
where \(L(t) = \sum_{i=1}^{3} h_i A_i, \quad dH = \sum_{i=1}^{3} \frac{d}{dt} A_i.\) It is easy to show by differentiation that
\[
g(t) L(t) g(t)^{-1} = \text{constant} \quad (24)
\]
It follows that if we define the conserved quantity \(K^2 = h_1^2 + h_2^2 + \varepsilon h_3^2\) then (24) can be conjugated such that
\[
g(t) L(t) g(t)^{-1} = \sqrt{\varepsilon} K A_3
\]
then defining \(g(t) \in G\) in the convenient form:
\[
g(t) = \exp(\phi_1 A_3) \exp(\phi_2 A_2) \exp(\phi_3 A_3)
\]
where $\phi_1, \phi_2, \phi_3$ are local coordinates then

$$L(t) = \sqrt{\varepsilon K g(t)^{-1} A_3 g(t)} \quad (27)$$

comparing with $L(t)$ gives:

$$\begin{pmatrix}
0 & -\varepsilon h_3 & h_1 \\
\varepsilon h_3 & 0 & h_2 \\
-h_1 & -h_2 & 0
\end{pmatrix} = \sqrt{\varepsilon K}(\hat{x} | \hat{y}) \hat{z} \quad (28)$$

where $\hat{x}, \hat{y}, \hat{z}$ are the vectors

$$\hat{x} = \begin{pmatrix} 0 & \cos(\sqrt{\varepsilon} \phi_2) & \sqrt{\varepsilon} \cos \phi_3 \sin(\sqrt{\varepsilon} \phi_2) \end{pmatrix}^T$$

$$\hat{y} = \begin{pmatrix} -\cos(\sqrt{\varepsilon} \phi_2) & 0 & -\sqrt{\varepsilon} \sin \phi_3 \sin(\sqrt{\varepsilon} \phi_2) \end{pmatrix}^T$$

$$\hat{z} = \begin{pmatrix} -\cos \phi_3 \sin(\sqrt{\varepsilon} \phi_2) & 0 & \varepsilon \sin \phi_1 \sin(\sqrt{\varepsilon} \phi_2) \end{pmatrix}^T \quad (29)$$

which yields

$$h_1 = -K \cos \phi_3 \sin(\sqrt{\varepsilon} \phi_2)$$

$$h_2 = K \sin \phi_3 \sin(\sqrt{\varepsilon} \phi_2)$$

$$h_3 = \frac{\alpha}{\varepsilon \sqrt{\varepsilon}} \cos(\sqrt{\varepsilon} \phi_2) \quad (30)$$

it follows that

$$\cos(\sqrt{\varepsilon} \phi_2) = \frac{\alpha h_3}{K}, \quad \sin(\sqrt{\varepsilon} \phi_2) = \frac{K^2 - \varepsilon h_3^2}{K} \quad (31)$$

and

$$\cos \phi_3 = -\frac{b_1}{\sqrt{K^2 - \varepsilon h_3^2}}, \quad \sin \phi_3 = \frac{b_3}{\sqrt{K^2 - \varepsilon h_3^2}} \quad (32)$$

these solutions will be used in conjunction with the following. First, we substitute equation (26) into $g(t)^{-1} \frac{dg(t)}{dt} = dH$ from (23) which yields:

$$\begin{pmatrix}
\frac{b_3}{c_3} = \cos(\sqrt{\varepsilon} \phi_2) \phi_1 + \phi_3 \\
\frac{b_2}{c_2} = \sin(\sqrt{\varepsilon} \phi_2) \phi_1 - \cos \phi_3 \phi_2 \\
\frac{b_1}{c_1} = -\sin \phi_3 \cos(\sqrt{\varepsilon} \phi_2) \phi_1 - \sin \phi_1 \phi_2
\end{pmatrix}$$

which on substitution of (31) and (32) simplifies to

$$\begin{pmatrix}
\frac{b_3}{c_3} = \frac{\alpha h_3}{K} \phi_1 + \phi_3 \\
\frac{b_2}{c_2} = \frac{\alpha \phi_3}{K^2} \phi_1 - \frac{b_3}{\sqrt{K^2 - \varepsilon h_3^2}} \phi_2 \\
\frac{b_1}{c_1} = \frac{\alpha \phi_3}{K^2} \phi_1 + \frac{b_3}{\sqrt{K^2 - \varepsilon h_3^2}} \phi_2
\end{pmatrix} \quad (33)$$

it follows that:

$$\phi_1 = \frac{K \sqrt{\varepsilon} \left( \frac{b_3}{c_1} + \frac{b_2}{c_2} \right)}{h_1^2 + h_2^2} \quad (34)$$

noting that the projection of $g(t) \in G$ (26) onto $S$ given by $g(t) | 1 \ 0 \ 0 \ 0 |^T$ is:

$$x = \cos \phi_1 \cos \phi_3 \cos \phi_2 - \cos \sqrt{\varepsilon} \phi_2 \sin \phi_3 \sin \phi_1$$

$$y = \cos \phi_1 \sin \phi_3 \cos \phi_2 + \cos \sqrt{\varepsilon} \phi_2 \sin \phi_3 \sin \phi_1$$

$$z = -\sin \sqrt{\varepsilon} \phi_2 \sin \phi_1 \sin \phi_3 \sin \phi_2 \quad (35)$$

then substituting (31), (32) and (35) into (36) gives (20). □

**Lemma 3:** The solution to the integral

$$\phi_1 = \frac{K \sqrt{\varepsilon} \left( \frac{b_3}{c_1} + \frac{b_2}{c_2} \right)}{h_1^2 + h_2^2} \quad (37)$$

with the extremal curves defined by (16) $(b_3 k_3 < a_3 d_3)$ is

$$\phi_1 = \frac{\alpha \sqrt{\varepsilon} k_3}{\sqrt{2} k_3} \frac{\gamma}{\sqrt{2} k_3} \Pi \left[ \frac{\gamma}{2 k_3}, \varepsilon a_3 d_3 + \beta_3, \frac{b_3 k_3}{a_3 d_3}, \frac{b_3 k_3}{a_3 d_3} \right] \quad (38)$$

where $\Pi[\cdot, \cdot, \cdot]$ is the incomplete elliptic integral [12] and $am(\cdot, \cdot)$ the Jacobi amplitude function with constants:

$$\alpha = 2HK \sqrt{\varepsilon}, \quad \Gamma = \frac{k_3 \sqrt{\varepsilon} b_3}{a_3 d_3}, \quad \gamma = \frac{\varepsilon b_3}{\varepsilon}$$

and with the extremal curves defined by (18) $(b_3 k_3 > a_3 d_3)$ is

$$\phi_1 = \frac{\alpha \sqrt{\varepsilon} k_3}{\sqrt{2} k_3} \frac{\gamma}{\sqrt{2} k_3} \Pi \left[ \frac{\gamma}{2 k_3}, \varepsilon a_3 d_3 + \beta_3, \frac{b_3 k_3}{a_3 d_3}, \frac{b_3 k_3}{a_3 d_3} \right]$$

with constants:

$$\alpha = 2HK \sqrt{\varepsilon}, \quad \Gamma = \frac{k_3 \sqrt{\varepsilon} b_3}{a_3 d_3}, \quad \gamma = \frac{\varepsilon b_3}{\varepsilon}$$

Proof: rearranging the differential equation (37) as an integral and using the Hamiltonian (5) and Casimir function (7), $\phi_1$ can be expressed in terms of $h_3$ as:

$$\phi_1 = \int \frac{K \sqrt{\varepsilon} \left( 2H - \frac{h_3^2}{c_3} \right)}{K^2 - \varepsilon h_3^2} \frac{1}{dt} \quad (42)$$

then substituting $h_3 = \frac{\alpha}{\sqrt{\varepsilon}} \sin(\pm a_3 d_3 t + \beta_3, \frac{b_3 k_3}{a_3 d_3})$ from (16) into (42) and integrating yields (38). Equation (40) is obtained in an analogous manner. □

**Lemma 4:** Riemannian and sub-Riemannian curves on non-Euclidean spaces of constant curvature $\varepsilon \in (-\infty, 0) \cup (0, \infty)$ are of the analytic form:

$$x = -\frac{\alpha \sqrt{\varepsilon} k_3}{\sqrt{2} k_3} \sin z_1 \sin \phi_3 \sin z_3 \sin \phi_1$$

$$y = -\frac{\alpha \sqrt{\varepsilon} k_3}{\sqrt{2} k_3} \sin \phi_1 + \sqrt{\varepsilon} b_3 \frac{b_3 k_3}{a_3 d_3} \sin z_3 \cos \phi_1$$

$$z = -\frac{\alpha \sqrt{\varepsilon} k_3}{\sqrt{2} k_3} \sin z_2 \quad (43)$$

where $z_i$ is the Jacobi amplitude function (17) for $b_3 k_3 < a_3 d_3$ and (19) for $b_3 k_3 > a_3 d_3$ and where $\phi_1$ is defined by (38) for $(b_3 k_3 < a_3 d_3)$ and (40) for $(b_3 k_3 > a_3 d_3)$.

**III. Conclusion**

In this paper a closed form solution for Riemannian and sub-Riemannian curves on non-Euclidean spaces of arbitrary curvature are derived. The projection of the curves onto the base space are expressed in terms of Jacobi elliptic functions and trigonometric functions of the sum of a secular term and an incomplete elliptic integral.

**References**


