A dual-terminal set based robust tube MPC for switched systems

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Abstract: This article considers the robust regulation problem for a class of constrained linear switched systems with bounded additive disturbances. The proposed solution extends the existing robust tube based model predictive control (RTBMPC) strategy for non-switched linear systems to switched systems. RTBMPC utilizes nominal model predictions, together with tightened sets of state and input constraints, to obtain a control policy that guarantees robust stabilization of the dynamic systems in presence of bounded uncertainties. Similar to RTBMPC for non-switched systems, a disturbance rejection proportional controller is used to ensure that the closed loop trajectories of the switched linear system are bounded in a tube centered on the nominal system. To account for the switching dynamics, the gain of this controller is chosen to simultaneously stabilize all switching dynamics. The RTBMPC for the switched system requires an on-line solution of a Mixed Integer Quadratic Program (MIQP). To reduce the complexity of the MIQP, a sub-optimal design is proposed, which considers the notion of a pre-terminal set in addition to the usual terminal set used to ensure stability. The RTBMPC design with the pre-terminal set aids in tuning the trade-off between the complexity of the control algorithm with the optimal performance of the closed-loop system while ensuring robust stability. Examples are presented to illustrate features of the proposed MPC.

Keywords: Hybrid dynamic systems, Switched systems, Robust tube based model predictive control (RTBMPC), Piece wise affine (PWA) systems

1. INTRODUCTION

Control of switched systems is challenging since they are characterized by simultaneous interactions between continuous dynamics and discrete events. Upon occurrence of an event, the system transits from one mode of the state space to another, wherein the state trajectory evolves based on a new dynamic map. The fact that these discrete modes of operations can be expressed as constraints, makes Model predictive Control (MPC) a natural choice for optimal control of switched systems.

Stabilizing MPC design for switched systems under nominal conditions is well studied (Grieder et al., 2005; Lazar et al., 2006). However, plant-model mismatch due to parametric variations, unmeasured disturbances as well as under-modelled dynamics, makes control of uncertain plants a challenging task. Min-max robust MPC formulations provide a straightforward but conservative solution to this problem. One way to reduce the conservativeness is by enabling feedback or closed loop in MPC, wherein the control moves are parameterized as a sequence of control laws that reduce the band of uncertainty (Kothare et al., 1996). However, the computational burden is prohibitive in this case since the resulting optimization problem becomes non-convex and may require enumeration strategies (Scokaert and Mayne, 1998). Other robust MPC design methods in literature, which are relevant in context of switched systems include Lyapunov-based bounded robust controllers, which explicitly characterize the region of robust closed-loop stability (Mhaskar et al., 2005) and robust formulations of MPC based on tightened constraint sets (Lazar, 2006).

Mayne et al. (2005) proposed a strategy where the optimization problem is separated from robustness related issues to reduce conservativeness and aid computational tractability. This essentially involves optimizing the cost corresponding to the evolution of the nominal dynamics, which is free from disturbances and bound its deviation from the uncertain dynamics by a sequence of invariant state space regions called robust reachability/invariant tubes. Such an approach and its variants ensure robust stability and are collectively called RTBMPC algorithms (Chisci et al., 2001; Mayne et al., 2005). However, for RTBMPC to be an acceptable solution for control of switched systems, it should be computationally tractable in addition to ensuring robust stability. Stabilizing formulations of RTBMPC parameterize the inputs over a sufficiently large but finite horizon to ensure that the state trajectory reaches a tightened terminal constraint set, beyond which a pre-determined state feedback control law becomes active. In case of switched systems, a large horizon necessitates the use of numerous binary variables, needed to track modes of the switched system in the on-line optimization problem, leading to a Mixed Integer Program (MIP). It is well-known that the worst-case com-

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utputational complexity of a combinatorial search algorithm used to solve the MIP, such as in Branch and Bound, has the order of $O(2^{n_{\text{binary}}})$, where $n_{\text{binary}}$ is the total number of binary variables involved in the optimization problem (Nemhauser and Wolsey, 1988). It is, therefore, of interest to limit the complexity of the control problem so as to make it practically implementable, while ensuring robust stability of the closed-loop system.

To address computational tractability and robust stability simultaneously for linear switched systems, the current study proposes two RTB MPC formulations: (i) a direct extension of RTB MPC, developed in literature in context of non-switched systems (Mayne et al., 2005), to switched systems; (ii) a sub-optimal formulation involving a dual terminal set, which can tune the computational complexity of the problem resulting in a trade-off between performance and computational burden. Novel elements of the proposed work to the best of the authors’ knowledge are as follows: (i) extension of RTB MPC design for linear and non-linear systems (Mayne et al., 2005; Cannon et al., 2011) to switched systems; (ii) the proposed RTB MPC brings in a notion of a pre-terminal set in context of RTB MPC of switched systems. The pre-terminal set corresponds to a fixed mode of the switched system and is that part of the state space where inputs have enough power to retain the state trajectory within its confines and hence within the given mode. This allows omitting the binary variables associated with the trajectory as soon as it enters the pre-terminal set. This, in turn, reduces the binary decision variables in the proposed RTB MPC optimization problem, thereby reducing the computational complexity.

The price paid is that the trajectory of evolution is constrained to lie in the pre-terminal set thereby making the performance sub-optimal. The notion of multiple terminal sets in not new and has been used in context of move blocking MPC (Gondhalekar et al., 2009) where a controlled invariant feasible set is used as a pre-terminal set to achieve feasibility of MPC formulation. The proposed RTB MPC with dual terminal set uses three invariant sets, namely a terminal set to ensure stability, a pre-terminal set to achieve computational benefits and a robust invariant tube to aid bounded evolution of the trajectories in presence of bounded uncertainties.

This article is organized as follows: preliminaries of RTB MPC are presented in section 2, section 3 presents the details of the proposed RTB MPC formulation; section 4 presents simulation examples to illustrate features of the present work; and section 5 gives concluding remarks.

**Notation:** Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\emptyset$ denote the set of real numbers, non-negative real numbers, integers, non-negative integers, and the null set, respectively. $\mathbb{Z}_{\geq c}$ and $\mathbb{Z}_{[a,b]}$ are used to denote the sets $\{k \in \mathbb{Z}|k \geq c\}$ and $\{k \in \mathbb{Z}|a \leq k \leq b\}$, respectively. Given two sets $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^m$, then (i) their Minkowski sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v | u \in \mathcal{U}, v \in \mathcal{V}\}$; and (ii) their Pontryagin set difference is defined as $\mathcal{U} \odot \mathcal{V} \triangleq \{u | u \in \mathcal{U}, v \notin \mathcal{U}\}$. The spectral radius of a matrix $A$ is represented by $\rho(A)$. For any arbitrary set $S$, its $i^{th}$ Cartesian power is defined as, $S^i \triangleq S \times \cdots \times S$ and $\text{int}(S)$, represents set interior.

## 2. PRELIMINARIES

Let the state space be partitioned into non-overlapping polyhedral regions (not necessarily closed) called modes $M_j$, $j \in J \subset \mathbb{Z}_+$, an index set representing the modes of operation. Let $J_0 \subset J \subset \mathbb{Z}_+$ be the index set that represents the collection of modes containing the origin $0$, which is the equilibrium state. The evolution of trajectories of the uncertain switched system in mode $M_j$ is given as,

$$x_{k+1} = A_j x_k + B_j u_k + f_j + w_k, \quad x_k \in M_j, \quad j \in J \quad (1)$$

$$x_k \in \mathcal{X} \subset \mathbb{R}^n, \quad u_k \in \mathcal{U} \subset \mathbb{R}^m, \quad w_k \in \mathcal{W} \subset \mathbb{R}^m \quad (2)$$

where $x_k$, $u_k$ and $w_k$ represent state, input and disturbance, respectively at the $k^{th}$ sampling instant, $k \in \mathbb{Z}_+$. The sets $\mathcal{U}$ and $\mathcal{W}$ are assumed to be compact and polyhedral while the set $\mathcal{X}$ is assumed to be closed and polyhedral. Furthermore, the origin $0 \in \text{int}(\mathcal{X})$, $\text{int}(\mathcal{U})$, $\text{int}(\mathcal{W})$. Note that if $j \in J_0$ then the corresponding $f_j = 0$. The assumption of additive uncertainty is not restrictive as it can be used to represent unmodelled dynamics and multiplicative uncertainties. To characterize switching, a switching function is defined in the following way: $\sigma_j : \mathcal{X} \rightarrow J$ such that,

$$\sigma_k \triangleq j, \quad \text{if} \quad x_k \in M_j$$

$$(A_j, B_j) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \text{ is assumed to be stabilizable } v_j \in J.$$ As the hybrid state space spans over different modes of operation, $\mathcal{X}$ can be partitioned into $\mathcal{X}_j \triangleq \left(\mathcal{X}(M_j), j \in J \right)$ and $\bigcup_j \mathcal{X}_j = \mathcal{X}$. For any $N \in \mathbb{Z}_{\geq 1}$, let $x_N^N(x_k, u_k, w_k) \triangleq (x_{k+1}/k, \ldots, x_{k+N}) \in \mathcal{X}^N$ represent the state sequence generated by applying the admissible input sequence $u_N^N \triangleq (u_{k/1}, \ldots, u_{N-1/k}) \in \mathcal{U}^N$ to the system Eqs. (1,2) with initial state $x_k \triangleq x_{k/k}$ under an admissible disturbance sequence $w_N^N \triangleq (w_{k/k}, \ldots, w_{N-1/k}) \in \mathcal{W}^N$.

### 3. RTB MPC FOR SWITCHED SYSTEMS

The main objective of this study is to obtain a stabilizing, robust MPC policy for the uncertain switched system Eqs. (1,2) with a horizon $N$. The main idea in RTB MPC is to obtain a control law for a system affected by disturbances using nominal predictions such that the uncertain system trajectory is bounded by a tube around the nominal trajectory.

**Definition 1. Nominal switched system:** The nominal switched system in mode $M_j$ corresponding to Eqs. (1,2) is defined by ignoring effect of uncertainties as follows,

$$x_{k+1} = A_j x_k + B_j u_k + f_j, \quad x_k \in M_j, \quad j \notin J_0$$

$$x_k \in \mathcal{X} \subset \mathbb{R}^n, \quad u_k \in \mathcal{U} \subset \mathbb{R}^m \quad (5)$$

Sets $\mathcal{X}$ and $\mathcal{U}$ are defined later. For any $N \in N_{\geq 1}$, $x_N^N(x_k, u_k) \triangleq (x_{k+1}/k, \ldots, x_{k+N}) \in \mathcal{X}^N$ represents the state sequence generated by applying the admissible control sequence $u_N^N \triangleq (u_{k/1}, \ldots, u_{N-1/k}) \in \mathcal{U}^N$ to the nominal system Eqs. (4,5), with initial state $x_k \triangleq x_{k/k}$.

Injecting the admissible control sequence, $u_N^N$, into the corresponding uncertain system will result in system trajectories that differ from nominal predictions even for identical initial conditions due to the presence of disturbances. To compensate the effect of this mismatch, it is desirable that the injected control law $u_k \triangleq \kappa(x_k)$ forces the uncertain system trajectories to lie close to the nominal trajectory.
This can be achieved by augmenting the nominal system control with a disturbance rejection controller as follows,

\[ u_k = u_k + K(x_k - \mathbf{x}_k) \]  

(6)

under which the uncertain system trajectory evolves as follows,

\[ x_{k+1} = (A_j + B_j K)x_k + w_k, \quad x_k \in M_j, j \in J \]  

(7)

where the disturbance rejection controller \( K \in \mathbb{R}^{m \times n} \) is chosen to ensure that the closed loop system in Eq. (7) is stable. In order to characterize the bounds of the uncertain system trajectory, a set namely, minimal Robustly Positively Invariant set (mRPI), is defined that determines whether the state trajectory of Eq. (7) satisfies constraints Eq. (2) under all admissible disturbance realizations.

**Definition 2. minimal Robustly Positively Invariant (mRPI) Set** \( (F_\infty) \) (Rakovic et al., 2005) : The set \( \mathcal{F} \subset \mathcal{X} \subset \mathbb{R}^n \) is a Positively Invariant (RI) set of Eq. (7), if \( (A_j + B_j K)x + w \in \mathcal{F} \) for all \( x \in \mathcal{F} \) and all \( w \in \mathcal{W}, j \in J \). Then the mRPI set of Eq. (7), \( F_\infty \), is the RI set in \( \mathbb{R}^n \), that is contained in every closed RI set \( \mathcal{F} \) of Eq. (7).

**Remark 3.** Asymptotic stability of the origin of the dynamic system Eq. (7) cannot be established due to the effect of disturbances. Since the zero initial condition response of Eq. (7) is bounded in \( F_\infty \), hence it may be regarded as the origin set of the uncertain system.

The Lemma below helps us obtain the disturbance rejection controller \( K \).

**Lemma 4.** Suppose \( \exists K \in \mathbb{R}^{m \times n} \) such that \( \rho(A_j + B_j K) < 1 \), \( \forall j \in J \), that is, it is simultaneously Hurwitz. Let \( A_j = A_j + B_j K') \) such that \( \rho(A_j' + B_j K') = \max \rho(A_j + B_j K) \), \( j', j \in J \). Furthermore, let \( F_\infty \) be the mRPI set for the dynamics \( x_{k+1} = A_k x_k + w_k \). If \( x_k \in x_k + F_\infty \) with \( u_k \equiv K(x_k) = u_k + K(x_k - \mathbf{x}_k) \), then \( x_{k+1} \in x_{k+1} + F_\infty \), \( \forall w_k \in \mathcal{W} \).

**Proof** The design of \( F_\infty \) is based on stable dynamics with the largest eigenvalue strictly in the unit disk. Thus, the disturbance rejection controller \( K \) can stabilize the dynamics corresponding to all the modes of operation. Hence \( F_\infty \) admits the worst-case disturbance invariance corresponding to all modes. So, using the Proposition 1 in Ref. (Mayne et al., 2005), the above claim is obvious. □

Since the switched system has different dynamics corresponding to different modes of operation, the set \( (x_k + F_\infty) \) that captures the uncertainty at the \( k^{th} \) time instant, can lie in different modes. This is because a disturbance realization can perturb the trajectory to the new modes. Thus, by choosing \( K \) based on worst-case dynamics, \( F_\infty \) admits disturbance invariance for all possible disturbance realizations.

**Remark 5.** In view of the constraint \( x_k \in x_k + F_\infty \) and control policy, \( u_k = u_k + K(x_k - \mathbf{x}_k) \), the state and input constraint sets of the nominal system should be tightened as: \( \mathbf{u}_k \in \mathcal{U} \supseteq (U \cup K F_\infty) \) and \( \mathbf{x}_k \in \mathcal{X} \supseteq (X \cup K \mathcal{F}_\infty) \), respectively (Mayne et al., 2005). This ensures that the uncertain dynamics do not violate the constraints \( x_k \in X \) and \( u_k \in \mathcal{U} \) for all possible realizations of \( w_k \in \mathcal{W} \).

Next, we define the maximal controlled positively invariant set for the origin containing modes, which is the terminal set of the nominal system Eqs. (4,5):

**Definition 6. Maximal Controlled Positively Invariant Set** of the nominal system \( (X_\infty) \) (Blanchini, 1999): A set \( X_\infty \subset (X_\infty \cup F_\infty) \) is said to be controlled positively invariant set of mode \( M_i \) for the nominal system Eqs. (4.5) if \( x_k \in X_\infty \), \( \exists K \in (U \cup K F_\infty) : (A_j + B_j K)x_k \in x_j, j \in J_0 \), where \( K \mathcal{X} \) is the state admissible stabilizing feedback law. Then \( X_\infty \) is defined as the union of all such \( X_{j} \), that is, \( X_\infty \supseteq \cup_{j \in J_0} X_j \).

Since the control law, \( K \mathcal{X} \) is stabilizing, the matrix \( A_K = (A_j + B_j K) \) is Hurwitz. Hence all trajectories that enter inside \( X_\infty \) eventually converge to the the equilibrium point. Note that for a linear, non-switched system one can use the optimal linear quadratic control law in place of \( K \mathcal{X} \). In this work, we use a linear state feedback law that simultaneously stabilizes all the modes \( j \in J_0 \), which is obtained as a solution of a common Lyapunov function corresponding to all modes of operation (Lazar et al., 2006). Next, the \( N- \) step stabilizable set and a feasible input sequence set corresponding to nominal system are defined.

**Definition 7. N-Step stabilizable set of the nominal system** \( (S_N(\mathcal{X}, \Gamma)) \) (Kerrigan and Maciejowski, 2000): This set is defined as: \( S_N(\mathcal{X} \cup F_\infty, \Gamma) \equiv \{ x_0 \in (X \cup F_\infty) : \exists \mathbf{u}_N \in (U \cup K F_\infty)^N, x_N(x_k, \mathbf{u}_N) \in (X \cup F_\infty), x_{k+1} \in \Gamma, i \in \mathbb{Z}[0,N]\} \), where \( \Gamma \) is a control invariant subset of \( (X \cup F_\infty) \). This set consists of those \( x_0 \in (X \cup F_\infty) \), which can be steered to \( \Gamma \) in \( N \) steps or less.

**Definition 8. Feasible input sequence set** \( \mathcal{U}_N(\mathcal{X}_k) \): The set of feasible input sequences \( \mathcal{U}_N(\mathcal{X}_k) \), for the system Eq. (4.5) is defined as: \( \mathcal{U}_N(\mathcal{X}_k) \equiv \{ \mathbf{u}_N \in (U \cup K F_\infty)^N : x_N(x_k, \mathbf{u}_N) \in (X \cup F_\infty), x_{k+1} \in \mathcal{X}_k \} \).

Based on the ingredients described above RTB MPC formulation for the switched system Eqs.(1.2), which is a direct extension of the RTB MPC proposed by (Mayne et al., 2005) for linear systems, is described as follows:

**Problem 9. RTB MPC Problem P1:** For \( N \in \mathbb{Z}_{\geq 2} \), let \( F : \mathbb{R}^n \to \mathbb{R}_+, \quad L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+ \) and let \( x_k \in \mathcal{X} \) for \( k \in \mathbb{Z}_+ \) be given. Then the MPC problem minimizes the following objective function:

\[ \Phi^N_{\text{mpc}}(x_k, \mathbf{u}_N) = \sum_{i=0}^{N-1} L(x_{k+i}, \mathbf{u}_{k+i}) + F(x_{k+N}) \]  

(8)

over all possible input sequences \( \mathbf{u}_N \in \mathcal{U}_N(\mathcal{X}_k) \) subject to \( x_k \in x_k + F_\infty \), where \( L(\cdot) \) is the stage cost and \( F(\cdot) \) is the terminal cost with \( F(0) = 0 \) and \( L(0,0) = 0 \). We consider, \( L(x, \mathbf{u}) = (1/2)(x^T Q x + 1/2)(\mathbf{u}^T R \mathbf{u}) \) and \( F(x) = (1/2)(x^T P x) \), where \( Q, R \) and \( P \) are positive definite matrices. To achieve stability, terminal cost \( F(x) \) and terminal set \( X_\infty \) satisfy the following axioms (Mayne et al., 2005): (A1) \( (A_j + B_j K)x_k \subset X_c, X_c \subset (X \cup F_\infty) \neq \emptyset, KX_c \subset (U \cup K F_\infty) \neq \emptyset \). (A2) \( L(x, K x) + F(A_K x) - F(x) \leq 0, \forall x \in X_\infty \).
Remark 10. If \( \mathbf{x}_k \in S_N((x_k \oplus (-F_\infty)) \cap (X \ominus F_\infty), \mathcal{X}_k) \neq \emptyset \), then \( \mathbf{u}^N_k(\mathbf{x}_k) \neq \emptyset \) and thus MPC problem \( P_1 \) is feasible.

Definition 11. RTB MPC value function of \( P_1 \) \((V^N_{(P1)}): \)

The value function corresponding to \( P_1 \) is defined as

\[
V^N_{(P1)}(\mathbf{x}_k) : = \inf_{\mathbf{u}_k \in x_k \oplus (\mathcal{X} \ominus F_\infty)} \Phi^N_{(P1)}(\mathbf{x}_k, \mathbf{u}_k)
\]

This infimum yields the optimal input sequence \( \mathbf{u}^N_k(\mathbf{x}_k) \).

Thus, at the \( k \)th instant, the implicit RTB MPC law injects the control law \( \mathbf{u}(\mathbf{x}_k) = \mathbf{u}^N_k(\mathbf{x}_k) + \mathbf{K}(\mathbf{x}_k - \mathbf{x}_k) \) into the plant. Note that, during state propagation, the binary variables that capture modes of the switched system must also be defined over the horizon \( N \), that is, until the state trajectory reaches inside the terminal set \( \mathcal{X}_\infty \).

Based on Remark 13, it is proposed to split the horizon \( N \) of problem \( P_1 \) into two parts, namely (i) \( N_1 \in \mathbb{Z}_{\geq 1} \), the number of control moves needed for the trajectory to reach \( \mathcal{M}_{j_1}^x, j \in J_0 \) from a feasible initial condition \( \mathbf{x}_k \), and (ii) \( h \in \mathbb{Z}_{\geq 1} \), the number of samples needed subsequently to reach the terminal set \( \mathcal{X}_\infty \). By Definition 12, once the state trajectory enters inside \( \mathcal{M}_{j_1}^x \), the inputs have enough power to confine the trajectory to that particular mode’s maximal controllable set, thereby eliminating the need for binary variables to track mode changes beyond \( k + N_1 \) and thus reducing the on-line complexity of the optimization problem. In the proposed formulation, once trajectories are inside \( \mathcal{M}_{j_1}^x \), \( j \in J_0 \), constraints corresponding to \( \mathcal{M}_{j_\infty}^x \) are invoked. As a result, all future course of the trajectories are constrained to lie in \( \mathcal{M}_{j_\infty}^x \) until they reach \( \mathcal{X}_\infty \).

Definition 14. Robust pre-terminal set: The set \( S_h((\bigcup_{j \in J_0} \mathcal{M}_{j_1}^x), \mathcal{X}_\infty) \) is defined as the robust pre-terminal set.

Definition 15. Feasible input sequence \( \mathbf{u}^{N_1+h}_{+h}(\mathbf{x}_k) \):

The set of feasible input sequences \( \mathbf{u}^{N_1+h}_{+h}(\mathbf{x}_k) \), for the system Eqs. (4,5) is defined as follows:

\[
\mathbf{u}^{N_1+h}_{+h}(\mathbf{x}_k) = \{ \mathbf{u}^{N_1+h}_{+h}(\mathbf{x}_k) \in (\bigcup_{k \in \mathcal{X} \ominus F_\infty})^{N_1+h} : \mathbf{x}^{N_1+h}_{+h}(\mathbf{x}_k, \mathbf{u}^{N_1+h}_{+h}(\mathbf{x}_k)) \subseteq (X - F_\infty)^{N_1+h}, \mathbf{x}_{k+N_1+i+k} \in \mathcal{S}_h((\bigcup_{j \in J_0} \mathcal{M}_{j_1}^x), \mathcal{X}_\infty), \mathbf{x}_{k+N_1+h+j/k} \in \mathcal{X}_\infty, \ i, j \in \mathbb{Z}_{\geq 1} \}
\]

The MPC problem \( P_1 \) is modified as follows:

Problem 16. MPC Problem \( P_2 \) (Modified Problem \( P_1 \)): For \( N_1, h \in \mathbb{Z}_{\geq 1}, \) let \( F : \mathbb{R}^n \mapsto \mathbb{R}^n, \) \( L : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n \), and the MPC problem \( P_2 \) minimizes the following objective function:

\[
\min_{\mathbf{u}} \\Phi_{(P2)}(\mathbf{x}_k, \mathbf{u}^{N_1+h}_{+h}(\mathbf{x}_k)) = \sum_{i=0}^{N_1+h-1} \sum_{j=i}^{N_1} L(\mathbf{x}_{k+i/j}, \mathbf{u}_{k+i/k}) + F(\mathbf{x}_{k+N_1+h/k})
\]

The value function \( V^{N_1+h}_{(P2)} \) for \( P_2 \), is defined analogous to \( V^{N}_{(P1)} \). The introduction of the maximal mode controllable set \( \mathcal{M}_{j_\infty}^x \) and the subsequent splitting of the horizon are the key distinguishing features of the \( P_2 \) formulation over \( P_1 \), which enables a trade-off between closed-loop performance and computational efficiency. The use of two terminal sets in \( P_2 \) formulation results in possible performance degradation only in the robust pre-terminal set, where the constraints \( \mathbf{x}_{k+N_1+i+k} \in \mathcal{S}_h((\bigcup_{j \in J_0} \mathcal{M}_{j_1}^x), \mathcal{X}_\infty), \ i \in \mathbb{Z}_{\geq 1} \) are enforced. Note that if a single horizon is used in MPC problem \( P_2 \) with the terminal set \( \mathcal{X}_\infty \), the formulation will reduce to that of \( P_1 \). Given a fixed initial condition, a practical notion of feasibility consists of determining sufficiently large horizon lengths \( N \) and \( N_1 + h \) for \( P_1 \) and \( P_2 \), respectively. In \( P_2 \), due to the introduction of the pre-terminal set, we are assured of feasibility once \( N_1 \) is sufficiently large. On the other hand, in \( P_1 \), \( N \) must be sufficiently large so that the trajectory enters the terminal set \( \mathcal{X}_\infty \). Thus, for comparison purposes, one may set \( N_1 = N \), that is, the number of stages needed to achieve feasibility of \( P_2 \) for a fixed initial condition. Under such circumstances, the set of feasible initial conditions of \( P_2 \) is larger than that of \( P_1 \). The proof is omitted for brevity.

Lemma 18. If \( P_2 \) is feasible for an initial condition \( \mathbf{x}_k \) at \( k \)th instant, then \( P_2 \) is also feasible for the initial condition \( \mathbf{x}_{k+1} = \mathbf{A}_1 \mathbf{x}_k + \mathbf{B}_1 \mathbf{u}^{N_1+h}_{+h}(\mathbf{x}_k) + \mathbf{w} \), at \( k + 1 \) time instant.

Theorem 19. Suppose there exists a feasible initial condition for \( P_2 \), that is, \( \mathbf{x}_k \in S_{V_1}(\mathbf{x}_k \oplus (\mathcal{X} \ominus F_\infty)) \cap (\bigcup_{j \in J_0} \mathcal{X}_{\infty}), \), \( \mathcal{S}_h((\bigcup_{j \in J_0} \mathcal{M}_{j_1}^x), \mathcal{X}_{\infty}) \) with \( N_1, h \in \mathbb{Z}_{\geq 1} \) and axioms A1 and A2 hold. Then the origin of the closed-loop system, using the MPC law resulting from Problem \( P_2 \), is exponentially stable.

Proofs of Lemma 18 and Theorem 19 are omitted for brevity. Since \( P_1 \) is a direct extension of RTB MPC presented in (Mayne et al., 2005), similar stability arguments hold for \( P_1 \). The mode through which the trajectory enters
the pre-terminal set, $M_{j\infty}$, is determined online. Alternatively, one can use a user specified $M_{j\infty}$, $M_{j\text{user}}$, $\ldots$ times for solving $P_2$ with two different algorithms, namely B&B and B&C is shown in Fig. (2b) for various $N_1$ values.

4. ILLUSTRATIVE EXAMPLES

We illustrate some features of the proposed RTBMPMC using two examples of linear switched systems, which are modeled as piecewise affine systems. In the examples, sets $\mathcal{F}_\infty$ and $\mathcal{X}_\infty$ are computed using algorithm presented in Refs. (Rakovic et al., 2005; Lazar et al., 2006), respectively. The stabilizing controller gain ($K$) and terminal weight ($P$) are computed by solving a common Lyapunov function whose domain of attraction characterizes the terminal set (Lazar et al., 2006). The set $M_{j\infty}$ is computed using the algorithm given in Blanchini (1999). Disturbance rejection controller $K$ is obtained by solving a nonlinear optimization problem such that it is simultaneously stabilizes dynamics of all system modes.

4.1 Example 1: Origin in the interior of a mode

Consider the following 2-dimensional PWA affine system with four modes of operation which is a modification of the example reported in Bemporad and Morari (1999):

$$x_{k+1} = A_j x_k + B u_k + f + w$$

subject to $x_k \in X = [-20,20] \times [-20,20] \; u_k \in U = [-3,3] \; \text{and} \; w \in W = [-0.1,0.1]$, where $A_1 = A_3 = \begin{bmatrix} 0.35 & -0.602 \\ -0.602 & 0.35 \end{bmatrix}$, $A_2 = A_4 = \begin{bmatrix} 0.35 & 0.602 \\ -0.602 & 0.35 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2 = f_3 = f_4 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$, $f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The state space partition of the system is given by $E_1 = -E_3 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $E_2 = -E_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

It is clear that the origin lies in the interior of mode 1 since $f_1$ is zero. The MPC cost is chosen as the quadratic norm with weights $Q = 10I(2 \times 2)$, $R = 1$, which makes the MPC optimization problem a Mixed integer quadratic program (MIQP). Soft constraints on input moves ($\mathbf{u}_k - \mathbf{u}_{k-1}$) are also added to restrict aggressive input moves, with a weighting matrix $100I(2 \times 2)$. Sets $\mathcal{X}_\infty$ and $\mathcal{M}_\infty$ for the the nominal system are computed using the state space matrices $A_1$ and $B$, and shown in Fig. (1) in green and magenta colors, respectively. $\mathcal{F}_\infty$ is represented in black color. To highlight the reduction in the feasible space due to disturbances, set difference between ($\mathcal{X}_\infty \oplus \mathcal{F}_\infty$) and $\mathcal{X}_\infty$ as well as ($\mathcal{M}_\infty \oplus \mathcal{F}_\infty$) and $\mathcal{M}_\infty$ are represented as yellow areas. After entering inside the pre-terminal set binary variables are dropped. Once the state trajectory is inside terminal set $\mathcal{X}_\infty$, stabilizing linear parameterized state feedback law, which is computed as the unconstrained LQR state feedback solution of the state transition matrices $A_1$ and $B$, is employed until the origin is reached and is found to be: $K = [0.4123 \; -0.5131]$. The disturbance rejection controller is computed by finding $K$ that simultaneously stabilizes the dynamics of all four modes and obtained as $K = [0.69 \; 0.45]$. Terminal weighting matrix $P$ that stabilizes the MPC is found by solving the Lyapunov equation corresponding to $\mathcal{X}_\infty$ as $P = \begin{bmatrix} 10.3212 & 1.0044 \\ -1.0044 & 15.4763 \end{bmatrix}$.

Fig. 1. Response of $P_2$, with $N_1 = 1, h = 5$.

Fig. 2. Comparison of value functions of $P_2$ and $P_1$. PWA affine model, given in Eq. (11) is converted into Mixed Logical Dynamical Model using the HYSDEL Toolbox (Torrisi and Bemporad, 2004). It is found that 12 binary variables are needed to characterize the mode evolution. For the initial condition vector $x_k = [1 \; -18]^T$, minimum number of steps ($N_1$) needed to reach the pre-terminal set is found to be 1. The number of moves ($h$) required to reach the terminal set $X_\infty$ from the pre-terminal set is found to be 5. For a prediction horizon of 6, this amounts to $6 \times 12 = 72$ binary decision variables for $P_1$. Since the problem $P_2$ is feasible with $N_1 = 1$ itself, the number of binary decision variables are needed only 12, which reduces the combinatorial complexity of the integer node fathoming strategy of the search tree in the MIP algorithm. Both MIQP problems $P_1$ and $P_2$ are solved using Tomlab using CPLEX solver (branch and cut) and MIQPBB (branch and bound) integrated with MATLAB 7.4.0. To illustrate the sub-optimality of $P_2$ compared to $P_1$, the value functions for the same initial condition is shown in Fig. (2a) for various values of $N_1$. Comparison of CPU times required to solve $P_2$ shows that the computation time for solving $P_1$ is higher than that for solving $P_2$, on a machine with Intel Quad-core Processor with 4GB RAM. The computation times for solving $P_2$ with two different algorithms, namely B&B and B&C is shown in Fig. (2b) for various $N_1$ values.
4.2 Example 2: Origin at the intersection of modes

This example consists of a non-trivial problem, in which the origin lies at the intersection of various modes. Let

\[
\begin{align*}
A_1 &= \begin{bmatrix} 0.6324 & 0.2785 \\ 0.0975 & 0.5469 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.6555 & 0.7060 \\ 0.1712 & 0.0318 \end{bmatrix},
\end{align*}
\]

with \( E_{b_i} = 0, f_i = 0, i = 1, 2, 3, 4 \) for the PWA system Eq. (11), with all other parameters identical to Example 1. The stabilizing state feedback law \( K \) and the terminal weight \( P \) are computed based on a common Lyapunov function for all modes \( M_j, j \in J_0 \) to yield a common \( P = \begin{bmatrix} 5.3678 & 0.4886 \\ 0.4886 & 6.5621 \end{bmatrix} \) and the common \( K = \begin{bmatrix} 0.5958 & 0.4091 \end{bmatrix} \). The disturbance rejection controller gain is obtained as \( R = [-0.57 1.65] \). The MIQP problem is solved using CPLEX B&C solver integrated with MATLAB. For the initial condition vector \( x_k = [-18 -17]^T \in M_2 \), one move \( (N_1 = 1) \) is sufficient to reach the pre-terminal set corresponding to mode 2 from where corresponding constraints get activated. The number of moves needed to reach the terminal set from the pre-terminal set is found to be four \( (h = 4) \) to obtain a feasible solution. This trajectory is represented by dash-dotted lines with diamond in Fig 3. The optimal trajectory for another initial condition \( x_k = [16 17]^T \in M_1 \), is also shown (solid line with triangles). Here, the system trajectory reaches the pre-terminal set corresponding to mode 4 in one step \( (N_1 = 1) \) and then takes one more step to reach the terminal set.

5. CONCLUSIONS

This paper presents two RTBMPC formulations for linear switched systems with bounded additive uncertainties (i) a direct extension of RTBMC of linear systems to switched systems and (ii) a formulation that integrates the computational aspects of RTBMPC in the control design itself. This gives the user an additional handle to control the complexity of the algorithm rather than rely on the optimization solver, while guaranteeing stability.

REFERENCES


