Robust control for a multi-stage evaporation plant in the presence of uncertainties

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Abstract—A Multi Stage Flash evaporation plant is investigated as a partly unknown process represented by structured uncertainties concerning several model parameters. Robust control designs in form of $H_\infty$, loop-shaping and $\mu$ controllers are applied to the plant. The control objective has been obtained from previous work, where the optimal operating point has been shown to be in form of a reference liquid level profile throughout all the tanks of the plant. The incorporation of uncertainties and the controller into the generalized plant is done via Linear Fractional Transformations. The nominal as well as robust stability and performance are investigated for the controllers.

I. INTRODUCTION

One of the most popular and established sea water desalination technology is the multi-stage flash evaporator plant (MSF) for conversion of sea water into potable water. Lots of work has been invested in developing models, which are utilized in optimization of the plant design and control for instance. In [1] and [2], steady-state and dynamic mathematical models of a Multi Stage Flash Evaporator plant (MSF) have been reported accounting for known stage geometries and considering the physico-chemical properties of salty sea water by the means of correlations.

In [3], a comprehensive overview is provided on the numerous control tasks and industrial control methods for desalination processes in MSF plants, where the global control task has been divided into numerous sub-problems and investigated independently.

In [4], a nonlinear 18-stage MSF in once-through configuration is investigated based on deterministic models. The optimal operating point regarding total water production and thermal efficiency of the plant is obtained by backwards motion-planning resulting in a reference liquid height profile throughout the stages of the plant. Different controllers - Feedback linearization, LQR and PI - are implemented for the nonlinear process in order to control the entire height profile of the plant. Based on defined evaluation criteria, the performance of the controllers is elaborated.

Few works have been dealing with uncertain MSF processes. In [5], a conventional PI controller is applied for the top brine temperature control of the nominal plant for uncertain time delay and time constants due to operation point changes. It has been shown that the fixed PI controller was able to deal with the parametric deviations from the nominal value and provides good results for temperature stabilization.

This paper is a further development of [4]; here the cascaded MSF plant is regarded as an uncertain process due to the existence of such as incomplete physical models or unknown model parameters. The parameter uncertainties are incorporated into the plant via Linear Fractional Transformations (LFTs) and are assumed to be structured. Three robust controllers - $H_\infty$, loop-shaping and $\mu$ synthesis controller - are designed and applied for the liquid profile reference control throughout the MSF plant; validation is done with respect to the nonlinear model. Further order reduction is carried out for the obtained controllers and nominal as well as robust stability and performance of the closed-loop system are investigated.

II. PROCESS DESCRIPTION

A MSF desalination plant is shown in Fig. 1 in a once-through configuration, where the brine is fed through the plant only once before being discharged back to the sea. In a series of flashing chambers the evaporation and distillate formation process take place when the saturated seawater undergoes a reduction in pressure and is evaporated at lower temperatures due to the design vacuum conditions. In this paper, the main focus...
lies on the linearized liquid level dynamics presented in the next section.

III. LINEARIZED LIQUID LEVEL DYNAMICS AND CONTROL TASK

A linear approximation of the nonlinear liquid level dynamics $\dot{x} = f(x, u)$ from [4], see (1), has been obtained. The operation point $x_0, u_0$ has been determined by backwards motion-planning and the system is locally linearized utilizing the Jacobians defined as: $A = \frac{df}{dx}(x_0, u_0)$ and $B = \frac{df}{du}(x_0, u_0)$. Eventually the local linear state space model is obtained in standard form:

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (1)

with $x \in \mathbb{R}^{18}$, $u \in \mathbb{R}^1$, $y \in \mathbb{R}^{18 \times 1}$, $A \in \mathbb{R}^{18 \times 18}$ and $B \in \mathbb{R}^{18 \times 1}$. The system matrix $A$ is sparse due to the cascaded structure of the MSF plant:

$$A(k_j, \tau) = \begin{bmatrix}
A_{1,1} & A_{1,2} & 0 & 0 & 0 & 0 \\
0 & A_{j,j} & A_{j,j} & A_{j,j+1} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{18,17} & A_{18,18}
\end{bmatrix}_{x_0, u_0} \hspace{1cm} (2)

The input matrix $B(\tau)$ has the following structure:

$$B(\tau) = \begin{bmatrix}
B_{1,1} & 0 & \cdots & 0 & 0
\end{bmatrix}_{x_0, u_0} \hspace{1cm} (3)

with the only non-zero entry of the input matrix $B$ being $B_{1,1} = \frac{\lambda_1}{\rho}$. The following substitute term $\chi_j$ is introduced:

$$\chi_j = \begin{cases}
1 - \frac{\lambda_1}{\rhoA}(\tau - T_{s,1}) & \text{for } j = 1 \\
1 - \frac{\lambda_1}{\rhoA}(T_{s,j-1} - T_{s,j}) & \text{for } j = 2 \ldots 18
\end{cases} \hspace{1cm} (4)

Note how the dependence on the uncertain parameters are introduced for the system matrix $A(\{k_j, \tau\})$ and input matrix $B(\tau)$; the single elements are presented in detail in Appendix I.

IV. UNCERTAIN GENERALIZED MSF PLANT

The main uncertainties occur at the MSF inlet in the form of an unknown control feed rate due to unknown discharge coefficient of the inlet orifice and the deviating top brine temperature of the entering fluid, which is corrupted by the deteriorating performance of the heat exchangers in the pre-heating phase. Inside the plant, the stages are connected via valves, which are subject to analogous parametric uncertainties as the inlet orifice.

A. Structured uncertainties

Parametric uncertainties are incorporated by using Linear Fractional Transformations (LFT’s), [6]. The following variables are considered as uncertain: a) discharge coefficient $c$ of the inlet orifice, b) top brine temperature $\tau$ and c) valve coefficients $k_j$ connecting the tanks and the following structured model is used:

$$\dot{\bar{c}} = \bar{c} + p_c \delta_c, \hspace{0.5cm} \dot{\bar{\tau}} = \bar{\tau} + p_{\tau} \delta_{\tau} \hspace{0.5cm} \text{and} \hspace{0.5cm} \bar{k}_j = k_j + p_{k,j} \delta_k$$ \hspace{1cm} (5)

with $p_c, p_{\tau}$ and $p_{k,j}$ being the known range of the corrupted variables, the accented (‘) indicates the uncertain variables, the accented (‘) is the nominal value and the subscript (j) is the location of the stage. Parameter $\bar{c}$ causes the control variable to be uncertain defined as:

$$\bar{u} = u (\bar{c} + p_c \delta_c), \hspace{1cm} (6)

where $u$ is the nominal computed control law by the respective controller. The components of the uncertainty blocks satisfy the following condition: $-1 \leq \delta_c, \delta_{\tau}, \delta_k \leq 1$. The uncertain variables and parameters can be expressed as an upper LFT in.

$$F_U(M, \Omega) := \left[ M_{22} + M_{21} \Omega (I - M_{11} \Omega)^{-1} M_{12} \right] \hspace{1cm} (7)

with

$$M_c = \left[ \begin{array}{cc}
p_c & 1 \\
\bar{c} & \tau
\end{array} \right], \hspace{0.5cm} M_{\tau} = \left[ \begin{array}{cc}
p_{\tau} & 1 \\
\bar{\tau} & \tau
\end{array} \right], \hspace{0.5cm} M_{k,j} = \left[ \begin{array}{cc}
p_{k,j} & 1 \\
\bar{k}_j & k_j
\end{array} \right] \hspace{1cm} (8)

B. General model of the first stage $S_1$

The uncertain liquid level dynamics of the first stage - first row of (2) - incorporate structured parameter uncertainties and are expressed by:

$$\dot{x}_1 = \bar{A}_{1,1} (x_1 - x_2) + \bar{B}_{1,1} \bar{u}, \hspace{1cm} (9)

where $\bar{A}_{1,2} = -\bar{A}_{1,1}$ holds and the uncertain coefficients are given as

$$\bar{A}_{1,1} = -(k_1 + p_{k,1} \delta_k) aAy \left[ A \sqrt{\frac{2}{p} (p_1 - p_2 + p_9 (x_1 - x_2))} \right]^{-1},$$

$$\bar{B}_{1,1} = \frac{1}{pA} \left(1 - \frac{c_{\tau}}{\lambda_1} (\tau + p_{\tau} \delta_{\tau} - T_{s,1})\right), \hspace{0.5cm} \bar{u} = (\bar{c} + p_c \delta_c) u.$$

The entries of the linearized system matrices $A$ and $B$ are evaluated in the operating point $x_0, u_0$. Defining $\bar{k}_1 = \bar{k} \bar{A}_{1,1}$, the generalized model of the first stage $S_1$ is treated as an exogenous input; it reads:

$$\begin{bmatrix}
x_1 \\
\vdots \\
\vdots \\
x_7 \\
x_8 \\
x_9 \\
\end{bmatrix} = \begin{bmatrix}
\bar{k} A_{1,1}' & Q_{13} & p_{c} A_{1,1}' & -\bar{k} A_{1,1}' & Q_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & M_{22} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
\vdots \\
\vdots \\
x_7 \\
x_8 \\
x_9 \\
\end{bmatrix} + \begin{bmatrix}
\bar{w}_c \\
\bar{w}_\tau \\
\bar{w}_{k,1} \\
x_2 \\
x_r \\
d_u \\
\end{bmatrix} \hspace{1cm} (10)

with

$$A_{1,1} = \frac{A_{1,1}}{k}, \hspace{0.5cm} a_1 = \frac{1}{pA}, \hspace{0.5cm} a_2 = \frac{c_{\tau}}{pA}, \hspace{0.5cm} a_3 = a_2 T_{s,1},$$

$$Q_{11} = a_1 M_{22} + a_2 \left(T_{s,1} M_{22} - M_{22} M_{22}\right), \hspace{0.5cm} Q_{13} = -a_2 M_{21},$$

$$Q_{12} = a_1 M_{22} + a_2 \left(T_{s,1} M_{22} - M_{22} M_{22}\right).$$

Later the generalized first stage $S_1$ is connected to the remaining downstream stages, which makes $x_2$ an internal state as supposed to.
C. General model of the downstream stages DS

The cascaded plant consists of a sequential arrangement of flashing chamber, which requires the coupling of the downstream stages DS to the general model of the first stage $S_1$. All the uncertainties of the interconnecting orifices $k_j$ are accounted in the modelling. For any downstream stage $j > 1$, each uncertainty block $M_{k,j}$ is modelled as:

$$w_{k,j} = \delta_k z_{k,j}, \quad z_{k,j} = v_{k,j}, \quad \dot{x}_j = p_k w_{k,j} + k_0 v_j,$$

where $v_{k,j} = A_{k,j-1}^* x_{j-1} + A_{k,j}^* x_j + A_{k,j+1}^* x_{j+1}$ and note that $A_{k,j+1}^* = 0$ for $j = 18$.

D. Global general plant MSF

The generalized global plant MSF is presented next using the following definitions:

$$\dot{X} = \begin{bmatrix} x_1 & x_2 & \cdots & x_j & \cdots & x_{18} \end{bmatrix}^T,$$

$$Z = \begin{bmatrix} z_c & z_r & z_{k,1} & z_{k,2} & \cdots & z_{k,j} & \cdots & z_{k,18} \end{bmatrix}^T,$$

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_j & \cdots & x_{18} \end{bmatrix}^T,$$

$$W = \begin{bmatrix} w_c & w_r & w_{k,1} & w_{k,2} & \cdots & w_{k,j} & \cdots & w_{k,18} \end{bmatrix}^T,$$

$$Y = \begin{bmatrix} y_{1,p} \ e \end{bmatrix}^T$$

and

$$U = \begin{bmatrix} r & d_y & u \end{bmatrix}^T,$$

where $X, \in \mathbb{R}^{18}$ are the internal states, $Z, \in \mathbb{R}^{20}$ are the outputs and inputs of the structured uncertainty block, $Y, \in \mathbb{R}^2$ is the exogenous output vector consisting of the perturbed output $y_{1,p}$ and control error $e$. The exogenous input vector $U, \in \mathbb{R}^3$ consists of the reference signal $r$, output disturbance $d_y$ and control signal $u$. Its state space representation is:

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} X \\ W \\ Y \end{bmatrix},$$

where MSF is the generalized global plant displayed as an interconnection block in Fig. 2. Detailed description of the submatrices is presented in Appendix II.

The uncertainty blocks is:

$$\Delta = \text{diag}\{\delta_c, \delta_r, \delta_{k,1}, \delta_{k,2}, \cdots, \delta_{k,j}, \cdots, \delta_{k,18}\}.$$  \hspace{1cm} (14)

Further it is assumed that $\|\Delta\|_\infty < 1$. The obtained generalized model MSF is of large dimensions: 40 $\times$ 41.

**Fig. 2:** Generalized plant $S_1$ coupled with the general model of the downstream stages MSF along with the structured uncertainty blocks in each stage.

V. REQUIREMENTS ON THE CONTROL SYSTEM

A. Performance criteria

The controller to be designed shall stabilize the system internally accounting for the given parameter uncertainties in (5) while meeting defined performance specifications. In this case study a mixed sensitivity optimization task is formulated

$$\left\| \begin{bmatrix} W_p (I + G K)^{-1} \\ W_u (I + G K)^{-1} \end{bmatrix} \right\|_\infty < 1,$$  \hspace{1cm} (15)

where $S = (I + G K)^{-1}$ is the sensitivity function based on the nominal plant $G$ and $K$ is the linear feedback controller yet to be found. The performance criteria is known as the $S$ over $K$&s design and it presents a trade-off solution between reference tracking, disturbance rejection and utilized control effort.

B. Weighting functions

In order to meet the mixed-sensitivity performance criteria (15), the weighting functions $W_p(s)$ and $W_u(s)$ are to be designed appropriately. It is known that the sensitivity $S(s)$ is desired to be small for low frequencies and 1 for frequencies larger than the bandwidth, which ensures reasonable tracking capability and disturbance rejection of the system. Avoiding saturation or damage of the actuator a constant upper limit is set over all frequencies achieved by $W_u(s)$. The following weighting functions have been chosen in this case study:

$$W_p(s) = 0.65 \frac{s^2 + 8.8s + 1}{0.8s^2 + 8s + 0.01}$$

and

$$W_u(s) = 10^{-3}.$$  \hspace{1cm} (16)

VI. CONTROL DESIGN

For the reference control of the liquid profile throughout all the stages of the MSF plant three controllers have been applied: $H_\infty$, loop-shaping and $\mu$ controller. All have been designed based on the linearized dynamics; (2) and (3), where the control signal $u$ is the inflow towards the first stage.

A. $H_\infty$ controller

Usually the optimization task of (15) is relaxed by setting a finite positive number $\gamma_0$ for which the following must hold:

$$\left\| \begin{bmatrix} W_p (I + G K_\infty)^{-1} \\ W_u (I + G K_\infty)^{-1} \end{bmatrix} \right\|_\infty < \gamma_0.$$  \hspace{1cm} (16)

This is known as the suboptimal $H_\infty$ problem, which minimizes the infinite-norm of the performance criteria over all stabilizing controllers $K_\infty$. The resulting controller is a dynamic system itself and is of 20th order and has the same number of states as the general plant including all the states from the performance weights.
B. $H_\infty$ loop-shaping controller (LSHC), [7]

The LSHC approach is formulated by the following optimization problem:

$$
\| [K_{\text{LSHC}}] (I - G K_{\text{LSHC}})^{-1} M^{-1} \|_\infty = \gamma_0.
$$

(17)

where $G = M^{-1} \tilde{N}$ is represented by a normalized left coprime factorization:

$$
[\tilde{N} \ M] := \begin{bmatrix}
A + HC & B + HD & H \\
R^{-1/2}C & R^{-1/2}D & R^{-1/2}
\end{bmatrix}.
$$

The parameter $\gamma_0$ is the lowest achievable value of the infinite norm above for all stabilizing controllers $K_{\text{LSHC}}$. Further, the LSHC design is similar to the conventional loop-shaping design, which is well-known and is attractive in control applications. Here, a pre-compensator $W_1(s) = 30 \, e^{-s + 1}$ with quasi integral behaviour is utilized to alter the frequency response of the plant $G$.

C. $\mu$ synthesis, [6]

Due to the known diagonal structure of parametric uncertainties - (14) - the synthesis of the controller is done by means of the structured singular value $\mu$. Robust performance can be formulated as a robust stabilization problem with respect to the augmented block

$$
\tilde{\Delta} := \begin{bmatrix}
\Delta & 0 \\
0 & \Delta_P
\end{bmatrix} : \Delta \in \mathbb{R}^{20 \times 20}, \Delta_P \in \mathbb{R}^{1 \times 2},
$$

which augments the parametric uncertainty block $\Delta$ by the performance block $\Delta_P$. The optimization problem is formulated in terms of the upper bound of the scaled structured singular value:

$$
\mu(N(K_{\mu})) \leq \min_{\Delta_P} \sigma(D N(K_{\mu}) D^{-1}).
$$

The controller shall minimize the infinite-norm of the following performance cost over frequency of this upper bound:

$$
\min_{K_{\mu}} \min_{D} \| D N(K_{\mu}) D^{-1} \|_\infty,
$$

(18)

where $D$ is a constant scaling matrix and $N(K_{\mu})$ is the lower LFT, which incorporates the controller into the generalized plant. The algorithm alternates between minimizing (18) with respect to $D$ and $K_{\mu}$ while keeping the other respective parameter constant. For detailed explanations on the method and notations the reader is referred to [6].

D. Order reduction of controllers

The resulting controllers are a separate dynamic system itself of high order, which is inconvenient in implementation for instance. A significant order reduction of the controllers is achieved by means of the Hankel singular values and the results are listed in Table I. For the frequency range of $[10^{-3} \ldots 10^2]$ rad it has been verified that the reduced controllers accurately replicate the behaviour over the original high-order controllers as depicted in Fig. 3. Therefore the low-order approximations are applied in the following.

TABLE I: Order reduction of the initial controllers.

<table>
<thead>
<tr>
<th>Order #</th>
<th>Initial design</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_\infty$ controller</td>
<td>21</td>
<td>5</td>
</tr>
<tr>
<td>Loop-shaping</td>
<td>23</td>
<td>8</td>
</tr>
<tr>
<td>$\mu$ controller</td>
<td>23</td>
<td>4</td>
</tr>
</tbody>
</table>

Fig. 3: Comparison of the reduced (red) against the full (blue) order controllers over a frequency band of $[10^{-3} \ldots 10^2]$ rad.

VII. NUMERICAL RESULTS

A. Robust stability

Robust stability analysis is carried out in terms of structured $\mu$ values. The upper and lower bound of the $\mu$ values are shown for all three controllers in Fig. 4. Robust stability is achieved for all the investigated controllers as the maximum value of $\mu$ is around 0.02. Furthermore the maximum singular value is depicted in black, which characterizes the robust stability with respect to unstructured perturbations; for $\omega < 0.4$ rad robust stability cannot be preserved. It is emphasized how further knowledge of the uncertainty structure results in an improved and less conservative design.

B. Nominal and robust performance

For all three controllers the nominal and robust performance criteria is investigated, [6]; the nominal performance condition is given in terms of a lower LFT:

$$
\| F_L(MSF, K) \|_\infty < 1,
$$

(19)

where where MSF is provided in 13 and $K$ is one of the designed controllers. The robust performance criteria is
that denoted with $η[\%]$. A RMS value is defined for each single stage and subsequently summed up over the entire simulation as an indicator for the accuracy of the profile control. The transition time $t^* [s]$ is defined as the time instant when all the liquid levels have been driven to their respective reference value with a tolerance of $5\%$.

At terminal time $T$ the achieved liquid profiles of the robust controllers are visualized in Fig. 7 along the reference profile and previously tested controllers such as Feedback linearization, LQR and PI controllers. The evaluation of the characteristic numbers are summarized for all the robust controllers in Table III along with the previously tested controllers for the sake of comparison.

VIII. CONCLUSION

In this paper, an uncertain model has been derived for the linearized liquid level dynamics of a cascaded MSF plant by using Linear Fractional Transformations in order to incorporate multiple parametric uncertainties into a deterministic system, which has been derived in earlier work. Based on the given reference profile throughout the stages and defined mixed-sensitivity performance criteria three robust controllers have been designed: $H_\infty$, loop-shaping and $\mu$ synthesis controllers. The high order of the obtained controllers could be reduced by using Hankel singular values without compromising the performance. The controllers have been tested and successfully verified with respect to nominal and robust stability and performance criteria. Moreover the controllers have been implemented in the original
TABLE III: Elaboration of the characteristic numbers for the FBL, LQR, PI, loop-shaping and µ controllers.

<table>
<thead>
<tr>
<th></th>
<th>FBL</th>
<th>LQR</th>
<th>PI</th>
<th>Loop-shaping</th>
<th>µ controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>u[ϵ]</td>
<td>196.82</td>
<td>193.96</td>
<td>194.35</td>
<td>196.72</td>
<td>196.31</td>
</tr>
<tr>
<td>η [%]</td>
<td>5.12</td>
<td>5.20</td>
<td>5.18</td>
<td>5.14</td>
<td>5.23</td>
</tr>
<tr>
<td>RMS</td>
<td>0.39</td>
<td>0.43</td>
<td>0.48</td>
<td>0.40</td>
<td>0.39</td>
</tr>
<tr>
<td>$t^*$[s]</td>
<td>880</td>
<td>880</td>
<td>2000</td>
<td>1000</td>
<td>1000</td>
</tr>
</tbody>
</table>

nonlinear MSF model in order to validate their performance based on defined characteristic numbers such as total distillate production, thermal efficiency of the plant, control accuracy by means of RMS and transition time. The main outcome is that the robust controllers perform very similar along with the previously tested controllers such as feedback linearization, LQR and PI controllers while accounting for the model uncertainties, which has been inherited by design.

APPENDIX I

ELEMENTS OF SYSTEM MATRIX $A(k_j, \tau)$

The elements of the system matrix $A(k_j, \tau)$ from (2) are presented next.

- **First stage ($j = 1$):**
  \[ A_{11} = -k \, a \, g \, \left( A \, \sqrt{\frac{2}{\rho}} \, (p_1 - p_2 + \rho g (x_1 - x_2)) \right)^{-1} \]
  \[ A_{12} = -A_{11} \]

- **Intermediate stages ($2 \leq j \leq 17$):**
  \[ A_{jj-1} = k \, a \, g \, \chi_j \left( A \, \sqrt{\frac{2}{\rho}} \, (p_{j-1} - p_j + \rho g (x_{j-1} - x_j)) \right)^{-1} \]
  \[ A_{jj+1} = k \, a \, g \left( A \, \sqrt{\frac{2}{\rho}} \, (p_j - p_{j+1} + \rho g (x_j - x_{j+1})) \right)^{-1} \]
  \[ A_{jj} = -\left( A_{jj-1} + A_{jj+1} \right) \]

- **Last stage ($j = 18$):**
  \[ A_{18,17} = k \, a \, g \, \chi_{18} \left( A \, \sqrt{\frac{2}{\rho}} \, (p_{17} - p_{18} + \rho g (x_{17} - x_{18})) \right)^{-1} \]
  \[ A_{18,18} = -k \, a \, g \left( A \, \sqrt{\frac{2}{\rho}} \, (p_{18} - \rho \, u_{mb} + \rho g F_{18}) \right)^{-1} \]

APPENDIX II

SUBMATRICES OF GENERALIZED PLANT MSF

The composition of the submatrices of the general global system MSF, (13), are presented next. The general plant MSF consists of known variables making it deterministic, the uncertainty block $\Delta$ and controller $K$ can be incorporated by upper and lower LFTs respectively. It is recalled that the MSF has the following structure:

\[
\begin{bmatrix}
\dot{X} \\
\dot{Z} \\
Y
\end{bmatrix} = \begin{bmatrix}
A_1 & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} \begin{bmatrix}
X \\
W \\
U
\end{bmatrix}.
\]

with the submatrices defined as:

\[
A = \begin{bmatrix}
A_{11}^{11} & -A_{11}^{11} & 0 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 0 \\
0 & 0 & A_{11}^{11} & A_{11}^{11} & 0 \\
0 & 0 & 0 & A_{11}^{11} & A_{11}^{11} \\
0 & 0 & 0 & 0 & A_{11}^{11}
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
Q_{11} & Q_{11} & \cdots & \cdots & \cdots \\
O_{17} & O_{17} & \cdots & \cdots & \cdots
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
O_{2x18} \\
Q_{11} & 0 & \cdots & \cdots & 0
\end{bmatrix}^T,
\]

\[
C_1 = \begin{bmatrix}
O_{2x18} \\
Q_{11} & 0 & \cdots & \cdots & 0
\end{bmatrix}^T,
\]

\[
C_2 = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & 0
\end{bmatrix},
\]

where $E \in \mathbb{R}^{17 \times 18}$ is a submatrix of $A$ by eliminating its first row.

$D_{11} = \begin{bmatrix}
M_{1}^{C} \\
M_{11}^{C} \\
0
\end{bmatrix},
\]

$D_{22} = \begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 0
\end{bmatrix},
\]

$D_{12} = \begin{bmatrix}
O_{2x3} & O_{2x17} \\
M_{12}^{C} & M_{12}^{C} \\
0 & O_{17x17}
\end{bmatrix}^T,
\]

$D_{21} = O_{2x20}.$

REFERENCES