A Risk based Approach to Estimate Key Uncertainties


* School of Chemical & Biomolecular Engineering
Georgia Institute of Technology, Atlanta, GA 30332 USA
Tel: 404-642-9151; e-mail: farminder.anand@chbe.gatech.edu).
** School of Chemical & Biomolecular Engineering
Georgia Institute of Technology, Atlanta, GA 30332 USA
(e-mail: matthew.realff@chbe.gatech.edu)
*** School of Chemical & Biomolecular Engineering
Georgia Institute of Technology, Atlanta, GA 30332 USA
(e-mail: jay.lee@chbe.gatech.edu)

Abstract: Decisions during the early stages of R&D are often made under substantial uncertainty. Evaluation of R&D alternatives under uncertainty generally does not provide a clear choice that is best under all possible scenarios. For optimal investment of R&D resources, it is important to identify the key uncertainty contributors from a decision maker’s perspective. Global sensitivity analysis (GSA) is a tool that can be used to determine key uncertainties that contribute the most to the variance of the bottom-line objective. It is often the case, however, that GSA is not able to distinguish between the uncertainties. Motivated by this, we propose a new tool called conditional – global sensitivity analysis, which further considers the decision-maker’s risk preference. The conditional sensitivity measures \((cGSA_{up}/cGSA_{down})\) quantify the contributions of different individual uncertainty factors to the upper and lower halves of the distribution function of the objective function. It is argued that the use of \(cGSA_{up}\) may appeal to a risk-aversive decision maker as it leads to a lower rate of false acceptance decisions at the expense of a higher rate of false rejection decisions, whereas the use \(cGSA_{down}\) does the opposite.

Keywords: GSA, c-GSA, variance, Lower Partial Moment, R&D.

1. INTRODUCTION

Evaluation of the key uncertainty contributors in the R&D process is an essential step to reduce the cost of the whole process. Global sensitivity analysis (GSA) and expected value of information (EVOI) are two tools that can be used to estimate the key uncertainty contributors. Global sensitivity analysis considers variance as a measure of risk and the importance of various uncertainties is measured as the fraction of the overall variance contributed by each uncertain factor. On the other hand, EVOI directly measures the value of reducing uncertainty in each factor.

The evaluation of GSA is significantly simpler than that of EVOI since the calculation of EVOI requires more detailed information about the effect of experimentation on the uncertainty. Though, EVOI provides more direct and therefore useful information than GSA from a decision-maker’s standpoint, the computational load associated with the EVOI limits its usefulness. In this work we supplement the traditional GSA approach with a new tool called conditional global sensitivity analysis (cGSA), which provides further insight into the selection of the key uncertainty.

For a R&D investment decision, there are two sides to the problem. First is the R&D investment cost, e.g., the experimentation cost. Second is the return or profits when the R&D project becomes successful. In order to make the overall process more profitable, one can either focus on the reduction in the R&D cost or on maximizing the probability of success of chosen projects. To address the two different strategies we consider two different risk behaviours of the decision maker:

(a). Decision Delayer / Just A Few Alternatives: This type of decision maker wants to ensure a high probability of success and is ready to accept a relatively high R&D expense. If there are just a few alternatives, one is more focused on maximizing the probability of success. In such cases, one is more concerned with not removing a potentially successful alternative than reducing the cost of higher R&D investment. Hence such a decision maker would prefer an approach that will lead to a lower FALSE REJECTION rate, even if it requires that the decision maker has to bear a higher FALSE ACCEPTANCE rate.
(b). Decision Maker / Many Alternatives: This type wants to minimize the R&D investment and is ready to accept a relative lower probability of success. If there are large numbers of alternatives, it is needed to trim down the options to just a few alternatives. Hence a decision maker in such a situation would prefer an approach that will lead to a lower FALSE ACCEPTANCE rate, even if it entails a higher FALSE REJECTION rate.

When GSA is not able to distinguish between the uncertainties, cGSA (either cGSA<sub>down</sub> or cGSA<sub>up</sub>) can be used for deciding the key uncertainty factor, depending on the situation and the decision maker’s risk attitude.

The rest of the paper is structured as follows: Section 2 presents the background about GSA. Section 3 presents the new approach, cGSA, along with the algorithm to calculate cGSA and discusses the risk behaviours cGSA<sub>up</sub> and cGSA<sub>down</sub> support. Various examples to illustrate the utility of cGSA are presented in Section 4 and Section 5 concludes the chapter.

2. BACKGROUND

2.1 Global Sensitivity Analysis (GSA)

GSA calculates the relative importance of input variables or factors (<i>x<i><sub>1</i>,<i>x<i><sub>2</sub>,...<i>x<i><sub>k</i></sub>) in determining the value of the output variable <i>y</i>. Assume a model <i>y</i> =<i>f</i>(<i>x</i><sub>1</sub>,<i>x</i><sub>2</sub>,...<i>x</i><sub>k</sub>) is composed of independent random variables <i>x</i><sub>1</sub>,<i>x</i><sub>2</sub>,...<i>x</i><sub>k</sub>. Moreover assume that the probability density function of <i>x</i><sub>1</sub>,<i>x</i><sub>2</sub>,...<i>x</i><sub>k</sub> is p<sub>j</sub>(<i>x</i><sub>1</sub>), p<sub>j</sub>(<i>x</i><sub>2</sub>),...,p<sub>j</sub>(<i>x</i><sub>k</sub>). Given the above, the joint distribution of <i>x</i><sub>1</sub>,<i>x</i><sub>2</sub>,...<i>x</i><sub>k</sub> is

\[ P(x_1, x_2, ..., x_k) = \prod_{i=1}^{k} p_j(x_i) \]  

(1)

The mean and variance of <i>y</i> can be calculated as:

\[ E(y) = \int \int ... \int f(x_1, x_2, ..., x_k) \prod_{i=1}^{k} p_j(x_i) dx_i \]  

(2)

\[ V(y) = \int \int ... \int (f(x_1, x_2, ..., x_k) - E(y))^2 \prod_{i=1}^{k} p_j(x_i) dx_i = \int \int ... \int f^2(x_1, x_2, ..., x_k) \prod_{i=1}^{k} p_j(x_i) dx_i - E^2(y) \]  

(3)

The first order sensitivity index is defined as:

\[ S_j = \frac{V(E[y|x_j])}{V(y)} (U_j - E^2(y)) \]  

(4)

\[ U_j = \int E^2(y|x_j = \tilde{x}_j) p_j(\tilde{x}_j) d\tilde{x}_j \]  

(5)

Ishigami and Homma (1990) makes the calculation of the above using a single Monte Carlo loop. (Sattelli, Andres et al. 1993) suggested the following Monte Carlo based procedure for calculation of global sensitivity indexes. Consider the two input sample matrices <i>M</i><sub>1</sub> and <i>M</i><sub>2</sub> as follows:

\[ M_1 = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1k} \\ x_{21} & x_{22} & ... & x_{2k} \\ ... \\ x_{n1} & x_{n2} & ... & x_{nk} \end{pmatrix}, \quad M_2 = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1k} \\ x_{21} & x_{22} & ... & x_{2k} \\ ... \\ x_{n1} & x_{n2} & ... & x_{nk} \end{pmatrix} \]  

(6)

where ‘<i>n</i>’ is the sample size used for the Monte Carlo estimation. <i>E</i>(<i>y</i>) can be calculated based on the values of <i>y</i> computed on matrices samples either from <i>M</i><sub>1</sub> or <i>M</i><sub>2</sub>, whereas <i>U</i><sub>j</sub> can be calculated from values of <i>y</i> computed from sample from matrices <i>M</i><sub>1</sub> and <i>N</i><sub>j</sub>.

\[ N_j = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1j} & ... & x_{1k} \\ x_{21} & x_{22} & ... & x_{2j} & ... & x_{2k} \\ ... & ... & ... & ... & ... & ... \\ x_{n1} & x_{n2} & ... & x_{nj} & ... & x_{nk} \end{pmatrix} \]  

(7)

The approximation for <i>U</i><sub>j</sub> is as follows:

\[ U_j = \frac{1}{n-1} \sum_{j=1}^{n} f(x_{11}, x_{12}, ..., x_{1n}) f(x_{21}, x_{22}, ..., x_{2n}) \]  

(8)

The computational cost associated with the calculation of the all first order sensitivity indexes is <i>S</i><sub>j</sub> is <i>n</i>*(<i>k</i>+1), where ‘<i>n</i>’ evaluations of ‘<i>f</i>’ are needed to calculate <i>E</i>(<i>y</i>) and <i>n</i> sets of ‘<i>f</i>’ for each first order sensitivity index.

3. CONDITIONAL – GLOBAL SENSITIVITY ANALYSIS

As explained in section 2.1, GSA provides information about the contribution by individual factors to the overall uncertainty, where uncertainty is measured in terms of variance. However, there are situations where GSA is unable to distinguish among several uncertain factors in terms of their importance, i.e. two or more factors have equal or nearly equal global sensitivity index (GSI). In such a case, it is difficult to differentiate between the factors and decide in which factor to reduce uncertainty through experimentation. In order to accommodate such a situation, a new tool called cGSA is presented, which may be able to differentiate between the factors having equal GSI based on the situation at hand and the decision maker’s risk preference.

3.1 Definition

Conditional global sensitivity analysis is the measure of the contribution of variance by individual factors to the conditional objective function. Here conditional distribution refers to the part of the distribution of model output <i>y</i> above or below its mean <i>E</i>(<i>y</i>) , for cGSA<sub>up</sub> or cGSA<sub>down</sub> respectively. The conditional (upside or downside) global sensitivity index (cGSI<sub>up</sub> or cGSI<sub>down</sub>) can be
calculated using (9) or (10) compared to GSI which is calculated using (4).

\[
c_{GSI}^{up} = \frac{V\left(E\left(x_j \mid y > E(y)\right)\right)}{V\left(y \mid y > E(y)\right)} \tag{9}
\]
\[
c_{GSI}^{down} = \frac{V\left(E\left(x_j \mid y < E(y)\right)\right)}{V\left(y \mid y < E(y)\right)} \tag{10}
\]

3.2 Algorithm

Calculation of \(c_{GSI}^{up}\)/\(c_{GSI}^{down}\) follows the same approach as GSI with the difference being that the random samples for the various factors conform to the conditionality constraint, i.e. the factors transformed by the model to the objective function value should lie above/below the mean of the model output \(E(y)\). So, in essence the problem is how to sample from the distribution of the factors so that the objective function values belong to the respective side of the distribution. This problem can be addressed by developing classification functions, which make sure that the model output will lie on the desired side of the mean of the model output distribution.

It is important to note, that the sampling from the factor distributions introduces correlation between the factors. Global sensitivity analysis problem for correlated factor distributions has been addressed in the literature (McKay 1995), (Saltelli 2002), (Jacques, Lavergne et al. 2006), (Xu and Gertner 2008). The generic idea is to follow the same approach for unconditional GSA but introduce correlation between the parameters by re-ordering the samples from the factor distributions by following the ordering scheme introduced in (Iman and Conover 1982). Moreover, the measure of uncertainty contribution for each factor is considered the same as for the traditional GSA approach, i.e., \(V(E(y|x_j))\) (Saltelli 2002).

The correlation introduced in cGSA cannot be calculated by using the re-ordering scheme of (Iman and Conover 1982), since the conditionality induces a relationship between the input factors that cannot be easily represented by either correlation or rank correlation. Hence in order to maintain the original distribution properties of the factors and account for the natural relationship introduced by the ‘conditionality’, we first sample the factors from their original distribution functions and then accept or reject the sample based on the result of the classification function separating the objective function values below and above the cut-off. (Note: this is very similar to the Acceptance-Rejection algorithm). The importance of using the classification function instead of the model is that the model can be computationally intensive and a high rejection rate can significantly increase the computational load.

Moreover, since the classification function would be an approximation, we would consider an enveloping classification function i.e. a classification function with high priority for acceptance of all possible valid combinations of factor values at the cost of relatively low priority for rejection of invalid combination of factor values. Though this would require additionally checking the validity of the sample point it would make the cGSI calculation more accurate. The complete procedure to calculate the cGSI given below assumes that GSI has been performed already and the user wants to perform cGSI for follow-up analysis:

3.3. Theoretical Interpretation

To understand the theoretical interpretation of the results of reducing uncertainty in the key cGSA\(^{up}\) and cGSA\(^{down}\) uncertainty and its application consider the following model output \(y\) given by (11).

\[
y = x_1 + x_2 \tag{11}
\]
\[
x_1 = \text{beta}(2, 10) \tag{12}
\]
\[
x_2 = \text{beta}(10, 2) \tag{13}
\]

The input uncertainties \((x_1\) and \(x_2\)) are given by (12) and (13). The distributions of \(x_1\) and \(x_2\) are positively and negatively skewed respectively as shown in figure 1. The distribution of model output \(y\) is shown in figure 2. Properties of the input uncertainties along with the model output are shown in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>0.167</td>
<td>0.104</td>
<td>0.92</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0.833</td>
<td>0.104</td>
<td>-0.92</td>
</tr>
<tr>
<td>(y)</td>
<td>1.0</td>
<td>0.147</td>
<td>0</td>
</tr>
</tbody>
</table>

Consider that the true decision criterion (14) is based on a cutoff, i.e. if the correct value of the model output lie above the cutoff the decision maker would keep the R&D alternative and if the model output lie below the cutoff the decision maker would reject the R&D alternative. For simplicity, assume the cutoff to be at the mean of the model output \(y (= 1)\). This means that the correct decision is to accept the technology 50% of the sampled cases and reject it in the other 50% of the cases. Consider the decision criterion under uncertainty (15) is driven by just the mean value, i.e., if the posterior mean \((\mu_{y, \text{posterior}})\) for realization i), after the uncertainty reduction, is below the cutoff, the alternative is rejected, and if the posterior mean \((\mu_{y, \text{posterior}})\) is above the
cutoff, then the alternative is accepted. Furthermore, assume that if the decision maker tries to reduce uncertainty in any input factor, then the correct value of that factor would be known exactly, i.e. the posterior value of that input factor would be a point estimate, though the actual point estimate that results would be a sample from the prior distribution.

\[
\text{True Decision} = \begin{cases} 
\text{Accept,} & \text{if } y > \text{cutoff} \\
\text{Reject,} & \text{if } y < \text{cutoff} \\
\text{No Decision,} & \text{otherwise}
\end{cases}
\]  

(14)

\[
\text{Decision Criterion} = \begin{cases} 
\text{Accept,} & \text{if } \mu_{y,\text{posterior}} > \text{cutoff} \\
\text{Reject,} & \text{if } \mu_{y,\text{posterior}} < \text{cutoff} \\
\text{No Decision,} & \text{otherwise}
\end{cases}
\]

(15)

Additionally, \(x_1\) will be identified as the key uncertainty if \(c\text{GSA}^\text{up}\) is used, whereas \(x_2\) will be if \(c\text{GSA}^\text{down}\) is used. If the key \(c\text{GSA}^\text{up}\) uncertainty is reduced, we will obtain a negatively skewed posterior distribution (\(y_{\text{posterior}}\)) for model output \(y\) for any particular realization \(i\), but the mean of these posterior distributions (\(\mu_{y,\text{posterior}}\)) obtained over all possible realizations would be positively distributed. To clarify, if uncertainty \(x_1\) is reduced then a posterior point estimate (\(\hat{x}_1\)) of \(x_1\) is obtained, which is sampled from the positively skewed prior distribution of \(x_1\) (12). Then, as in (16), the shape of the posterior distribution is mainly driven by the input uncertainty \(x_2\), and hence \(y_{\text{posterior}}\) is a negatively skewed distribution. The mean of the posterior distribution (\(\mu_{y,\text{posterior}}\)) for any realization \(i\) is given by (17), the distribution of which overall all possible realizations is positively skewed as can be seen from (17). The expected value of the posterior mean of \(y\) (i.e. \(\mu_{y,\text{posterior}}\)) is constant (18), where \(\mu_1\) and \(\mu_2\) are the mean of the prior distributions of \(x_1\) and \(x_2\), respectively.

\[
y_{\text{posterior}}^{i} = \hat{x}_1^{i} + x_2
\]

(16)

\[
\mu_{y,\text{posterior}}^{i} = \hat{x}_1^{i} + \mu_2
\]

(17)

\[
E(\mu_{y,\text{posterior}}) = \mu_1 + \mu_2
\]

(18)

Since the two input uncertainties have similar distributions and the model structure is simply the addition of the two uncertainties, it is obvious that both the uncertainties will equally contribute to the uncertainty in the model output, which means they would have an equal GSI value.

Figure 1: (a) Frequency distribution of random variable \(x_1\), (b) Frequency distribution of random variable \(x_2\).

Figure 2: Frequency distribution of model output \(y\).

\[
y_{\text{posterior}} = \hat{x}_1 + x_2
\]

(16)

\[
\mu_{y,\text{posterior}} = \hat{x}_1 + \mu_2
\]

(17)

\[
E(\mu_{y,\text{posterior}}) = \mu_1 + \mu_2
\]

(18)

In order to evaluate the pros and cons of the \(c\text{GSA}^\text{up}\) and \(c\text{GSA}^\text{down}\), false acceptance and false rejection is defined by (29) and (30) respectively, where \(N\) symbolizes the frequency. False acceptance is defined as the percentage of reject decisions which are falsely concluded to be acceptance
decision. Similarly false rejection is defined as the percentage of the correct acceptance decisions which are falsely rejected.

\[ \text{False Acceptance} = \frac{100}{N} \frac{\text{Correct Decision} = \text{Reject}}{\text{Predicted Decision} = \text{Accept}} \]  
\[ N = \text{\ldots} \]  
(19)

\[ \text{False Rejection} = \frac{100}{N} \frac{\text{Correct Decision} = \text{Accept}}{\text{Predicted Decision} = \text{Reject}} \]  
\[ N = \text{\ldots} \]  
(20)

Consider the scenario, when cGSA\textsuperscript{up} is used to determine which factor’s uncertainty should be reduced. Posterior mean that one obtains in this case would be positively skewed, as shown in Figure 3. For a positively skewed distribution the median lies below the mean, as indicated in the same figure. Hence, the mean-based decision criterion shown in (15) would lead to a higher percentage of rejection instances. This is confirmed in the numerical results shown in Table 2. On the other hand, if cGSA\textsuperscript{down} is used instead, the resulting posterior mean would be negatively skewed as shown in Figure 4. Hence, it leads to a higher percentage of acceptance decisions, which is confirmed in the numerical results in Table 2.

In addition, the results presented in Table 2 indicate that the reduction in uncertainty in the factor identified by cGSA\textsuperscript{up} not only leads to a higher percentage of rejection decisions but also a higher rate of false rejection decisions. Again, the opposite is true if cGSA\textsuperscript{op} is used instead.

4. TEST CASE STUDIES

To further demonstrate the utility of cGSA, we consider a generic model \( y \) composed of two skewed model inputs \( (x_1, x_2) \) and two symmetric distributed model inputs \( (h_1, h_2) \) with parameters \( (y_0, a, b, c, d, e, f) \). The values and prior distributions of the various underlying parameters \( (T_1, T_2, \theta_1, \theta_2) \) of \( x_1, x_2, h_1 \) and \( h_2 \) are given by (29) – (33). The frequency distributions of the two model input factors \( x_1 \) and \( x_2 \) are shown in Figure 5 and those for model inputs \( h_1 \) and \( h_2 \) are shown in Figure 6. For reducing uncertainty of parameters \( \theta_1 \) and \( \theta_2 \), experiments are conducted to collect measurements for \( g_1 \) and \( g_2 \) given by (26) and (27) respectively. These measurements are corrupted by Gaussian noises of (34) and (35) with their variances given by (36).

\[
y = y_0 + a.x_1 + b.x_2 + c.x_1.x_2 + \ldots \]  
(21)

\[
d.h_1 + e.h_2 + f.h_1.h_2 \]  
(26)

\[ x_1 = (z_1^T \theta_1 - T_1)^2 \]  
(22)

\[ x_2 = (z_2^T \theta_2 - T_2)^2 \]  
(23)

\[ h_1 = z_1^T \theta_1 \]  
(24)

\[ h_2 = z_2^T \theta_2 \]  
(25)

\[ g_1 = z_1^T \theta_1 + \epsilon_1 \]  
(26)

\[ g_2 = z_2^T \theta_2 + \epsilon_2 \]  
(27)

\[ z_1 = z_2 = [1 \ 1 \ 1]^T \]  
(28)

\[ \theta_1 \sim N(\mu_{\theta_1}, \Sigma_{\theta_1}) \]  
(29)

\[ \theta_2 \sim N(\mu_{\theta_2}, \Sigma_{\theta_2}) \]  
(30)

\[ \mu_{\theta_1} = \begin{bmatrix} -5 \\ 10 \\ 5 \end{bmatrix} \]  
(31)

\[ \Sigma_{\theta_1} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \]  
(32)

\[ T_1 = T_2 = 5 \]  
(33)

\[ \epsilon_1 \sim N(0, \sigma_1^2) \]  
(34)

\[ \epsilon_2 \sim N(0, \sigma_2^2) \]  
(35)

\[ \sigma_1^2 = \sigma_2^2 = 0.05 \]  
(36)

To evaluate the performance of the cGSA, both cGSI\textsuperscript{up} (9) and cGSI\textsuperscript{down} (10) are calculated, along with it the rates of false acceptance (19) and false rejection (20).
Procedure to calculate the probabilities are explained as follows:

Consider the scenario when cGSA (either cGSA\textsuperscript{up} or cGSA\textsuperscript{down}) identifies $x_1$ as the key uncertainty. This means more information should be obtained to reduce the uncertainty in $x_1$. In order to acquire more information, one conducts an experiment, with $z_1$ and $z_2$ obtained through D-optimal design (denoted by $\eta_1^{\text{exp}}$) on the model $g_1$ (see (26)), and after obtaining the experimental measurement ($g_1^{\text{exp}}$) of $g_1$, the posterior distribution, $\theta_1^{\text{posterior}} \sim N\left(\mu_{\theta_1^{\text{posterior}}}, \Sigma_{\theta_1^{\text{posterior}}}\right)$, of $\theta_1$ is obtained.

In order to systematically compare the cGSA\textsuperscript{up} and cGSA\textsuperscript{down}, $N$ ($= 8000$) realizations of $\theta_1$, $\theta_2$, $\varepsilon_1$ and $\varepsilon_2$ are generated and stored in $\Theta_1^i, \Theta_2^i, \varepsilon_1^i$ and $\varepsilon_2^i$ (where $i = 1, 2, \ldots, N$). To generate the experimental realization $g_1^{\text{exp}}$, a sample $\Theta_1^i$ from the prior distribution of $\Theta_1$ and a sample noise, $\varepsilon_1^i$ from the noise distribution $\varepsilon_1$ are used. The posterior distribution for $\Theta_1$ can be calculated using (37) and (38).

$$
\mu_{\theta_1^{\text{posterior}}} = \left(\eta_1^{\text{exp}}\right)^T \left(\eta_1^{\text{exp}} + \left(\Sigma_{\theta_1}/\sigma_1^2\right)^{-1}\right)
$$

(37)

$$
\Sigma_{\theta_1^{\text{posterior}}} = \sigma_1^2 \left(\left(\left(\eta_1^{\text{exp}}\right)^T \left(\mu_{\theta_1^{\text{posterior}}} + \left(\Sigma_{\theta_1}/\sigma_1^2\right)^{-1}\right)\right)^T \left(\eta_1^{\text{exp}}\right) + \left(\Sigma_{\theta_1}/\sigma_1^2\right)^{-1}\right)^{-1}
$$

(38)

Now given this posterior distribution ($\theta_1^{\text{posterior}}$) of $\theta_1$, the posterior distribution of $x_1$ is easily obtained using (22). Next using the posterior distribution of $x_1$ and prior distribution of $x_2$ the updated distribution of $y$ is obtained. The required properties of the posterior distribution are calculated and the decision is predicted based on the decision criterion of (15).

In the above, one sample of experimental realization $g_1^{\text{exp}}$ is considered and it leads to one updated distribution of $y$ and then one decision (either selection or rejection) is made; this constitutes of one scenario. Similarly $N$ ($N = 8000$) scenarios are generated and using the definition of false acceptance (19) and false rejection (20) the values are reported.

Consider the model structure in (39). Given the input uncertainties the distribution of the model output $y$ is symmetric. The GSA results show that both the uncertainties are equally important. Further cGSA\textsuperscript{up} suggests that $x_1$ is the main uncertainty while cGSA\textsuperscript{down} suggests that $x_2$ is the key uncertainty. Results for two different cutoff values of 12 and 88 are shown in Tables 4 and 5 respectively. Results support the argument that the uncertainty reduction in the key uncertainty indicated by cGSA\textsuperscript{up} leads to a higher rate of false rejection and a lower rate of false acceptance, and vice versa.

$$
y = 50 + x_1 - x_2
$$

(51)

Table 4: Comparison of cGSA\textsuperscript{up} and cGSA\textsuperscript{down} for the cutoff value of 12

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GSA</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>cGSA\textsuperscript{up}</td>
<td>0.88</td>
<td>0.00</td>
</tr>
<tr>
<td>cGSA\textsuperscript{down}</td>
<td>0.01</td>
<td>0.76</td>
</tr>
<tr>
<td>False Acceptance (%)</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>False Rejection (%)</td>
<td>41</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5: Comparison of cGSA\textsuperscript{up} and cGSA\textsuperscript{down} for the cutoff value of 88

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GSA</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>cGSA\textsuperscript{up}</td>
<td>0.88</td>
<td>0.00</td>
</tr>
<tr>
<td>cGSA\textsuperscript{down}</td>
<td>0.01</td>
<td>0.76</td>
</tr>
<tr>
<td>False Acceptance (%)</td>
<td>0</td>
<td>41</td>
</tr>
<tr>
<td>False Rejection (%)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We have conducted many more numerical studies with different parameter values and decision criterion but these are not presented here due to the space limitation.

REFERENCES


Ishigami, T. and T. Homma (1990). An importance quantification technique in uncertainty analysis for computer models Proceedings of the ISUMA'90, First International Symposium on Uncertainty Modelling and Analysis University of Maryland


