A nonlinear control law is presented for stable, multiple-input, multiple-output processes, whether their delay-free part is minimum- or non-minimum-phase. It is derived by exploiting the connections between continuous-time model-predictive control and input-output linearization. The differential-geometric, control law induces a linear closed-loop response approximately. It has a few tunable parameters (one for each controlled output), and thus, is easily tuned.

Keywords
Nonlinear control, State feedback design, Input-output linearization, Non-minimum-phase systems, Model-based control

Scope and Mathematical Preliminaries
Consider the class of MIMO, nonlinear processes of the form:

\[
\begin{align*}
\frac{d\bar{x}(t)}{dt} &= f(\bar{x}(t), u(t)), \quad \bar{x}(0) = \bar{x}_0 \\
\frac{d\bar{y}_i(t)}{dt} &= h_i(\bar{x}(t - \theta_i)) + d_i, \quad i = 1, \ldots, m
\end{align*}
\]

where \(\bar{x} = [\bar{x}_1 \cdots \bar{x}_m]^T \in X\) is the vector of the process state variables, \(u = [u_1 \cdots u_m]^T \in U\) is the vector of manipulated inputs, \(\bar{y} = [\bar{y}_1 \cdots \bar{y}_m]^T \in Y\) is the vector of process outputs, \(\theta_1, \ldots, \theta_m\) are the measurement delays, \(d = [d_1 \cdots d_m]^T \in D\) is the vector of constant unmeasured disturbances, \(f(\cdot, \cdot)\) is a smooth vector field on \(X \times U\), and \(h_1(\cdot), \ldots, h_m(\cdot)\) are smooth functions on \(X\). Here \(X \subset \mathbb{R}^n\) is a connected open set that includes \(\bar{x}_{ss}\) and \(\bar{x}_0\), \(U \subset \mathbb{R}^m\) is a connected open set that includes \(u_{ss}\), and \(D \subset \mathbb{R}^m\) is a connected set, where \((\bar{x}_{ss}, u_{ss})\) denotes the nominal steady-state (equilibrium) pair of the process; that is, \(f(\bar{x}_{ss}, u_{ss}) = 0\).

The system:

\[
\begin{align*}
\frac{d\bar{x}(t)}{dt} &= f(\bar{x}(t), u(t)), \quad \bar{x}(0) = \bar{x}_0 \\
\frac{d\bar{y}_i(t)}{dt} &= h_i(\bar{x}(t)) + d_i, \quad i = 1, \ldots, m
\end{align*}
\]

is referred to as the delay-free part of the process. The relative orders (degrees) of the controlled outputs \(y_1, \ldots, y_m\) with respect to \(u\) are denoted by \(r_1, \ldots, r_m\), respectively, where \(r_i\) is the smallest integer for which \(\frac{d^{r_i}y_i}{dt^{r_i}}\) explicitly depends on \(u\) for every \(x \in X\) and every \(u \in U\). The relative order (degree) of a controlled output \(y_i\) with respect to a manipulated input \(u_j\) is denoted by \(r_{ij}\) (\(i = 1, \ldots, m, j = 1, \ldots, m\)), where \(r_{ij}\) is the smallest integer for which \(\frac{d^{r_{ij}}y_i}{dt^{r_{ij}}}\) explicitly depends on \(u_j\) for every \(x \in X\) and every \(u \in U\). The set-point and the set of acceptable set-point values are denoted by \(y_{sp}\) and \(Y\), respectively, where \(Y \subset \mathbb{R}^m\) is a connected set.

The following assumptions are made:

(A1) For every \(y_{sp} \in Y\) and every \(d \in D\), there exists an equilibrium pair \((\bar{x}_{ss}, u_{ss}) \in X \times U\) that satisfies \(y_{sp} - d = h(\bar{x}_{ss})\) and \(f(\bar{x}_{ss}, u_{ss}) = 0\).
The nominal steady-state (equilibrium) pair of the process, \((\bar{x}_{ss}, u_{ss})\), is hyperbolically stable; that is, all eigenvalues of the open-loop process evaluated at \((\bar{x}_{ss}, u_{ss})\) have negative real parts.

For a process in the form of (1), a model in the following form is available:

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(x(t), u(t)), \quad x(0) = x_0 \\
y_i(t) &= h_i[x(t - \theta_i)], \quad i = 1, \ldots, m
\end{align*}
\]

(3)

where \(x = [x_1 \cdots x_n]^T \in X\) is the vector of model state variables, and \(y = [y_1, \ldots, y_m]^T\) is the vector of model outputs.

The following notation is used:

\[
\begin{align*}
h_i^1(x) &= \frac{dy_i^s}{dt} \\
&\vdots \\
h_i^{r_i-1}(x) &= \frac{d^{r_i-1}y_i^s}{dt^{r_i-1}} \\
h_i^{r_i}(x, u) &= \frac{d^{r_i}y_i^s}{dt^{r_i}} \\
h_i^{r_i+1}(x, u(0), u(1)) &= \frac{d^{r_i+1}y_i^s}{dt^{r_i+1}} \\
&\vdots \\
h_i^{p_i}(x, u(0), u(1), \ldots, u(p_i-r_i)) &= \frac{d^{p_i}y_i^s}{dt^{p_i}}
\end{align*}
\]

where \(p_i \geq r_i\) and \(u^{(\ell)} = d^\ell u/dt^\ell\).

**Input-Output Linearization**

For a process in the form of Equation 1, responses of the closed-loop process outputs are requested, having the linear form:

\[
\begin{bmatrix}
(\epsilon_1 D + 1)^{r_1} \tilde{y}_1(t + \theta_1) \\
\vdots \\
(\epsilon_m D + 1)^{r_m} \tilde{y}_m(t + \theta_m)
\end{bmatrix} = y_{sp},
\]

(5)

where \(D\) is the differential operator (i.e., \(D \equiv \frac{d}{dt}\)), and \(\epsilon_1, \ldots, \epsilon_m\) are positive, constant, adjustable parameters that set the speed of the response of the closed-loop process outputs \(\tilde{y}_1, \ldots, \tilde{y}_m\) respectively. Substituting for the process output derivatives from the model in Equation 5, one obtains:

\[
\begin{bmatrix}
h_1(\bar{x}) + \binom{r_1}{1} \epsilon_1 h_1^1(\bar{x}) + \cdots + \binom{r_1}{r_1} \epsilon_1^{r_1} h_1^{r_1}(\bar{x}, u) \\
\vdots \\
h_m(\bar{x}) + \binom{r_m}{1} \epsilon_m h_m^1(\bar{x}) + \cdots + \binom{r_m}{r_m} \epsilon_m^{r_m} h_m^{r_m}(\bar{x}, u)
\end{bmatrix}
\]

(6)

\[
y_{sp} - d
\]

Under the assumption of the nonsingularity of the characteristic (decoupling) matrix:

\[
\begin{bmatrix}
h_1(\bar{x}, u) \\
\vdots \\
h_m(\bar{x}, u)
\end{bmatrix}
\]

on \(X \times U\), Equation 6 represents a feedforward/state feedback. When the process delay-free part exhibits non-minimum-phase behavior, the input-output behavior of the closed-loop system under the feedforward/state feedback of Equation 6 is governed by the linear response of Equation 5, but the internal dynamics (unobservable modes) of the closed-loop system are unstable.

The dynamic feedforward/state feedback

\[
\Phi_p(\bar{x}, u, U) = y_{sp} - d
\]

(7)

where the \(i\)th component of \(\Phi_p(\bar{x}, u, U)\):

\[
\begin{bmatrix}
[\Phi_p(\bar{x}, u, U)]_i = h_i(\bar{x}) + \binom{p_i}{1} \epsilon_i h_i^{r_i}(\bar{x}) + \cdots + \\
\binom{p_i}{r_i} \epsilon_i^{r_i} h_i^{r_i}(\bar{x}) + \binom{p_i}{r_i + 1} \epsilon_i^{r_i+1} h_i^{r_i+1}(\bar{x}, u(1)) + \cdots + \binom{p_i}{r_i}
\end{bmatrix}
\]

\[
\epsilon_i^{p_i} h_i^{p_i}(\bar{x}, u(1), \ldots, u(p_i-r_i)) = u = [u^{(1)} \cdots u^{(\max[p_1-r_1, \ldots, p_m-r_m])}]^T, \quad p = [p_1 \cdots p_m]
\]

\[
\partial \Phi_p / \partial \left[ \begin{bmatrix}
\max(p_1-r_1, \ldots, p_m-r_m) \\
\vdots \\
\max(p_1-r_1, \ldots, p_m-r_m)
\end{bmatrix}
\right]^T
\]

(8)

nonsingular, \(\forall x \in X\), also induces a linear, closed-loop, output response of the form:

\[
\begin{bmatrix}
(\epsilon_1 D + 1)^{p_1} \tilde{y}_1(t + \theta_1) \\
\vdots \\
(\epsilon_m D + 1)^{p_m} \tilde{y}_m(t + \theta_m)
\end{bmatrix} = y_{sp},
\]

(9)

Similarly, the dynamic feedforward/state feedback of Equation 7 cannot ensure asymptotic stability of the closed-loop system when the delay-free part of the process exhibits non-minimum-phase behavior. Consequently, an objective of this study is to design a feedback control law that ensures asymptotic stability of the closed-loop system, whether the delay-free part of the process is minimum- or non-minimum-phase.
Nonlinear Feedforward/State Feedback Design

Assume that for every \( x \in X \), every \( d \in D \), and every \( y_{sp} \in Y \), the algebraic equation:

\[
\phi_p(\bar{x}, u) = y_{sp} - d
\]

where

\[
\phi_p(\bar{x}, u) = \Phi_p(\bar{x}, u, 0)
\]

has a real root inside \( U \) for \( u \), and that for every \( \bar{x} \in X \) and every \( u \in U \), \( \partial_{\phi_p(\bar{x}, u)} \) is nonsingular. The corresponding feedforward/state feedback that satisfies Equation 10 is denoted by

\[
u = \Psi_p(\bar{x}, y_{sp} - d)
\]

Note that the preceding feedforward/state feedback was obtained by setting all the time derivatives of \( u \) in Equation 7 to zero.

Theorem 1 For a process in the form of Equation 1, the closed-loop system under the feedforward/state feedback of Equation 12 is asymptotically stable, if the following conditions hold:

(a) The nominal equilibrium pair of the process, \((\bar{x}_{ss}, u_{ss})\), corresponding to \( y_{sp} \) and \( d \), is hyperbolically stable.

(b) The tunable parameters \( p_1, \ldots, p_m \) are chosen to be sufficiently large.

(c) The tunable parameters \( \epsilon_1, \ldots, \epsilon_m \) are chosen such that for every \( \epsilon_\ell (\ell = 1, \ldots, m) \) and for every \( \lambda_i (i = 1, \ldots, n) \), \( \epsilon_\ell \lambda_1 + 1 < 1 \), where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( J_{cl} \). For an overdamped, stable process, \( \epsilon_1, \ldots, \epsilon_m \) should be chosen such that for every \( \epsilon_\ell \), \( \ell = 1, \ldots, m \), and for every \( \tau_i \), \( i = 1, \ldots, n \), \( 0 < \epsilon_\ell \tau_i < 2 \), where \( \tau_1, \ldots, \tau_n \) are the open-loop time constants of the process. In other words, \( \epsilon_1, \ldots, \epsilon_m \) should be chosen to be less than \( 2\tau_{min} \), where \( \tau_{min} \) is the smallest time constant of the process [i.e., \( \tau_{min} = min(\tau_1, \ldots, \tau_n) \)].

The proof can be found elsewhere (Kanter et al., 2000). Condition (c) states that \( \epsilon_1, \ldots, \epsilon_m \) should be chosen such that for every \( \epsilon_\ell (\ell = 1, \ldots, m) \) and for every \( \lambda_i (i = 1, \ldots, n) \), \( \epsilon_\ell \lambda_1 + 1 < 1 \), where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( J_{cl} \). For an overdamped, stable process, \( \epsilon_1, \ldots, \epsilon_m \) should be chosen such that for every \( \epsilon_\ell \), \( \ell = 1, \ldots, m \), and for every \( \tau_i \), \( i = 1, \ldots, n \), \( 0 < \epsilon_\ell \tau_i < 2 \), where \( \tau_1, \ldots, \tau_n \) are the open-loop time constants of the process. In other words, \( \epsilon_1, \ldots, \epsilon_m \) should be chosen to be less than \( 2\tau_{min} \), where \( \tau_{min} \) is the smallest time constant of the process [i.e., \( \tau_{min} = min(\tau_1, \ldots, \tau_n) \)].

Note that the feedforward/state feedback of Equation 12 does not induce the linear, closed-loop response of Equation 9, since in the derivation of the feedforward/state feedback the time derivatives of \( u \) were set to zero. The nonlinearity of the resulting delay-free output response is the price of ensuring closed-loop stability for processes with a non-minimum-phase delay-free part.

Example 1 Consider a linear process without delaytime in state space form with

\[
A = \begin{bmatrix} -2 & 5 & -3 \\ 0 & -1 & 3 \\ 0 & 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

It is non-minimum-phase (has a right-half plane [RHP] transmission zero at \( s = 7.28 \)) and hyperbolically stable (has three left-half plane [LHP] eigenvalues at \( s = -1, s = -2 \) and \( s = -10 \)). Its relative orders \( r_1 = 1 \) and \( r_2 = 1 \). With \( \epsilon_1 = 0.1 \) and \( \epsilon_2 = 0.05 \), the eigenvalues of \( \epsilon_1 A \) are \(-0.2, -0.1 \) and \(-1 \), while the eigenvalues of \( \epsilon_2 A \) are \(-0.5, -0.1 \) and \(-0.05 \). These eigenvalues are inside the unit circle centered at \((-1, 0)\).

Furthermore, as \( p_1, \ldots, p_m \rightarrow \infty \), the state feedback places the \( n \) eigenvalues of the Jacobian of the closed-loop system evaluated at the nominal equilibrium pair at the \( n \) eigenvalues of the Jacobian of the open-loop process evaluated at the nominal equilibrium pair.

The proof can be found elsewhere (Kanter et al., 2000).

Theorem 2 For a process in the form of Equation 1 with incomplete state measurements, the closed-loop sys-
tem under the error-feedback control law

\[ \frac{dx}{dt} = f[x, u] \]

\[ u = \Psi_p \begin{bmatrix} x, e + \begin{bmatrix} h_1[x(t - \theta_1)] \\ \vdots \\ h_m[x(t - \theta_m)] \end{bmatrix} \end{bmatrix} \] (13)

where \( e = y_{sp} - \bar{y} \), is asymptotically stable, if the following conditions hold:

(a) The nominal equilibrium pair of the process, \((\bar{x}_{ss}, u_{ss})\), corresponding to \(y_{sp}\) and \(d\), is hyperbolically stable.

(b) The tunable parameters \( p_1, \ldots, p_m \) are chosen to be sufficiently large.

(c) The tunable parameters \( \epsilon_1, \ldots, \epsilon_m \) are chosen such that for every \( \ell = 1, \ldots, m \), all eigenvalues of \( \epsilon_\ell J_\ell = \epsilon_\ell \frac{\partial}{\partial x} f(x, u) \mid (\bar{x}_{ss}, u_{ss}) \) lie inside the unit circle centered at \((-1, 0)j\).

Furthermore, the error-feedback control law of Equation 13 has integral action.

The proof can be found elsewhere (Kanter et al., 2000).

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