Abstract— The output feedback control of uncertain fuzzy models is one of the most challenging problems in the fuzzy control field. In this paper an approach to design an observer-based robust fuzzy control of uncertain fuzzy models is proposed. Based on quadratic Lyapunov function and Linear Matrix Inequalities (LMI) formulation, sufficient conditions are derived for robust asymptotic output stabilization. An example is given to illustrate the proposed results.

I. INTRODUCTION

There have been several recent studies concerning the stability and the synthesis of controllers and observers for nonlinear systems described by Takagi-Sugeno fuzzy (T-S) models [10]. Based on quadratic Lyapunov function some papers have discussed the feedback control and the state estimation for fuzzy systems [5][7][1][3][13][19]. For example in [7], stability conditions for fuzzy control systems are reported to relax the conservatism of the basic conditions. To less of conservatism another method to relaxed quadratic stability conditions are also proposed in [19]. More recently, in [2] new stability conditions are obtained by relaxing the stability conditions of the previous works and particularly these derived in [19]. Motivated by the aforementioned works, sufficient conditions are derived for robust asymptotic output stabilization using quadratic Lyapunov function. These stability conditions are solved using LMI optimization techniques. A T-S model of four-wheel drive vehicles is used to compare the proposed result and to show the effectiveness of the derived conditions.

Notation: symmetric definite positive matrix $P$ is defined by $P > 0$, the set $\{1, 2, \ldots, n\}$ is defined by $I_n = \{1, 2, \ldots, n\}$, the symbol $*$ denotes the transpose elements in the symmetric positions and $\sum_{i<j} x_i x_j = \sum_{i=1}^{n} \sum_{i<j} x_i x_j$.

II. PRELIMINARY

The uncertain fuzzy model considered is described as:

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i (z(t)) \left( (A_i + \Delta A_i) x(t) + (B_i + \Delta B_i) u(t) \right)$$

and

$$y(t) = \sum_{i=1}^{n} \mu_i (z(t)) C_i x(t)$$

With the properties:

$$\sum_{i=1}^{n} \mu_i (z(t)) = 1, \mu_i (z(t)) \geq 0 \quad \forall \ i \in I_q$$

Where $q$ is the number of sub-models, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^l$ is the output vector, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{l \times n}$ are the $i^{th}$ state matrix, the $i^{th}$ input matrix and the $i^{th}$ output matrix respectively. Vector $z(t)$ is the premise variables depending
on measurable variables. \( \Delta A_i \) and \( \Delta B_i \) are time-varying matrices representing parametric uncertainties in the plant model. These uncertainties are admissibly norm-bounded and structured.

Assumption 1: The considered parameter uncertainties are norm-bounded:
\[
\Delta A_i = D_i A_i(t) E_{di}, \quad \Delta B_i = D_i B_i(t) E_{bi}
\]
where \( D_i, E_{di}, E_{bi} \) are known real constant matrices of appropriate dimension, \( A_i(t) \) is an unknown matrix function with Lebesgue-measurable elements and \( I \) is the identity matrix.

The following lemma, commonly used in several papers, is recalled.

Lemma 1[1]: Given constant matrices \( D \) and \( F \), symmetric constant matrix \( S \) and unknown constant matrix \( F_i \) of appropriate dimension satisfying the constraint \( F^TF < R \). The following two propositions are equivalent:

1) \[
S + DFE + E^TF^TD^T < 0
\]
2) \[
S + \left( F^T D \right) \left( \begin{array}{cc}
\epsilon I & 0 \\
0 & \epsilon I
\end{array} \right) \left( \begin{array}{cc}
E & D^T
\end{array} \right) < 0 \text{ for some } \epsilon > 0.
\]

III. QUADRATIC OUTPUT STABILISATION

The fuzzy observer is defined as follows
\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{n} \mu_i(z(t)) (A_i x(t) + B_i u(t) + G_i y(t) - \hat{y}(t)) \\
\dot{\hat{y}}(t) &= \sum_{i=1}^{n} \mu_i(z(t)) C_i \hat{y}(t)
\end{align*}
\]
where \( G_i \in \mathbb{R}^{n \times l} \) are the observer gains to be determined. The estimation error is defined as
\[
e(t) = x(t) - \hat{x}(t)
\]
The objective is to design an observer-based controller of model (1), in the form:
\[
u(t) = -\sum_{i=1}^{n} \mu_i(z(t)) K_i \hat{y}(t)
\]
where \( K_i \in \mathbb{R}^{m \times n} \) are the feedback gains to be determined.

Tacking into account (1), (5) and (6), we obtain:
\[
\dot{x}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(z(t)) \mu_j(z(t)) \left( A_i + \Delta A_i - (B_i + \Delta B_i) K_j \right) x(t) + \\
\sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(z(t)) \mu_j(z(t)) (B_i + \Delta B_i) K_j e(t)
\]
\[
\dot{\hat{y}}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(z(t)) \mu_j(z(t)) \left( A_i - B_i K_j \right) \hat{y}(t) + \\
\sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(z(t)) \mu_j(z(t)) L_i C_j e(t)
\]
\[
\dot{e}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(z(t)) \mu_j(z(t)) \left( A_i - L_i C_j - \Delta B_i K_j \right) e(t) + \\
\sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(z(t)) \mu_j(z(t)) (\Delta A_i - \Delta B_i K_j) x(t)
\]
\[
\Theta_i = \begin{bmatrix}
\Theta_i \\
E_iK_i \\
(PD_i) \\
(PD_i)
\end{bmatrix} = \begin{bmatrix}
-(\epsilon_i^{+}+1)I \\
-(\epsilon_i^{+}+1)I \\
0 \\
0
\end{bmatrix} < 0
\]

with

\[
\Phi_a = QA_t + AQ_t + A_iQ_i - M_iB_i^T - B_iM_i + (r-1)YY + I
\]

\[
\Psi_y = QA_t + AQ_t + A_iQ_i - M_iB_i^T - B_iM_i - M_iB_i^T - B_iM_i + D_iD_i^T + D_iD_i^T - 2Y + 2I
\]

\[
T_0 = A_i^T P + PA_i - C_i^T N_i - N_iC_i + K_i B_i^T B_i K_i + (r-1)\Gamma
\]

\[
\Theta_y = A_i^T P + PA_i + A_i^T P + PA_i - C_i^T N_i - N_iC_i - C_i^T N_i + N_iC_i - K_i B_i^T B_i K_i - 2\Gamma
\]

and \(r\) is the number of sub-models simultaneously activated.

Then, fuzzy model (1) is globally asymptotically stable via control law (6) based on estimated state (4) with

\[
K_i = M_iQ_i^{-1}
\]

\[
G_i = P_i^T N_i
\]

Proof: Considering the Lyapunov function candidate:

\[
V(x(t),e(t)) = V_1(x(t)) + V_2(e(t))
\]

with

\[
V_1(x(t)) = x(t)^TP_1x(t), \quad P_1 = Q_i^{-1} > 0;
\]

\[
V_2(e(t)) = e(t)^TP_2e(t), \quad P > 0
\]

The time-derivative of \(V_1(x(t))\) along the trajectory of (7) is

\[
\dot{V}_1(x(t)) = \sum_{j=1}^n \mu_j(z(t))^2 (x(t))^T (H_j^TP_j + P) x(t) +
2\sum_{i<j} \mu_i(z(t))\mu_j(z(t)) x(t)^T \left( \frac{H_i^T + H_j^T}{2} P + P \left( \frac{H_i^T + H_j^T}{2} \right) \right) x(t)
\]

\[
+2\sum_{i=1}^n \mu_i(z(t)) x(t)^T (P_i (B_i + \Delta B_i) K_i e(t)
\]

\[
+2\sum_{i<j} \mu_i(z(t))\mu_j(z(t)) x(t)^T (P_i (B_i + \Delta B_i) K_i e(t)
\]

Using lemma 1 and assumption 1, the following inequality holds:

\[
2x(t)^T P_i (B_i + \Delta B_i) K_i e(t) \leq x(t)^T (P_i^T + P_iD_iD_i^TP_i) x(t) +

\]

\[
\dot{V}_2(e(t)) = \sum_{i,j} \mu_i(z(t))\mu_j(z(t)) e(t)^T K_i^T (B_i B_i^T + E_i E_i^T) K_j e(t)
\]

Thus, the first and the second sum of (18) are bounded respectively as follows:

\[
\sum_{i,j} \sum_{j=1}^n \mu_i(z(t))\mu_j(z(t)) x(t)^T (P_i^T + P_iD_iD_i^TP_i) x(t) \leq
\]

\[
\sum_{i,j} \sum_{j=1}^n \mu_i(z(t))\mu_j(z(t)) x(t)^T (P_i^T + P_iD_iD_i^TP_i) x(t) +

2\sum_{i,j} \mu_i(z(t))\mu_j(z(t)) x(t)^T \left( \frac{2P_i^T + P_iD_iD_i^TP_i + P_iD_iD_i^TP_i}{2} \right) x(t)
\]

Taking account of (19) and (20) in (16), the time-derivative of \(V_1(x(t))\) is as follows:

\[
\dot{V}_1(x(t)) \leq \sum_{i=1}^n \mu_i(z(t))^2 x(t)^T \left( H_i^T P_i + P_i H_i + P_i^2 + P_i D_i D_i^T P_i \right) x(t)
\]

\[
+2\sum_{i=1}^n \mu_i(z(t)) \mu_i(z(t)) x(t)^T \left( H_i^T + H_j^T \right) P_i + P_i \left( H_i^T + H_j^T \right) x(t)
\]

\[
+2\sum_{i<j} \mu_i(z(t))\mu_j(z(t)) e(t)^T K_i^T (B_i B_i^T + E_i E_i^T) K_j e(t)
\]

The time-derivative of \(V_2(e(t))\) along the trajectory of (9) is
\[ \dot{V}_2(e(t)) = \sum_{i=1}^{q} \mu_i(z(t))^2 e(t)^T \left( \sum_i P + P \sum_j P \right) e(t) + \\
2 \sum_{i=1}^{q} \mu_i(z(t))^2 e(t)^T \left( \sum_i + \sum_j \right) P \sum_j \left( \sum_i + \sum_j \right) e(t) + \\
+ 2 \sum_{i=1}^{q} \sum_{j=1}^{m} \mu_i(z(t)) \mu_j(z(t))^2 e(t)^T \left( \sum_i + \sum_j \right) P \left( \Delta \alpha - \Delta B K_j \right) x(t) \]

(22)

with

\[ \sum_i = A_i - G_i C_j + \Delta B K_j \]

Using lemma 1 and assumption 1, the following inequality holds:

\[ 2 e(t)^T P \left( \Delta A_i - \Delta B K_j \right) x(t) \leq e(t)^T \left( P D_i D_i^T P \right) e(t) + \\
x(t)^T \left( E_{ii} - E_{ii} \left( q T \right) \right) x(t) \]

(24)

The same technique is used to bound (24) as the one used in (18). The derivative-time of \( V_2(e(t)) \) is then as follows:

\[ \dot{V}_2(e(t)) \leq \sum_{i=1}^{q} \mu_i(z(t))^2 e(t)^T \left( \sum_i P + P \sum_j P D_i D_i^T P \right) e(t) + \\
2 \sum_{i=1}^{q} \mu_i(z(t))^2 e(t)^T \left( \sum_i + \sum_j \right) P \sum_j \left( \sum_i + \sum_j \right) e(t) + \\
+ 2 \sum_{i=1}^{q} \sum_{j=1}^{m} \mu_i(z(t)) \mu_j(z(t))^2 e(t)^T \left( \sum_i + \sum_j \right) P \left( \Delta \alpha - \Delta B K_j \right) x(t) + \\
\sum_{i=1}^{q} \mu_i(z(t))^2 x(t)^T L_x x(t) + \\
2 \sum_{i=1}^{q} \sum_{j=1}^{m} \mu_i(z(t)) \mu_j(z(t))^2 x(t)^T \left( L_x + L_y \right) x(t) \]

(25a)

With

\[ L_y = \left( E_{ii} - E_{ii} \right) \left( E_{ii} - E_{ii} \right) \]

(25b)

Finally, by combining (21) and (25), the time derivative of \( V(x(t), e(t)) \) can be obtained.

The negativity of \( V(x(t), e(t)) \neq 0 \), is ensured by exploiting the basic result of [7] and lemma 1.

We note that the proposed result is less conservative than these proposed in [6] where the derived synthesis conditions are based on the negativity of each terms. Moreover the derived conditions introduce supplementary parameters \((\Gamma, \gamma, r)\) which lead to more relaxation.

In the objective of advantage of less conservatism, we propose in the following section to exploit the result of [2].

IV. RELAXED QUADRATIC OUTPUT STABILISATION

In this section a new stability conditions are obtained by relaxing the stability conditions derived in the above section. Theorem 2 is obtained by extending the result of [2] to the case of uncertain fuzzy models. The reduction of conservatism is obtained by introducing multiple matrices, instead of single matrix [7], where the off-diagonal matrices are allowed to be non-symmetric.

**Theorem 2**: suppose that there exist matrices \( Q > 0, P > 0, M_j, N_j, \gamma, \Gamma, \) and scalars \( e_{\gamma} \) satisfying the following conditions \( \forall (i, j) \in I_q, i \leq j \):

\[ \Phi_i < 0 \]

(26a)

\[ \Psi_i < 0 \]

(26b)

\[ \bar{\alpha}_i < 0 \]

(26c)

\[ \bar{\alpha}_i < 0 \]

(26d)

where \( \Phi_i, \Psi_i, \bar{\alpha}_i, \bar{\alpha}_i \) are defined respectively in (10a), (10b), (11a) and (11b) with

\[ \Phi_i = Q A_i^T + A_i Q - M_i^T B_i - B_i M_i + \gamma_i + I \]

(30a)

\[ \Psi_i = Q A_i^T + A_i Q + Q A_i^T + A_i Q - M_i^T B_i - B_i M_i - M_i^T B_i - B_i M_i + \gamma_i + \gamma_i + D_i \gamma_i + D_i \gamma_i + I \]

(30b)

\[ \Theta_i = A_i^T P + P A_i + A_i^T P + P A_i - C_i^T N_i - N_i C_i - K_i^T B_i K_i + \Gamma_i \]

(30c)

Then, fuzzy model (1) is globally asymptotically stable via control law (6) based on estimated state (4) with

\[ K_i = M_i Q^{-1} \]

(31a)
\[ G_i = P^{-1}N_i \]  \hspace{1cm} (31b)

Proof: To ensure the negativity of \( \dot{V}(x(t), e(t)), \forall (x(t)\,$$, \, e(t)^T) \neq 0 \), we consider the result [2]. The detail of proof is omitted due to lack of space. \( \blacksquare \)

Remarks and discussion:

- Comparing theorem 1 and theorem 2, we note that second result allows introducing multiple matrices and non-symmetrical off-diagonal matrices instead of single symmetric semi-definite positive matrix. Hence there are more variables in the second result than in the first one which leads to more great freedom in guaranteeing the stability of the output robust control of fuzzy systems. The same remark can be stated comparing the proposed constraints with previous results ([1][6] and references therein).

- We note that the obtained conditions are in the BMI (bilinear matrix inequality) forms which are impossible to solve directly by LMI tool (LMI toolbox of Matlab software for example). However to be able to use the LMI tool, we propose to solve sequentially synthesis conditions (26)-(28):

-First, we solve the LMI (26) and (28) in the variables \( Q, Y_{ij} \) and \( M_i \).

-Once gains \( K_i \) have been calculated from (31a), conditions (27) and (29) become linear in \( P, \Gamma_{ij} \) and \( N_i \), and can be easily resolved using the LMI tool to determine gains \( L_i \) from (31b).

V. Simulation

Considering the Takagi-Sugeno model of four-wheel drive vehicles represented by two sub-models [12][14]:

\[
\dot{x}(t) = \sum_{i=1}^{2} \mu_i(z(t))((A_i + \Delta A_i)x(t)) + \sum_{i=1}^{2} \mu_i(z(t))((B_{fi} + \Delta B_{fi})\delta_f(t) + (B_i + \Delta B_i)\delta_i(t)) \\
y(t) = Cx(t)
\]

(32)

Where

\[ A_1 = \begin{bmatrix} -4.7964 & -0.988 \\ 2.3982 & -3.7532 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.636 & -1.0054 \\ -1.0757 & -0.4907 \end{bmatrix} \]

(33a)

\[ B_{f1} = \begin{bmatrix} 2.3982 \\ 28.7784 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} 0.3738 \\ 4.4852 \end{bmatrix} \]

(33b)

\[ B_{i1} = \begin{bmatrix} 2.3982 \\ -31.1766 \end{bmatrix}, \quad B_{i2} = \begin{bmatrix} 0.2623 \\ -3.4095 \end{bmatrix} \]

(33c)

\[ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

(33d)

\[ \mu_i(r) = 1 - \mu_i^2(r) = \frac{1}{1 + abs \left( \frac{\beta_i - c_i}{a_i} \right)^{2h}} \]

(34)

and \( x(t) = (\beta, r)^T \) denotes respectively the side slip angle (\( \beta \)) and the yaw velocity (\( r \)), \( \delta_f \) is the front steer angle and \( \delta_r \) is the rear steer angle.

The parameters uncertainties are defined as follows:

\[ D_1 = D_2 = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \alpha \in \mathbb{R}^+ \]

(35)

\[ E_{si} = E_{s2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_{bi} = E_{b2} = \begin{bmatrix} 1.2 \\ 1.2 \end{bmatrix} \]

(36)

It is well known that the stability domain of vehicle is reduced when the speed and/or the steering angle increase, which is source of instability. So to improve the performances of the vehicle, active control systems must be developed and installed. For this, a control system operating on the rear wheel-axle of the vehicle to improve its stability is proposed.

Thus, the objective is to design a robust rear control \( \hat{z}(t) = \sum_{i=1}^{2} \mu_i(z(t))K_i\dot{x}(t) \) based on observer of the closed-loop uncertain fuzzy models (32) when the front steering angle \( \hat{\delta}_f \) is as given in figure 1.

The conditions proposed in [6] fail to design the observer and the control law \( \hat{\delta}_f \) guaranteeing the stability of the closed-loop uncertain fuzzy models (32) when \( \alpha > 0.2 \) whereas these derived in theorem 1 fails to give solution when \( \alpha > 0.3 \). However until \( \alpha = 0.35 \), the derived constraints of theorem 2 remain feasible. Thus the resolution of LMI conditions (26)-(28) allow to design the controller gains:

\[ K_1 = \begin{bmatrix} 1.3359 & -3.4860 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.3367 & -2.7140 \end{bmatrix} \]

\[ Q = \begin{bmatrix} 0.8325 & 0.2741 \\ 0.2471 & 0.2680 \end{bmatrix} \]

(37)

Then the resolution of the constraints (27)-(29) becoming linear in \( P, \Gamma_{ij} \) and \( N_i \) make it possible to design the observer gains:
The simulation shows that the designed controller based observer for the fuzzy model (32) is robust against norm-bounded parametric uncertainties (figure 1).

![Figure 1. Closed-loop response with initial conditions](image)

VI. CONCLUSION

This paper presents new stability conditions for robust asymptotic output stabilization. The derived results are based on the use of the quadratic Lyapunov function and solved using LMI optimization techniques. The proposed stability conditions are proved to be less conservative than previous results. An example is given to illustrate the utility of the proposed results.

VII. REFERENCES


