Finite Time Stability and Quasihomogeneous Control Synthesis of Uncertain Switched Systems with Application to Underactuated Manipulators

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Abstract—Switched control synthesis is developed for underactuated mechanical systems. In order to locally stabilize an underactuated system around an unstable equilibrium, its output is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the desired stability property of the closed-loop system is provided by applying a quasihomogeneous switched controller, driving the system to the zero dynamics manifold in finite time. Although the present synthesis exhibits an infinite number of switches on a finite time interval, it does not rely on the generation of sliding modes, while providing robustness features similar to those possessed by their sliding mode counterparts. Theoretical results are supported by an application to drive systems with backlash.

I. INTRODUCTION

Stabilization of underactuated systems, forced by fewer actuators than degrees of freedom, presents a challenging problem [5], [22]. As well known (see, e.g., [2], [28]), these systems possess nonholonomic properties, caused by nonintegrable differential constraints, and therefore, they cannot be stabilized by means of smooth feedback. With this in mind, the present investigation on stabilization of underactuated mechanical systems is focused on switched control methods and it is based on the results recently published in [17].

The switched control synthesis to be developed is as follows. In order to locally stabilize an underactuated system around an unstable equilibrium, an output of the system is specified to ensure that the corresponding zero dynamics is locally asymptotically stable. Once such an output is chosen, the desired stability property of the closed-loop system is provided by applying a switched controller that drives the system to the zero dynamics manifold in finite time. Synthesis of such a controller that maintains an underactuated system on the zero dynamics manifold, which is typically of co-dimension greater than the control space dimension (the singular case in sliding mode control), is a main contribution of the paper.

The structure of the switched controller constructed is inspired from the quasihomogeneous controller of [16]-[17], stabilizing a one-link manipulator in finite time. Although that controller exhibited a so-called Zeno behavior with an infinite number of switches on a finite time interval (see [12], [14] for Zeno modes), it did not rely on the generation of sliding modes, while providing robustness features similar to those possessed by their sliding mode counterparts. A second order sliding mode (see the original work [11] for second order sliding modes and [1], [7], [8] for advanced results in the area) appeared in the equilibrium point only. In contrast to standard sliding mode control algorithms which are capable of providing the closed-loop manipulator with the ultimate boundedness property only [13], the afore-mentioned quasihomogeneous controller stabilized the manipulator in finite time, thus constituting an interesting alternative to standard sliding mode controllers.

The quasihomogeneous synthesis procedure is subsequently generalized for underactuated systems and capabilities of the procedure are illustrated by a regulation problem for a drive system with backlash, presenting a simple underactuated system.

For the drive system, which consists of a motor part, driven by DC motor, and a reducer part (load), operating in the presence of backlash, we ensure that all the trajectories of the closed-loop system are bounded and the angular position of the load asymptotically decays to a desired position. Due to practical requirements [10], the motor angular position and velocity are assumed to be the only information available for feedback.

The rest of the paper is outlined as follows. Section 2 is focused on the quasihomogeneous stabilization of a simple one degree-of-freedom manipulator, operating under uncertainty conditions. In Section 3, the proposed quasihomogeneous synthesis is extended to underactuated mechanical systems and its effectiveness is then illustrated in Section 4 by an application to a drive system with backlash. Finally, Section 5 presents some conclusions.

II. ROBUST FINITE TIME STABILIZATION OF ONE-LINK MANIPULATOR

The quasihomogeneous synthesis is first illustrated with a simple one degree-of-freedom mechanical manipulator, operating under uncertainty conditions. The dynamics of the manipulator is governed by

\[ \dot{y} = \omega(y, \dot{y}, t) + u \] (1)

where \( y \) is the position of the manipulator, \( \dot{y} \) is the velocity of the manipulator, \( u \) is the controlled input, \( \omega(y, \dot{y}, t) \) is a piece-wise continuous nonlinearity that captures all forces (viscous and Coulomb frictions, gravitation, etc.), affecting the manipulator.
Solutions of the differential equation (1) with piece-wise continuous right-hand side, are defined in the sense of Filippov [6] as that of a certain differential inclusion with a multi-valued right-hand side.

Operating under uncertainty conditions implies imperfect knowledge of the nonlinearity \( \omega(y, \dot{y}, t) \). This possibly destabilizing term

\[
\omega(y, \dot{y}, t) = \omega^\text{nom}(y, \dot{y}, t) + \omega^\text{un}(y, \dot{y}, t)
\]  

(2)

typically contains an \textit{apriori} known nominal part \( \omega^\text{nom}(y, \dot{y}, t) \) to be handled through nonlinear damping and an uncertainty \( \omega^\text{un}(y, \dot{y}, t) \) to be rejected. It is assumed that \( \omega^\text{un}(y, \dot{y}, t) \) is locally bounded

\[
|\omega^\text{un}(y, \dot{y}, t)| \leq N \text{ for all } t \geq 0
\]  

(3)

by an \textit{apriori} known constant \( N > 0 \). Apart from this, both functions \( \omega^\text{nom}(y, \dot{y}, t) \) and \( \omega^\text{un}(y, \dot{y}, t) \) are assumed to be piece-wise continuous.

The following control law

\[
u = -\omega^\text{nom}(y, \dot{y}, t) - \text{asign}(y) - b\text{sign}(\dot{y}) - hy - p\dot{y}
\]  

(4)

subject to

\[
N < b < a - N, \ h, p \geq 0
\]  

(5)

appears to stabilize the uncertain system (1)-(3) in finite time. Apparently, the above controller (4), (5) consists of the nonlinear damping \(-\omega^\text{nom}(z, t)\), the linear gain \(-hy - p\dot{y}\), and the homogeneous switching part \(\varphi(y, \dot{y}) = -\text{asign}(y) - b\text{sign}(\dot{y})\) such that \(\varphi(cy, c\dot{y}) = \varphi(y, \dot{y})\) for all \(c > 0\).

Until recently, finite time stability of asymptotically stable homogeneous systems has been well-recognized for only continuous vector fields [3], [9]. Extending this result to switched systems has required proceeding differently [17] because a smooth homogeneous Lyapunov function, whose existence was proven in [23] for continuous asymptotically stable homogeneous vector fields, can no longer be brought into play.

The novel techniques that was developed in [17] has established that the finite time stability of a switched homogeneous system is saved regardless of some inhomogeneous perturbations. In particular, it has been shown that the inhomogeneous system (1)-(5) is globally finite time stable whenever condition (3) holds globally. The local version of this result is as follows.

\textbf{Theorem 1}: Let a one-link manipulator (1) operate under uncertainty conditions (2), (3) and let it be driven by the quasihomogeneous switched controller (4), (5). Then the closed-loop system (1)-(5) is locally finite time stable, uniformly in the admissible uncertainties (2), (3).

\textbf{Proof} of Theorem 1 is nearly the same as that of Theorem 4.2 of [17] and it is therefore omitted.

The qualitative behavior of the one link manipulator (1)-(3), driven by the switched controller (4), (5), is depicted in Fig. 1 and it is as follows. While approaching the origin \(y = \dot{y} = 0\), the system trajectories rotate around it. Since by Theorem 1, the closed-loop system is locally finite time stable, the switching times of the controller have a finite accumulation point.

Thus, system (1)-(5) does exhibit Zeno behavior with an infinite number of switches in a finite amount of time. This system does not generate sliding motions everywhere except the origin. If a trajectory starts there at any given finite time, the so-called sliding mode of the second order appears [11]. In a particular case, when the uncertainty \(\omega(y, \dot{y}, t) = \omega^\text{un}(y, \dot{y}, t)\) has no nominal part and the control gains \(h, p\) are set to zero, the proposed control law (4) degenerates to the well-known homogeneous twisting algorithm [7], [8].
under uncertainty conditions. The nonlinear gain is a locally minimum phase system. The role of this notion required for this gain. The destabilizing term destabilize the closed-loop system because of the minimum phase hypothesis, which is why no more information is required for this gain. The destabilizing term

\[ f(x, \xi, \dot{\xi}) = f^{n}(x, \xi, \dot{\xi}) + f^{b}(x, \xi, \dot{\xi}) \]

is partitioned into a nominal part \( f^{n} \) known \textit{a priori}, and an uncertain bounded gain \( f^{b} \) whose components \( f_{j}^{b} \), \( j = 1, \ldots, m \) are locally upper estimated

\[ |f_{j}^{b}(x, \xi, \dot{\xi})| \leq N_{j} \]

by \textit{a priori} known constants \( N_{j} > 0 \). Apart from this, both functions \( f^{n} \) and \( f^{b} \) are assumed to be piece-wise continuous.

Being inspired from the quasihomogeneous controller (4), (5), the following switched control law

\[ u(x, \xi, \dot{\xi}) = -f^{n}(x, \xi, \dot{\xi}) - \text{asgn} \xi - \beta \text{sign} \xi - H\xi - P\xi \]

with the parameter gains

\[ H = \text{diag}(h_{j}), \quad P = \text{diag}(p_{j}) \]

\[ \alpha = \text{diag}(\alpha_{j}), \quad \beta = \text{diag}(\beta_{j}) \]

subject to

\[ N_{j} < \beta_{j} < \alpha_{j} - N_{j}, \quad h_{j}, p_{j} \geq 0, \quad j = 1, \ldots, m \]

is proposed to locally stabilize the uncertain system (8), (11), (12) whose state \( (x, \xi, \dot{\xi}) \) is available for measurements. Hereafter, the notation \text{diag} is used to denote a diagonal matrix of an appropriate dimension; \text{sign} \xi with a vector \( \xi = (\xi_{1}, \ldots, \xi_{m})^{T} \) stands for the column vector \( (\text{sign} \xi_{1}, \ldots, \text{sign} \xi_{m})^{T} \).

In what follows, the switched control law (13), (16) is shown to drive the uncertain system (8) to the zero dynamics manifold \( \xi = \dot{\xi} = 0 \) in finite time thereby yielding desired stability properties of the closed-loop system.

**Theorem 2:** Let Assumptions 1-3 be satisfied and let the uncertain system (8), (11), (12) be driven by the state feedback (13) such that condition (16) holds. Then the closed-loop system (8), (13)-(16) is locally asymptotically stable, uniformly in the admissible uncertainties (11), (12).

**Proof:** The closed-loop system (8) driven by (13) is represented as follows

\[ \dot{x} = g(x, \xi, \zeta) \]
\[ \dot{\xi}_{j} = \zeta_{j}, \quad j = 1, \ldots, m \]
\[ \dot{\zeta}_{j} = f_{j}^{b}(x, \xi, \dot{\xi}) - \alpha_{j}\text{sign} \xi - \beta_{j}\text{sign} \zeta_{j} - h_{j}\zeta_{j} - p_{j}\zeta_{j}, \quad j = 1, \ldots, m. \]

Due to Assumption 1, Theorem 8 of [6, p. 85] is applicable to system (17), (18), and by applying this theorem, the system has a local solution for all initial data and uncertainties (12). Let us demonstrate that each solution of (17), (18) is globally continuatable on the right.

First, let us note that given \( j \in (1, \ldots, m) \), no motion appears on the axes \( \dot{\xi}_{j} = 0 \) and \( \dot{\zeta}_{j} = 0 \) except their intersection \( \zeta_{j} = \xi_{j} = 0 \). Indeed, if \( \dot{\zeta}_{j}(t) = 0 \) on a trajectory of (17), (18) then it follows from (17) that \( \zeta_{j}(t) = 0 \) along the trajectory. In turn, if \( \zeta_{j}(t) = 0 \) on a trajectory of (17), (18) then due to the parameter subordination (16), the second equation of (18) fails to hold for \( \xi_{j} \neq 0 \).

Next, let us compute the time derivative of the function

\[ V_{j}(\xi_{j}, \zeta_{j}) = \alpha_{j}\zeta_{j} + \frac{1}{2}(h_{j}\xi_{j}^{2} + \zeta_{j}^{2}), \quad j = 1, \ldots, m \]

along the trajectories of (18). Taking into account (12), one derives
that 
\[
\dot{V}_j(\xi_j(t), \zeta_j(t)) = \alpha_j \xi_j \text{sign} \xi_j + h_j \xi_j \zeta_j + \epsilon_j \{f_j^b(x, \xi, \zeta) - \alpha_j \text{sign} \xi_j - \beta_j \text{sign} \zeta_j \} - h_j \xi_j - p_j \zeta_j \} \leq -[\beta_j - f_j^b(x, \xi, \zeta) \text{sign} \zeta_j] \times (|\zeta_j| - p_j \zeta_j^2) \leq -(\beta_j - N_j)|\zeta_j(t)|
\]
(19)
everywhere but on the axis \(\xi_j = 0\) where the function \(V_j(\xi_j, \zeta_j)\) is not differentiable. Since no sliding motion appears on the axis \(\xi_j = 0\) except the intersection \(\xi_j = \zeta_j = 0\) where \(V_j(\xi_j(t), \zeta_j(t)) = 0\), inequality (19) remains in force for almost all \(t\).

By virtue of (16), it follows that each solution of subsystem (18) subject to (12) is uniformly bounded in \(t\). Coupled to Assumption 2, this ensures that all possible solutions of the over-all uncertain system (12), (17), (18) remain bounded on any finite time interval and by the property B of Theorem 9 of [6, p. 86], these solutions are globally continuable on \([0,\infty)\).

In order to complete the proof, it remains to note that by Assumption 3, the zero dynamics (10) is locally asymptotically stable. Coupled to the local uniform finite time stability of (18), this ensures that the closed-loop system (17), (18) is locally uniformly asymptotically stable. The proof of Theorem 2 is thus completed.

Summarizing, the following quasihomogeneity-based local stabilization procedure is proposed for underactuated systems. First, an output of the system is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the underactuated system is transformed into the normal form (8), whose stabilization is achieved by applying the quasihomogeneous controller (13), (16).

To this end we note that the closed-loop system (8), (13)-(16) becomes globally asymptotically stable if Assumptions 1-3 as well as the upper estimate (12) hold globally. The proof of this fact is similar to that of Theorem 2 and it is omitted because of space limitations. Instead, the capability of the global stabilization is subsequently illustrated with a simple example.

IV. OUTPUT REGULATION OF DRIVE SYSTEM WITH BACKLASH

To demonstrate the effectiveness of the proposed synthesis procedure it is now applied to an electrical drive, consisting of a motor part and a reducer part (load). The objective is to drive the load to a desired position while providing the boundedness of the system motion and attenuating external disturbances, including nonlinear backlash effects. To meet practical requirements the motor angular position and velocity are assumed to be the only information available for feedback.

A. State equations and backlash model

The dynamic models of the angular position \(q_i(t)\) of the DC motor and that \(q_o(t)\) of the load are given according to [15]:
\[
J_o N^{-1} \ddot{q}_o + f_o N^{-1} q_o = T
\]
\[
J_i \ddot{q}_i + f_i \dot{q}_i + T = \tau_m
\]
(20)
\[
y = (q_i, \dot{q}_i)^T
\]
(21)

Hereinafter, \(J_o, f_o, \dot{q}_o\) and \(q_o\) are, respectively, the inertia of the load and the reducer, the viscous motor friction, the motor acceleration and the motor velocity are denoted by \(J_i, f_i, \dot{q}_i\) and \(q_i\), respectively. The input torque \(\tau_m\) serves as a control action, \(y(t)\) is the measurement vector.

The transmitted torque \(T\) through a backlash with an amplitude \(j\), located within \([-0.6, 0.6]\), is modelled as follows [15]:
\[
T = K \Delta q + K_\eta(\Delta q)
\]
(22)
where
\[
\Delta q = q_i - N q_o \in [-2j, 2j],
\]
(23)
\[
\eta(\Delta q) = \frac{4j - e^{-\gamma_s \Delta q}}{1 - e^{-\gamma_s \Delta q}},
\]
(24)

\(K\) is the stiffness, \(N\) is the reducer ratio and \(\gamma_s = \frac{1}{2j}\) is the positive graphic parameter.

Since the control input \(\tau_m\) enforces the motor part only and it does not have a direct access to the load, the electrical drive presents a simple underactuated model.

B. Switched control synthesis

As mentioned above, our objective is to design a switched regulator, using the measurements of the motor angular position \(q_i(t)\) and velocity \(\dot{q}_i(t)\) only, so as to obtain the closed loop system in which all the trajectories are bounded and the output \(q_o(t)\) asymptotically decays to a desired position \(q_d\) as \(t \to \infty\) while also attenuating the influence of external disturbances.

To investigate the regulation problem, let us introduce the state deviation vector \(x = [x_1, x_2, x_3, x_4]^T\) with
\[
x_1 = q_o - q_d
\]
\[
x_2 = q_o
\]
\[
x_3 = q_i - N q_d
\]
\[
x_4 = \dot{q}_i
\]
(1This should be viewed as a restriction on the validity of the present backlash model.

2Implication (23) should be viewed as a physical constraint on the displacement \(\Delta q\) of the motor position \(q_i\) with respect to the reducer \(N q_o\).
where $x_1$ is the load position error, $x_2$ is the load velocity, 
$x_3$ is the motor position deviation from its nominal value, 
$x_4$ is the motor velocity. The nominal motor position value 
$Nq_d$ has been pre-specified in such a way to guarantee that 
the deviation system

$$
\dot{x} = Ax + \varphi(x) + B\tau_m 
$$

(25)

where

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-J_o^{-1}KN^2 & -J_o^{-1}f_o & 0 & 0 \\
0 & 0 & 0 & 1 \\
J_o^{-1}KN & 0 & -J_i^{-1}K & -J_i^{-1}f_i \\
\end{bmatrix},
$$

$$
\varphi(x) = \begin{bmatrix}
J_o^{-1}KN\eta(x_4 - Nx_1) \\
0 \\
-J_i^{-1}K\eta(x_3 - Nx_1) \\
0 \\
\end{bmatrix},
$$

$$
B = \begin{bmatrix}
0 \\
0 \\
0 \\
J_i^{-1} \\
\end{bmatrix},
$$

is minimum phase with respect to the output $x_3 = 0$. Indeed, 
according to [4] the zero dynamics of system (25) on the 
manifold $x_3 = 0$ is given by

$$
\begin{align}
\dot{x}_1 &= x_2 \\
x_2 &= J_o^{-1}[-KN^2x_1 - f_o x_2 + KN\eta(-Nx_1)] \\
\end{align}
$$

(26)

and its global asymptotic stability is guaranteed by the 
following.

**Theorem 3:** Let the nonlinearity $\eta$ be governed by (24). 
Then system (26) is globally asymptotically stable.

**Proof:** To begin with, let us consider a Lyapunov 
function of the form

$$
V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}KN^2J_o^{-1}x_2^2 + 
8jKNJ_o^{-1}\left[2\ln\frac{e^{\frac{N\eta}{J_o}}}2 + 1 - \frac{Nx_1}{2j}\right].
$$

(27)

Since

$$
2\ln\frac{e^{\frac{N\eta}{J_o}}}2 + 1 \geq \frac{Nx_1}{2j} \text{ for all } x_1 \in R
$$

by inspection, the function $V$, governed by (27), appears to 
be positive definite.

Then, let us compute the time derivative of the Lyapunov 
function on the trajectories of (26):

$$
\dot{V} = -J_o^{-1}(KN^2x_1 + f_o x_2 + 4jKN\eta\frac{N\eta}{J_o} - 1)x_2 + 
KN^2J_o^{-1}x_1 x_2 + 8j^2KNJ_o^{-1} \frac{4N\eta}{4j(e^{\frac{N\eta}{J_o}} + 1)}x_2 - \frac{Nx_2}{2j} \leq 0.
$$

(28)

Now let us observe that the zero dynamics system (26) has 
no trivial solutions on the manifold $x_2 = 0$ where the time 
derivative of the Lyapunov function equals to zero. Indeed, 
if $x_2 = 0$ then due to (26)

$$
\frac{Nx_1}{2j} + 2\frac{1 - e^{\frac{N\eta}{J_o}}}{1 + e^{\frac{N\eta}{J_o}}} = 0,
$$

(29)

thus concluding that $x_1 = 0$. To reproduce the latter 
conclusion it suffices to represent equation (29) in terms of 
$z = \frac{Nx_1}{2j}$

$$
z + 2\frac{1 - e^z}{1 + e^z} = 0
$$

(30)

and note that the left-hand side of (30) is a strictly increasing 
function of $z$ because its derivative is positive definite by inspection:

$$
1 - \frac{4e^z}{(1 + e^z)^2} = (1 - e^z)^2 > 0 \text{ for all } z \neq 0.
$$

(31)

In order to complete the proof it remains to apply the 
LaSalle-Krasovskii invariance principle to the system in 
question.

Our control strategy is to drive the system to the manifold 
$x_3 = 0$ in finite time, thereby imposing desired stability 
properties on the closed-loop system. The following switched 
control law

$$
\tau_m = -k_1\text{sgn}(x_3) - k_2\text{sgn}(x_4) - hx_3 - px_4
$$

(32)

with $h, p \geq 0$ and $k_1 > k_2 > 0$ such that

$$
6jK < k_2 < k_1 - 6jK
$$

(33)

is deduced from the quasihomogeneous controller (13) to 
globally drive the system to the zero dynamics manifold 
$x_3 = 0$ in finite time thereby providing global asymptotic 
regulation of system (25) to the desired position $q_d$.

**Theorem 4:** The servomechanism (20)-(24) driven by the 
switched regulator (32) is globally asymptotically stable at 
the desired point $x = 0$.

**Proof:** First, let us note that the physical domain of 
the servomechanism (20)-(24) is governed by implication 
(23) and hence the transmitted torque (22) satisfies the linear 
growth condition on this domain.

Moreover, being restricted to the physical domain (23), 
the transmitted torque is estimated similarly to (12):

$$
|T(\Delta q)| \leq 6jK,
$$

(34)

and by virtue of the parameter subordination (33), the gain 
condition (16) holds everywhere on the physical domain.

Since in addition, the zero dynamics (26) on the manifold 
$y = 0$ is globally asymptotically stable by Theorem 3, the 
closed-loop system (25), (32) satisfies all the conditions of 
Theorem 5.1 from [17].

By applying [17, Theorem 5.1], the closed-loop system 
(25), (32), representing the state of the servomechanism (20)-
(24) in terms of the error vector \( x \), is established to be globally asymptotically stable. Theorem 4 is thus proved.

It is worth of noting that applying Theorem 5.1 from [17] to the drive system (20)-(24) not only guarantees the stabilizing properties of the switched control law (32) but also ensures the desired robustness properties of the closed-loop system (25), (32) against matched external disturbances with norm bounds less than the maximum value of \( k_1 - k_2 \) and \( k_2 \).

Performance issues of the quasihomogeneous regulation of a drive system with backlash have been tested in a numerical study presented in [20] whereas an experimental support will appear elsewhere.

V. CONCLUSIONS

The quasihomogeneous control synthesis is developed to locally asymptotically stabilize underactuated mechanical systems. The stabilizing strategy is to drive the system to the zero dynamics manifold in finite time and maintain it there in spite of parameter uncertainties and external disturbances. Desired robustness properties and asymptotic stability of the closed-loop system are thus provided.

The proposed control synthesis presents an interesting alternative to the standard sliding mode control techniques. Although the resulting controllers do exhibit Zeno modes with an infinite number of switches on a finite time interval, however, they do not rely on the generation of sliding motions on the switching manifolds but on their intersections. Compared to sliding mode controllers, Zeno mode controllers exhibit additional attractive features recently illustrated in applications to fully actuated friction mechanical systems [16]-[17]. In those publications the controllers were demonstrated to be capable of providing the static position stabilization and the desired system performance in spite of significant uncertainties in the system description as is typically the case in control of electromechanical systems with complex hard-to-model nonlinear phenomena.

In the present work, the quasihomogeneity-based control synthesis is extended to underactuated mechanical systems. In order to locally stabilize an underactuated system around an unstable equilibrium, an output of the system is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the desired stability property of the closed-loop system is provided by applying a quasihomogeneous controller, driving the system to the zero dynamics manifold in finite time.

As an illustration, the proposed quasihomogeneous synthesis is shown to be capable of providing the global asymptotic regulation of a mechanical drive system with backlash to a desired load position in a practical situation where the motor angular position and velocity are the only information available for feedback.

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