On Fractional Systems $H_\infty$ -Norm Computation

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Abstract— Two methods are proposed in this paper for fractional system $H_\infty$-norm computation. These methods are extensions to fractional systems of well-known methods for integer systems. The first is based on singular value properties of a linear system and is applied on an academic example. In the second, two extensions of the real bounded lemma derived directly from Lyapunov’s theory are deduced. The first method is applied on an academic example.

I. INTRODUCTION

Fractional differentiation is now a well known tool for controller synthesis. Several presentations and applications of the fractional PID controller [1], [2], [3], [4] and of CRONE control [5] demonstrate their efficiency. Fractional differentiation also permits a simple representation of some high order complex integer systems [6]. Consequently, basic properties of fractional systems have been investigated these last ten years and criteria and theorems are now available in the literature concerning stability [7], observability, and controllability [8] of fractional systems.

Lyapunov based methods have also been developed for stability analysis and control law synthesis of integer linear systems, and for more complex systems such as nonlinear, time-varying, and LPV [1]. This has been possible, thanks to the development of efficient numerical methods to solve convex optimization problems [10], by resolving Lyapunov’s theory. The main point is that the methods presented here can easily be formulated in the form of Linear Matrix Inequalities (LMI), thus be easily solved thanks to the development of efficient numerical methods to solve convex optimization problems. The first tool is applied to an academic example.

II. NOTATIONS AND DEFINITIONS

A. Fractional Calculus

Riemann-Liouville fractional differentiation is used and the fractional integral of a function $f(t)$ is defined by

$$I^\nu_0 f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi)d\xi$$

(1)

where $\nu \in \mathbb{R}^+$ denotes the fractional integration order, and where

$$\Gamma(\nu) = \int_0^\infty e^{-x}x^{\nu-1}dx.$$  

(2)

The fractional derivative of a function $f$ of order $\nu \in \mathbb{R}^+$ can consequently be defined by [15]

$$D^\nu f(t) = D^m[D^{m-\nu}f(t)],$$

(3)

where $m$ is the smallest integer that exceeds $\nu$.

B. $H_\infty$ -Norm

The $H_\infty$-norm of a continuous, Linear, Time-Invariant (LTI) system whose transfer function is $G(s)$, is defined through the $L_2$-norm by:

$$\|G(s)\|_\infty = \sup_{\|U(s)\|_2 < 1} \frac{\|Y(s)\|_2}{\|U(s)\|_2},$$

(4)

where $Y(s)$ and $U(s) \in H_2$ denote respectively Laplace transform of the output signal and of the input signal. $L_2$-norm of signal $x(t)$ is

$$\|x(t)\|_2 = \left(\int_0^\infty x(t)^2 dt\right)^{1/2}$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)H(X(j\omega))d\omega\right)^{1/2}.$$  

(5)

Consequently, $H_2$ is the set of functions $f(s)$, analytic on $\Re(s) \geq 0$, and whose $L_2$-norm is bounded.
It can also be shown that
\[
\| G(s) \|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(G(j\omega)), \tag{6}
\]
where \( \sigma_{\text{max}} \) denotes the greatest value of the singular values of \( G(j\omega) \) defined as
\[
\sigma(G(j\omega)) = \sqrt{\text{spec}(G(-j\omega)^T G(j\omega))}. \tag{7}
\]

III. COMPUTATION OF FRACTIONAL SYSTEM \( H_\infty \)-NORM BASED ON STABILITY ANALYSIS

A. Method description

Let us consider a stable Multi-Input, Multi-Output (MIMO) LTI fractional system \( S_f \) whose input \( u(t) \in \mathbb{R}^p \) and output \( y(t) \in \mathbb{R}^m \) are linked by the fractional differential equation:
\[
(\bar{S}_f): \sum_{i=0}^{M} b_i(D^\nu)^i y(t) = \sum_{i=0}^{N} a_i(D^\nu)^i u(t). \tag{8}
\]

It is supposed that \( N \leq M, (N,M) \in \mathbb{N}^2, a_i \in \mathbb{R}^{m \times p}, b_i \in \mathbb{R}^{p \times m}, \) and that all the differentiation orders are multiples of the commensurate order \( \nu \).

It is also assumed that system \( S_f \) is relaxed at \( t=0 \), so the Laplace transforms of \( D^\alpha u(t) \) and of \( D^\alpha y(t) \) are respectively considered as \( s^\alpha U(s) \) and \( s^\alpha Y(s) \) for any \( \alpha \in \mathbb{R} \).

Given commensurate order hypothesis, system \( S_f \) also admits the state-space description [7]:
\[
(\bar{S}_f): \begin{cases}
D^\nu x(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\end{cases}, \tag{9}
\]

where \( \nu \in \mathbb{R}^+ \) denotes the fractional order of the system, \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times p}, C \in \mathbb{R}^{n \times m}, \) and \( D \in \mathbb{R}^{n \times p} \).

Based on this representation, transfer matrix \( G(s) \) is given by
\[
G(s) = C(s^\nu I - A)^{-1} B + D. \tag{10}
\]

For simplicity, the form \( (A,B,C,D,\nu) \) is used in the paper to refer to state space description of relation (9).

Let \( \gamma \) denotes a real positive number satisfying
\[
\gamma > \sigma_{\text{max}}(D). \tag{11}
\]

From relation (6), system \( S_f \) \( H_\infty \)-norm is bounded by \( \gamma \) if and only if
\[
\sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(G(j\omega)) < \gamma, \tag{12}
\]

namely, if and only if
\[
\forall \omega, \forall i \in \{1,...,\min(m,p)\}, \sigma_i(G(j\omega)) < \gamma, \tag{13}
\]
where \( \sigma_i(G(j\omega)) \) is the \( i^{th} \) singular value of matrix \( G(j\omega) \). Hence, from relation (7), \( H_\infty \)-norm of system \( S_f \) is bounded by \( \gamma \) if and only if
\[
\forall \omega, \forall i, \lambda_i(G(-j\omega)^T G(j\omega)) < \gamma^2, \tag{14}
\]

Due to eigenvalue properties, relation (14) can be rewritten as:
\[
\forall \omega, \forall i, \lambda_i(\gamma^2 I - G(-j\omega)^T G(j\omega)) > 0, \tag{15}
\]
which is equivalent to the Linear Matrix Inequality (LMI):
\[
\forall \omega, \gamma^2 I - G(-j\omega)^T G(j\omega) > 0. \tag{16}
\]

As
\[
\lim_{\omega \to \infty} \gamma^2 I - G(-j\omega)^T G(j\omega) = \gamma^2 I - D^T D, \tag{17}
\]
which is positive from condition (11), relation (16) is satisfied if and only if
\[
\forall \omega, \gamma^2 I - G(-j\omega)^T G(j\omega) \text{ is non-singular}, \tag{18}
\]
that is if and only if
\[
\gamma^2 I - G(-s)^T G(s) \text{ has no zero on the imaginary axis.} \tag{19}
\]

Hence the \( H_\infty \)-norm of system \( S_f \) is bounded by \( \gamma \) if and only if system \( S_f \) whose transfer matrix is
\[
G_\gamma(s) = \left[\gamma^2 I - G(-s)^T G(s)\right]^{-1} \tag{20}
\]
is asymptotically stable.

\( H_\infty \)-norm of fractional system \( S_f \) can thus be computed using a dichotomous algorithm on variable \( \gamma \), stability of system \( S_\gamma \) being analyzed for each value of \( \gamma \) produced by the algorithm. This stability analysis can be done using LMI tools recently developed [16]. Application of these tools requires a state space description for \( S_\gamma \).

According to [16], note that \( \gamma^{\alpha} \) stability is used to refer to the asymptotic stability of linear fractional systems.

From relation (10),
\[
G(-s) = C((-s)^\nu I - A)^{-1} B + D, \tag{21}
\]
or, using exponential form of \( -1 \),
\[
G(-s) = C(e^{s\nu} I - A)^{-1} B + D, \tag{22}
\]

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and then

\[ G(s) = C \left( s^\gamma I - e^{-\gamma s} A \right)^{-1} e^{-\gamma s} B + D. \quad (23) \]

State space description associated with \( G(-s)^T \) is thus

\[ \left( e^{-\gamma s} A^T, C^T, e^{-\gamma s} B^T, D^T, v \right). \]

State space description associated with transfer matrix

\[ \gamma^2 I - G(-s)^T G(s) \]

is thus \( (A', B', C', D', \nu) \), were

\[ A' = \begin{pmatrix} A & 0 \\ C^T & e^{-\gamma s} \end{pmatrix}, \quad B' = \begin{pmatrix} B \\ C^T D \end{pmatrix}, \quad D' = \begin{pmatrix} D^T C & e^{-\gamma s} B^T \end{pmatrix}, \]

and \( D' = D^T D \).

Finally, from dynamic inversion rule, state space description of system \( S_{\gamma} \) is \( (A_{\gamma}, B_{\gamma}, C_{\gamma}, D_{\gamma}, \nu) \) where

\[ A_{\gamma} = \begin{pmatrix} A + B(D^T D - \gamma^2 I)^{-1} D^T C & e^{-\gamma s} B(D^T D - \gamma^2 I)^{-1} B^T \\ C^T (I + D^T (D^T D - \gamma^2 I)^{-1} D^T)^{-1} & e^{-\gamma s} (D^T D - \gamma^2 I)^{-1} B^T \end{pmatrix} \]

\[ B_{\gamma} = \begin{pmatrix} B(D^T D - \gamma^2 I)^{-1} D^T C \\ C^T (D^T D - \gamma^2 I)^{-1} B^T \end{pmatrix}, \quad C_{\gamma} = \begin{pmatrix} \gamma^2 I - D^T D \end{pmatrix} \]

and \( D_{\gamma} = \gamma^2 I - D^T D \).

\[ (24) \]

Given conclusion directly after relation (20), and using Matignon’s stability theorem [7], the following theorem can hence be stated.

**Theorem 1:** \( H_\infty \) norm of fractional system \( S_f \), whose state space description is given by relation (9), is bounded by a real positive number \( \gamma \) if and only if the eigenvalues of matrix \( A \), given by relation (24) lie in the stable domain defined by \( \{ s \in \mathbb{C} \mid \arg(s) > \gamma \frac{\pi}{2} \} \).

\[ \square \]

**B. Application**

Consider a DC motor whose transfer function is

\[ G(s) = \frac{K}{s(1 + \tau_e s)(f_m + J_m s)}, \quad (25) \]

where \( K = 2.34\, Nm/V^{-1}, \tau_e = 4.7 \times 10^{-3}\, s \). The viscous forces are modeled by \( f_m = 2 \times 10^{-3}\, Nm.s.rad^{-1} \), and \( J_m \) is the system inertia. By adding masses on the rotated disk, \( J_m \) can vary such that \( J_m = J_0 + \Delta J \), where \( J_0 = 0.066\, kg.m^2 \) and \( |\Delta J| < 0.042\, kg.m^2 \).

The dynamic performances of the motor are imposed using a fractional controller whose transfer function is

\[ C(s) = K_c \left( \frac{s^\nu}{\omega_b} + 1 \right) \left( \frac{s^\nu}{\omega_f} + 1 \right), \quad (26) \]

where \( \omega_b = 0.1\, rad.s^{-1}, \omega_f = 10\, rad.s^{-1}, \nu = 0.6 \) and \( K_c = 2.82 \times 10^{-3} \) to ensure a nominal gain cross-over frequency \( \omega_{eg} = 1 rad.s^{-1} \) as can be seen in the open loop Bode diagrams in figure 1.

As controller \( C(s) \) has been designed using nominal plant \( G_0(s) \) defined by

\[ G_0(s) = \frac{K}{s(1 + \tau_e s)(f_m + J_0 s)}, \quad (27) \]

stability of the closed loop for the entire plant set defined by (25) is to be verified. Small gain theorem introduced in [17] can be used for this purpose.

![Fig. 1. Nominal open loop Bode diagrams](image)

**Theorem 2:** (Small gain theorem) If \( M(s) \) and \( \Delta(s) \) are stable, the system depicted on figure 2 is stable for every \( \Delta(s) \) such that \( \|\Delta(s)\|_\infty < \alpha \) if \( \|M(s)\|_\infty \leq \alpha^{-1} \).

Fig. 2. Standard system for small gain theorem application

Noticing that

\[ \frac{1}{f_m + J_0 s} = \frac{1}{f_m + J_0 s} \]

\[ \frac{1}{f_m + J_0 s} \]

\[ \frac{1}{f_m + J_0 s} \]

transfer function \( G(s) \) can be rewritten in the form

\[ G(s) = \frac{K}{s(1 + \tau_e s)} \frac{1}{f_m + J_0 s}, \quad (29) \]

The closed loop defined by

\[ \beta_c = \frac{C(s)G(s)}{1 + C(s)G(s)} \quad (30) \]
where $\gamma(t)$ and $u(t)$ respectively denote output and input of system $S$. $H_\infty$-norm of system $S$ is then defined as the lowest $\gamma$ satisfying relation (33).

Note that relation (33) holds for any system $S$ (linear or not), including fractional systems.

Among the various methods that can be used for the $H_\infty$-norm computation of a system, real bounded lemma permits its computation through an LMI resolution.

**Lemma 1** [17]: (real bounded lemma) Integer system $S_i$ whose state space description is given in (9) with $\nu = 1$, is stable and its $H_\infty$-norm is bounded by $\gamma$ if and only if there exists a symmetric positive definite matrix $P$ such that

$$
\begin{pmatrix}
A^T P + PA & PB & C^T \\
B^T P & -\gamma I & D^T \\
C & D & -\gamma I
\end{pmatrix} < 0.
$$

A rigorous proof of this theorem can be found in [17].

For a fractional system, real bounded lemma can not be directly applied to state space description (9) as $x(t)$ is not explicitly given in this state space description.

### B. Extension to general MIMO fractional systems

As shown by figure 4, it is supposed that the studied fractional system is decomposed into two sub-systems, a purely integer one $S_i$ (whose output is $y_i(t)$) and a fractional one $S_{ni}$ (whose output is $y_{ni}(t)$).

For instance, such a decomposition can be obtained using the methods presented in [18] and [19].

It is also supposed that the $L_2$ gain of the fractional subsystem $S_{ni}$ is bounded by a known real number $r_2$ (as it is in [18] and [19]),

$$\|S_{ni}\|_{L_2} \leq r_2,$$

and that no pure fractional integration appears in $S_{ni}$.

From (33), $H_\infty$-norm of a fractional system $S_f$ is bounded by $\gamma$ if and only if $S_f$ is stable and $\forall T \geq 0$,

$$
\int_0^T (y_i(t) + y_{ni}(t))^T (y_i(t) + y_{ni}(t)) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt,
$$

(36)
or \( \forall T \geq 0, \)
\[
\begin{align*}
\int_0^T y_i(t)^T y_i(t) + y_{ni}(t)^T y_{ni}(t) + 2 y_{ni}(t)^T y_i(t) dt & \leq \gamma^2 \int_0^T u(t)^T u(t) dt. \tag{37}
\end{align*}
\]

As
\[
(y_{ni}(t) - y_i(t))^T (y_{ni}(t) - y_i(t))
= y_{ni}(t)^T y_{ni}(t) + y_i(t)^T y_i(t) - 2 y_{ni}(t)^T y_i(t),
\]
and considering that
\[
(y_{ni}(t) - y_i(t))^T (y_{ni}(t) - y_i(t)) \geq 0,
\]

it is straightforward to note that
\[
y_{ni}(t)^T y_{ni}(t) + y_i(t)^T y_i(t) \geq 2 y_{ni}(t)^T y_i(t),
\]

and thus:
\[
\text{It is shown in [20] that the } L_1 \text{ norm of the impulse response of fractional subsystem } P \text{ is bounded by a calculable real number } \eta \text{ such that}
\]
\[
\|P\|_{L_1} \leq \eta .
\]

Under some conditions, this decomposition is said "structural". This decomposition is not presented here for brevity purposes but is exhaustively given in [20]. Note that this decomposition can be easily extended to MIMO systems described by transfer matrix. Demonstration is in MIMO case based on partial fraction expansion of each element of fractional system \( S_f \) transfer matrix \( G(s) \) given by (10).

It is shown in [20] that the \( L_1 \) – norm of the impulse response of fractional subsystem \( P \) is bounded by a calculable real number \( \eta \) such that

\[
\|P\|_{L_1} \leq \eta .
\]

From relation (46),
\[
\forall T \geq 0, \int_0^T y_{p}(t)^T y_{p}(t) dt \leq \eta^2 \int_0^T u(t)^T u(t) dt,
\]

where \( \eta \) is a bound of the \( L_1 \) – norm of the impulse response of fractional subsystem \( P \).

A procedure similar to the one used to obtain lemma 1 can then be employed to derive lemma 2.

\[\text{Lemma 2}: \text{(Extended lemma) Fractional system } S_f \text{ is stable and its } H_\infty \text{-norm is bounded by } \gamma \text{ if there exists a symmetric positive definite matrix } P \text{ such that}
\]
\[
\begin{pmatrix}
A_i^T P + PA_i & PB_i & C_i^T \\
B_i^T P & \left(\gamma^2 - 2 r_2^2\right)^{1/2} I & D_i^T \\
C_i & D_i & \left(\gamma^2 - 2 r_2^2\right)^{1/2} I
\end{pmatrix} < 0,
\]

where \( \begin{pmatrix} A_i, B_i, C_i, D_i, 1 \end{pmatrix} \) is the state space description of the integer subsystem \( S_{ni} \) and \( r_2 \) is a bound of the \( L_2 \) – norm of the fractional subsystem \( S_{ni} \) of figure 4.

\[\text{Lemma 3}: \text{(Extended lemma, a second version for SISO systems) SISO Fractional system } S_f \text{ is stable and its } H_\infty \text{-norm is bounded by } \gamma \text{ if there exists a symmetric positive definite matrix } P \text{ such that}
\]
\[
\begin{pmatrix}
A_E^T P + PA_E & PB_E & C_E^T \\
B_E^T P & \left(\gamma^2 - 2 r_2^2\right)^{1/2} I & D_E^T \\
C_E & D_E & \left(\gamma^2 - 2 r_2^2\right)^{1/2} I
\end{pmatrix} < 0,
\]

and considering that
\[
\begin{align*}
\int_0^T y_i(t)^T y_i(t) + y_{ni}(t)^T y_{ni}(t) & \geq 2 y_{ni}(t)^T y_i(t),
\end{align*}
\]

and thus, it is straightforward to note that
\[
y_{ni}(t)^T y_{ni}(t) + y_i(t)^T y_i(t) \geq 2 y_{ni}(t)^T y_i(t),
\]

namely if \( \forall T \geq 0, \)
\[
\int_0^T 2 y_i(t)^T y_i(t) dt + \int_0^T 2 y_{ni}(t)^T y_{ni}(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt.
\]

A procedure similar to the one used to obtain lemma 1 can then be employed to derive lemma 2.
where \((A_E, B_E, C_E, D_E, 1)\) is the state space description of the integer subsystem, and \(r\) is a bound of the \(L_1\) norm of the impulse response of fractional subsystem.

\[
\dot{y}(t) + \sum_{i=1}^{n} \frac{d^i a_i}{d t^i} y(t - \tau_i) + \sum_{i=1}^{m} \frac{d^i c_i}{d t^i} y(t - \tau_i) = \sum_{i=1}^{n} \frac{d^i b_i}{d t^i} u(t - \tau_i)
\]

Fig. 5. A decomposition of a SISO fractional system

V. CONCLUSION

Fractional PID regulator and CRONE robust regulators are now well known in the field of fractional differentiation application in control theory. Synthesis strategies for these two classes of regulators are usually done in the frequency domain and are mainly based on the application of Nyquist criterion and its extensions. Paradoxically, no method based on more powerful tools such as Lyapunov stability or small gain theorem has been investigated for fractional systems. However, such methods are now essential for the extension of the existing control methods to time-varying or nonlinear fractional systems. In order to develop control methods for more complex fractional systems than the linear one, this paper proposes two tools for the computation of a fractional system \(H_{\infty}\)-norm. The first one is based on tools for fractional linear system. Computing the \(H_{\infty}\)-norm with this method consists in stability analysis of a fractional system and can be easily implemented. The second is an extension of the real bounded lemma. It is based on decomposition of a fractional system into two sub-systems: an integer one and a fractional one. This method requires the computation of the fractional subsystem \(L_1\)-gain upper bound in the MIMO case or the computation of the fractional subsystem \(L_1\)-gain upper bound (see [20] for this gain estimation). Implementation of this method is thus complex but until now, no other real bounded lemma extension exists for fractional systems. Our goal is thus now to improve this second method in order to show its efficiency for CRONE control of time-varying systems.

REFERENCES