Wake stabilization using POD Galerkin models with interpolated modes

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Abstract—A principal challenge in the use of empirical proper orthogonal decomposition (POD) Galerkin models for feedback control design in fluid flow systems is their typical fragility and poor dynamic envelope. Closed loop performance and optimized sensor(s) location are significantly improved by use of interpolated POD modes from a succession of low dimensional models from sections of a controlled transient manifold. This strategy is demonstrated in the benchmark of stabilization of the wake flow behind a circular cylinder.

I. INTRODUCTION

The complexity of fluid dynamics and related computational (CFD) models, is a major hindrance to model based feedback control [1]: Hardware architecture optimization, feedback design and real time implementation are prohibitively expensive with such models. Effective low-dimensional flow models are therefore essential enablers.

Empirical proper orthogonal decomposition (POD) Galerkin models (GMs), based on a Karhunen-Loève approximation of flow data [2], offer efficient low-dimensional flow representations. Yet for control applications, PODs suffer from fundamental deficiencies: Dynamic fragility away from the reference orbit and flow conditions is particularly detrimental in a context where transients occupy center stage. Other shortcomings include truncated energy dynamics and a difficulty to incorporate boundary actuation. This note follows a succession of studies, e.g. [3]–[19], aiming to develop tools that make empirical GMs useful for control design.

One basic observation is that any low dimensional flow model is necessarily restricted to a dynamic manifold, formed by targeted families of transients. The issue at hand is therefore an effective modeling of the dynamics in subspace neighborhoods of such manifolds. In some cases, reasonable representations of natural transients from an unstable steady flow to an attractor are feasible with modes from both the start and end operating conditions are used, along with mean flow correction [8]. When system identification tools are used to correct system coefficients, modes extracted from the attractor and mean flow changes alone may suffice [17]. The representation of actuated transient manifolds appears to be considerably more challenging. For example whereas 3 modes suffice to capture the essence of natural transients in the laminar cylinder wake flow, some 40 POD modes were used to capture the actuated transient manifold when optimal control was sought in the same system [20]. This dimension proliferation is needed for two reasons: To compensate for the gradual deformation of dominant modes along transients and to assure sufficiently accurate prediction of actuation effects.

Here we explore an alternative modeling approach for systems where local characteristics of expansion modes geometry and model structure are preserved at intermediate operating points. The idea is to interpolate a series of local, similarly structured expansions, as a substitute for the use of a single, higher order global model. We illustrate this approach by the ubiquitous benchmark of laminar vortex shedding suppression behind a cylinder [21]–[24]. In this benchmark, instead of a single, 40-dimensional global model, a succession of 3 dimensional, similarly structured models will be used, where the 2 modes representing the oscillatory vortex shedding are varied as the system traverses actuated transients. The advantage over the traditional POD model are demonstrated both with respect to achievable closed loop performance and when sensor locations are optimized, to assure even performance throughout the transient range, rather than in a narrow neighborhood of the natural attractor.

II. EMPIRICAL GALERKIN MODELS

Empirical Galerkin models are based on experimental data or a direct numerical simulation (DNS) of the non-dimensionalized, incompressible, actuated Navier-Stokes equation (NSE)

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u}) = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{g}, \quad \nabla \cdot \mathbf{u} = 0 \quad (1)$$

where $\mathbf{u}(x, t)$ is the velocity field, $p$ the pressure, and $\mathbf{g}(x)$ a volume force modulated by the control command $\epsilon(t)$. The Reynolds number $Re = UD/\nu$ is based on the non-dimensionalization scales of velocity $U$, length $D$ and kinematic viscosity $\nu$. 
A. Standard models

Having fixed a spatial flow domain $\Omega$, velocity fields are embedded in the Hilbert space $L^2(\Omega)$ with the inner product

$$\langle u, v \rangle_\Omega := \int_\Omega dV \cdot u \cdot v, \quad u, v \in L^2(\Omega).$$

The Galerkin approximation of the flow is expressed by

$$u^{[N]}(x, t) = \sum_{i=0}^N a_i(t) \mathbf{u}_i(x),$$

where the coefficients $a_i(t)$ capture time dependence, $\mathbf{u}_0$ is the time-averaged field (so that $a_0 \equiv 1$, by definition [3]), and $\mathbf{u}_i$, $i \geq 1$, are orthonormal POD modes [2] that minimize the averaged energy residual in (3), in the reference trajectory. In this respect, (3) is an optimal energy residual in (3), in the reference trajectory. In this respect, (3) is an optimal kinematic approximation of the reference.

The standard [2] dynamic model for the Fourier coefficients is 

$$\frac{d}{dt} a_i = \frac{1}{Re} \sum_{j=0}^N l_{ij} a_j + \sum_{j,k=0}^N q_{j,k} a_j a_k$$

where the linear and quadratic terms represent the viscous and convective Navier-Stokes terms, respectively, with constant coefficients $l_{ij} := (\mathbf{u}_i, \Delta \mathbf{u}_j)_\Omega$, $q_{j,k} := (\mathbf{u}_i, \nabla \cdot (\mathbf{u}_j \mathbf{u}_k))_\Omega$. The pressure term may change the coefficients $q_{j,k}$, but not the form (4) [7].

The reference data and corresponding GM may describe a natural or forced flow. Forcing may enhance coherent structures and thus reduce the attractor’s GM dimension. Examples are the 4-dimensional model of Kelvin-Helmholtz vortices of a shear-layer [7] excited by periodic inlet condition and the 32-dimensional model of a transitional boundary layer, manipulated by a periodic upstream tripping wire [3]. The key issue is that the standard POD approximation is hardwired to the reference, including a specific actuation, and the Galerkin system (GS) has no free actuation input. This excludes the standard Galerkin modeling approach for control design.

B. Shift-mode

The shift-mode $\mathbf{u}_\Delta \propto \mathbf{u}_0 - \mathbf{u}_s$ is a mean-field correction (where $\mathbf{u}_s$ is the unstable steady NSE solution) and is orthogonal to the POD modes. Including the shift-mode $\mathbf{u}_\Delta$ as the $N+1^{st}$ mode in (3) is an enabler for non-equilibrium model for transient flow [8], [9], [13].

C. Volume force representation

The volume force in the NSE (1) may represent, e.g., a Lorentz force in magneto-hydrodynamical flows, a buoyancy term in the Boussinesq approximation, or an external pressure gradient in pipe flows. The control command, $\epsilon(t)$, modulates the fixed field $\mathbf{g}(x)$. This corresponds to a control term of the form $\epsilon(t) \mathbf{g}_i$, on the right-hand side of the $i^{th}$ GS equation (4), where $\mathbf{g}_i := (\mathbf{u}_i, \mathbf{g})_\Omega$ is the magnitude of the projection of $\mathbf{g}$ on $\mathbf{u}_i$. This basic textbook form of actuation is chosen for nomenclature simplicity. Alternative forcing may involve state dependent $\epsilon_i$ coefficients [18].

III. MODEL-BASED STABILIZATION OF THE CYLINDER WAKE

A. The System and a Simple Complete Information Control

The circular cylinder wake flow transition to instability and vortex shedding, forming a periodic attractor, at $Re \approx 47$, and is considered here at $Re = 100$. Figure 1 is a schematic of a planar flow with two forms of actuation: The vertical volume-force actuator used in this note, and an AFA experimental rig with vertical vibrations of the disk, as in [10], [11], [18]. Streamlines represent the natural flow. Optimized sensor(s) position are discussed in §III-F.

Vortex shedding is undesirable, as it causes mechanical vibrations and drag, and the design objective is the attenuation and delay of shedding to the far wake. A physically motivated control policy is based on dissipation: the energy extraction rate $-\epsilon(t)(\mathbf{g}, \mathbf{u}(t))_\Omega \approx -\epsilon(t) v_{vf}(t) A_g$, where $v_{vf}$ and $A_g$ are the respective vertical velocity field at the center of the supporting disk of the volume force, and the area of that disk. This gives rise to the feedback $\epsilon = -k(\mathbf{g}, \mathbf{u})_\Omega \approx -k A_g v_{vf}$, where the gain $k > 0$ determines the dissipation rate. Indeed, implementing this policy directly on the DNS model can be shown to completely attenuate vortex shedding.

B. A Standard POD Model of the Cylinder Wake

The natural attractor is dominated by two modes - $\mathbf{u}_1$ and $\mathbf{u}_2$ in Figure 2 - capturing some 95% of the perturbation energy. The structurally similar mean flow $\mathbf{u}_0$ (same figure) and the unstable, steady flow $\mathbf{u}_s$, are characterized by symmetry with respect to the $x$ axis and a near wake recirculation bubble which becomes shorter during transition to the attractor. The normalized difference $\mathbf{u}_0 - \mathbf{u}_s$ is the shift mode $\mathbf{u}_\Delta = \mathbf{u}_s$. (Thus the Fourier coefficient value $a_3 = 0$ represents the attractor.) The respective GS is of the form

$$\dot{a}_3 = A(a) a + B \epsilon + \eta, \quad s = Ca$$

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where \( a := [a_1, a_2, a_3]^T, \eta = [0, 0, \eta_3]^T \) is the constant term from (4), \( \epsilon \) is the actuation command, \( s \) is the sensor signal, and

\[
A(a) := \begin{bmatrix}
\sigma_r & -\omega & -\beta a_1 \\
(\omega + \gamma a_3) & \sigma_r & -\beta a_2 \\
\delta a_1 & \delta a_2 & -\rho
\end{bmatrix}, \quad B = \begin{bmatrix} g_1 \\ g_2 \\ 0 \end{bmatrix}
\]

This form\(^1\) clearly reveals the key features of a periodic attractor and a nearly parabolic attracting invariant manifold, formed by transients from perturbations of the steady solution \((a_1 = a_2 = 0, a_3 = -\eta_3/\rho)\) to the attractor \((a_3 = 0)\) in the unactuated flow \([8], [17]\).

\[\begin{aligned}
\dot{r} &= \sigma_r - \beta r - \rho a_3 + b \cos(\theta - \phi) \\
\dot{a}_3 &= \delta a_1 r + \delta a_2 - \rho a_3 \\
\dot{\phi} &= \omega + \gamma a_3 + \frac{b}{r} \sin(\theta - \phi) \epsilon
\end{aligned}\]

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\end{aligned}\]

where \( \theta = \angle (g_1, g_2) \) and \( b = \sqrt{g_1^2 + g_2^2} \). This form reveals two basic facts: First, however designed, an admissible and effective attenuating actuation must be in phase with \(-\cos(\theta - \phi)\) \([9], [12], [13]\). That is, such control policy is bound to (roughly) imitate the simple physics based, dissipative policy suggested in §III-A. A subsequent observation is that the angle \( \theta \), extracted from the low dimensional Galerkin approximation, is critical to correct orientation of the actuation force. This point will be revisited later.

A simple GM based counterpart of the dissipative control of §III-A is that where the inner product \((g, u(t))\) is substituted by the Galerkin approximation: Since the mean flow makes no contribution to the vertical velocity along \( x = 0 \), the Galerkin approximation (3) leads to \( a_1(t)g_1 + a_2(t)g_2 \). Equivalently, the actuation is set to be proportional to \(-\beta r \cos(\theta - \phi)\). A dynamic observer \([9], [12]\) could be used to dynamically estimate the Fourier coefficients, hence the values of \( r \) and \( \phi \), from sensor data (ignoring here the

\(^1\)The convention that \( a_2 = 0 \) on the attractor is different from some of our previous notes, and results here with \( \sigma_r \approx 0 \).
circles in Figure 6, were obtained by complete information feedback $\epsilon = -kv_{r f}$, for escalated values of the gain $k$. We are interested in transients along the manifold connecting these cycles. The dynamics near each of the cycles is dominated by the (same) shift mode and two locally extracted oscillatory modes $u_1$ and $u_2$. The Galerkin system is obtained by projecting the NSE (1) on these local modes, which are the same (6)–(8), albeit with different coefficients, which could be parameterized by the characteristic value of $a_3$ on the respective cycle. That is, the local values of $\beta$, $\delta$, $\omega$, $\gamma$, $\rho$, $\eta$, $g_1$ and $g_2$, are functions of the characteristic value $a_3$ (equiv. of the respective length $x_{Rec}$ of the recirculation bubble). This dependence can be easily parameterized or tabulated. If, in addition, the local $u_1$ and $u_2$ modes or oriented so that the local Fourier coefficients $a_1$ and $a_2$ of a flow field will transition continuously between neighboring models, a global interpolated model will be formed: It will retain the form (6) (with $a_3$ dependent coefficients) and the Fourier coefficients will be interpreted with respect to the local expansion modes that is associated with $a_3$. This model will be valid for slow vertical transitions along the dynamic manifold in Figure 6 (which prevents the need to include the dynamics of mode deformation). While the total number of modes used in constructing the interpolated model may be large, say, on par with the number of modes used in [20], the great advantage of the proposed model is that, at any given time, only a small number of Fourier coefficients - here 3 coefficients - are involved. The Galerkin system thus maintains a relatively simple, low-dimensional structure. These advantages make the interpolated model particularly suitable for practical feedback design and implementation.

While the technical details of the interpolation deserve well more than the space available here, the key relevant fact for control design, in our system, is that the local model provides both the instantaneous phase $\phi$ and amplitude $r$, of the flow. Indeed, these are the three key quantities needed for effective control.

**E. Control Design With Interpolated POD Models**

Demonstrating the advantage of the suggested modeling paradigm for control design, we implement the counterpart of the physics based control of §III-A, now, using the interpolated Galerkin model to estimate $v_{r f}$ in terms of the Fourier coefficients $a_1$ and $a_2$, as determined by the current local model. (Equiv., we extract local values of $r$, $\phi$ and $\theta$.) The dissipative control policy remains $\epsilon = -kv_{r f}$ (equiv. $\epsilon = -kr \cos(\theta - \phi)$). Figure 6 compares the natural attractor with limit cycles obtained by feedback control with a POD model extracted from the natural flow, control using the interpolated Galerkin model, and, as a benchmark, control with direct flow measurement. In all cases, the control policy and feedback gain are identical: $k = 0.3$. As can be seen, the attenuation achieved with the traditional POD model is much inferior to what is attained with the interpolated model, which, in turn is close to the response with direct flow measurement. This improvement is enabled by the fact that the high level of flow reconstruction by the interpolated model is maintained along trajectories, but lost when the natural attractor’s POD is used.

In closing this section it must be noted that the simplicity of low order description need not mask the intrinsic distributed nature of the flow. Here, holding transients on the targeted manifold becomes harder as vortex shedding is attenuated. The reason is the relative weight of the flow field within the volume force domain is approaching zero, making phase prediction especially difficult. Figure 5 depicts the trajectories of a GM based forcing and of the inner product $(g, u)_\Omega$, as well as representation of the phase prediction at two representative snapshots of that flow trajectory. As the flow field over the volume force domain attenuates, closed loop dynamics develops a periodic “ringing”, whereby the correct phase prediction deteriorates as the $(g, u)_\Omega$ approaches zero and is recovered when oscillations increase. As can be anticipated, the best phase predictions are associated with POD modes obtained on the shortest domain, which offers the tightest cover of the volume force.

**F. Optimizing Sensor Location and Observation with an Interpolated POD Model**

As in any feedback design, dynamic observers / estimators are intrinsic in closed loop flow control. The challenges posed by the distributive nature of the flow and the often strict physical limitations on hardware (i.e., space, weight, location, etc.) are as manifest in this context as they are in the context flow actuation. The viability of observer based feedback in the current system has been demonstrated in [9], [12]. Here we shall therefore be content with brief comments concerning observer design, and dedicate the remaining available space to the benefits of the interpolated model in optimizing sensor location(s).

In the context of the dissipative actuation, above, an observer is charged with two tasks: The estimation of the “vertical” flow state position along the manifold depicted in Figure 6, and, given that evaluation, the estimation of the Fourier coefficients, relative to the appropriate local Galerkin expansion. As long as actuation does not impose sharp “vertical” transients, the former task amounts to a determination of $a_3$ (relative to the unique shift mode) or an equivalent quantity. Examples of equivalent quantities include the oscillation amplitude and frequency [9], which stand in a monotonous relation with $a_3$. When the controlled changes in the operating condition, hence the frequency, are slow, the frequency can be easily tracked in real time from oscillatory sensor readings [26] whose primary targets are the Fourier coefficients $a_1$ and $a_2$, as detailed below. An alternative is the low pass filtered version of a stream-wise velocity sensor reading. A good location to minimize the harmonic component would be along the equator, downstream from the saddle point $(x_{Rec}, 0)$, say at $(5, 0)$. As seen in Figure
Fig. 6. Top: Phase portrait of the first three Fourier coefficients feedback with direct flow measurements, all with on the respective base modes. Four limit cycles are shown: i) the natural oscillation amplitude of the (actual) vertical velocity force, denoted \( v_f \) at the center of the volume force, denoted \( v_{f,\text{max}} \), b) the average length of the recirculation bubble \( x_{\text{Rec}} \), c) the perturbation (= turbulent kinetic energy (TKE)) in each limit cycle, and d) the Fourier coefficient \( a_3 \).

7, this quantity provides a good indicator of both the operating condition (hence of \( a_3 \)) and of the associated oscillation frequency.

Under slow transition in the operating condition (hence in \( a_3 \) and \( r \)) the dynamic estimation of the Fourier coefficients becomes equivalent to observer design in a pure oscillator with slow drifts in its periodic characteristics. The main task is then phase estimation, rather than full state estimation as in [9], [12]. We shall use this framework as a simple illustration of issues associated with optimizing sensor(s) location.

A simplified dynamic model is then
\[
\frac{d}{dt} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & -\bar{\omega} \\ \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + s C \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}
\]
(9)

where \( \bar{\omega} \) is the (known) instantaneous frequency, the known

\[ C_{i,j} = v_{i,j}^T u_{i,j}(x_i), \quad i = 1, \ldots, m, \quad j = 1, 2 \]

where \( \lambda \) parameterizes the operating conditions, \( p \) parameterizes the set of sensor locations \( x_i \), and respective orientation unit vectors \( v_i \). A good indicator of the sensitivity of the sensors to the state is the Observability Grammian. Since we are interested in short time behavior of a system with a drift, we shall consider the normalized finite Grammian over a period \( T = 2\pi/\bar{\omega} \):

\[
G = \frac{1}{T} \int_0^T e^{At} C' C e^{At} dt = \frac{1}{2} \|C\|_F^2 I_2
\]
(10)

where we use the “\( A \)” matrix from (9), \( \|C\|_F \) is the Frobenius norm, \( I_2 \) is the identity matrix, and where the second equality is obtained by straightforward calculation. Having fixed \( \lambda \), an optimal sensor placement is thus one that maximizes \( \|C\|_F \). This issue has already been addressed in the cylinder wake benchmark [27], albeit based on the physical insight that it is advantageous to place and orient velocity sensors at points of local extrema of \( u_1 \) and \( u_2 \), rather than on Grammian evaluation.

The true issue, however, is due to the location of such extrema varying markedly with the operating point, as easily predicted from Figure 2 (b), (c) and (d). A meaningful criterion is thus rather to maximize the worst case value of \( G \), over all operating points:

\[
\max_p \min_\lambda \|C(\lambda, p)\|_F^2
\]
(11)

The optimal set of sensors is defined by \( p_\star \), for which the maximum in (11) is attained. This is dependent, of course, on the range of operating conditions considered.

In Figure 8 we show plots of \( \|C(\lambda, p)\|_F^2 \) as a function of \( \lambda \), where \( p \) was optimized with respect to the first 1, 4, 9 or 18 (out of 18) equally spaced operating conditions, and for the case of a single and for three velocity sensors. As is clearly
observed, the performance of sensors that are selected for a single \(\lambda\) (i.e., for the natural attractor) is higher early on but deteriorates rapidly with the change of \(\lambda\), while those optimized over a wider range maintain an increasingly even performance during transitions. Multiple sensors improve the relative flatness (peak-to-peak ratio) of this performance measure.

IV. CONCLUSIONS

A framework of interpolated Galerkin models for fluid flow systems strikes a balance between the need for higher number of modes to represent actuation and transients and the desire to maintain model simplicity and minimize the number of dynamic variables that need to be estimated in real time, in feedback implementation. Advantages over traditional POD models have been illustrated in the context of vortex shedding suppression behind a circular cylinder, and are manifest by improved ability to suppress vortex shedding and an improved sensor performance over a wider transients range.

REFERENCES


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