Localized Adaptive Bounds for Online Approximation Based Control

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Abstract—This article develops new methods for adaptively bounding approximation accuracy with methods that involve localized forgetting. The existing results use global forgetting. The importance of local versus global forgetting is motivated by adding additional approximation resources in the regions where the approximation error bound is large.

Keywords: Adaptive control, nonlinear systems, adaptive bounds.

I. INTRODUCTION

Most nonlinear adaptive control methods are proposed to address model uncertainties that are assumed to be the multiplication of known nonlinearities and uncertain parameters [4], [10]. Since the first stability results appeared, adaptive robust nonlinear control has been extensively developed to retain closed-loop stability properties in the presence not only of large parametric uncertainty, but also modeling errors such as additive disturbances and unmodeled dynamics [4]. On-line approximation methods [1], [2], [5], [8], [9], [11], [12], [14], [15], [16] are designed to achieve stability and accurate reference input tracking for systems with partially unknown nonlinearities, by implementing approximations to the unknown nonlinear dynamics during the operation of the system.

Nonlinear close-loop systems which incorporate on-line approximators can be analyzed using Lyapunov stability methods. Both the feedback control law and the approximator parameter estimation equations are derived such that the time derivative of a Lyapunov function has some desirable properties (e.g., negative definiteness). The theory for approximation based nonlinear control is provided in [2], [5], [8], [9], [12], [14]. The design and analysis of adaptive systems have been extensively addressed in [2], [12], [14], including controller structure selection, automatic adjustment of the control law, and complete proofs of stability. Its application based on the feedback linearization method is developed in e.g., [8], [9]. On-line approximation based control by backstepping methods is considered in e.g., [5].

Since on-line approximation based control can never achieve an exact modeling of unknown nonlinearities, inherent approximation errors could arise even if optimal approximator parameters were selected. Usually, a restrictive assumption is made that a magnitude bound on the inherent approximation error is known. Articles [5], [12], [13] relax the assumption of a known bound on the inherent approximation errors. With a partially known bound, these articles discuss estimation of the bounding of adaptive robust controllers to guarantee global uniform ultimate boundedness.

However, the global features of the leakage modification for parameter updates in [5], [12], [13] causes each parameter estimate to drift toward certain design parameters as the operating point leaves a local region for which the parameter is applicable. Thus, both the approximated function and the bounding function will lose local accuracy and any knowledge learned from past experience will not be retained for future use. This issue of global forgetting was addressed in [17], by deriving localized leakage based adaptation algorithms for both the approximator parameters and bounding parameters. The analysis of [17] focused on the scalar single-input-single-output system: $\dot{x} = f(x) + g(x)u$ with $g(x) = 1$ and $x \in \mathbb{R}^1$.

In this paper, we use the backstepping extension proposed in [3] and develop an adaptive robust control scheme for higher order (i.e., $g_i(x) \neq 1, \forall i = 1, \cdots, n$) single-input-single-output systems by incorporating on-line approximation of the unknown bounding functions on approximation errors. The existing localized adaptation algorithms [17] for function approximator parameters and bounding parameters are extended to higher order systems with $g_i(x) \neq 1, \forall i = 1, \cdots, n$. Filtering techniques [3] are applied to calculate time derivatives of intermediate state commands for the backstepping approach. The stability and robustness results yield a smaller m.s.s. bound on the tracking error than those in the literature; in addition, the bounding function and function approximation information are retained as a function of the operating point even as the operating point moves around the operating envelope.

II. PROBLEM FORMULATION

Consider the following class of $n$th-order single-input-single-output nonlinear systems

$$\dot{x}_i(t) = f_i(x) + g_i(x)x_{i+1}(t), \quad 1 \leq i \leq n-1,$$  \hspace{1cm} (1)

$$\dot{x}_n(t) = f_n(x) + g_n(x)u(t)$$  \hspace{1cm} (2)

where $x = [x_1, \cdots, x_n]^T$ is the state vector and $u$ is the control signal. It is assumed that the system is strictly feedback passive (see p.46 in [7]). The functions $f_i(x), g_i(x), i = 1, \cdots, n$ represent nonlinear effects that are unknown at the design stage. Each of these functions is assumed to be continuous on a known compact set $D$. To ensure controllability, it is necessary to assume that each $g_i$ is bounded away from
zero and of known sign. Therefore, without loss of generality, we will invoke the following assumption:

**Assumption 1.** It is assumed that \( g_i(x) \) has lower bound such that \( g_i(x) \geq g_i(0) \geq g_i(x) > 0 \), \( \forall x \in D \), where \( g_i(x) \) is a known function and \( g_i(x) \) is a known constant.

### A. Reference Trajectory

There is a desired trajectory \( x_d(t) \) with derivative \( \dot{x}_d(t) \), both of which are available and lie in the region \( D \) for all \( t > 0 \). The region \( D \) is a compact domain of operation that is specified at the design stage. The region \( D \) contains all trajectories \( x = [x_1, x_2, \ldots, x_n]^T \) for which the system is expected to operate. In fact, we will assume existence of a constant \( \gamma > 0 \) such that

\[
\gamma \leq \min_{x \in [N-D]} (\|x_c(t) - x\|),
\]

for any \( t \geq 0 \). This condition states that the desired trajectory is at least a distance \( \gamma \) from the boundary of \( D \). The region \( D \) also defines the largest region over which approximations to \( f \) and \( g \) will be developed. Our goal is to design the control signal \( u \) to steer \( x_1(t) \) to track the reference input \( x_d(t) \) and to achieve boundedness for the states \( x_i \), for \( i = 2, \ldots, n \). Note that existing approaches in the literature (e.g., [6], [7]) would require knowledge of the first \( n \) derivatives of \( x_d(t) \).

The approach herein only requires knowledge of \( x_d(t) \) and its first derivative [3].

### B. Approximation Definition

For \( x \in D \), we define approximations to the unknown functions \( f_i(x) \) and \( g_i(x) \) as \( f_i(x) = \theta_i^f \Phi_i(x) \) and \( g_i(x) = \theta_i^g \Phi_i(x) \) for \( i = 1, \ldots, n \), where the parameter vectors \( \theta_i^f \) and \( \theta_i^g \) will be adapted on-line. For \( x \not\in D \), \( \hat{f}_i(x) = 0 \) and \( \hat{g}_i(x) = g_i(x) \). The vector \( \Phi_i(x) \) is a user specified regressor vector containing the basis functions for the approximation. Denote the support of the \( k \)-th basis function of \( \Phi_i(x) \) vector by \( S_{f,i,k} = \{ x \in D | \Phi_i(x) \neq 0 \} \). Let \( S_{f,i,k} \) denote the closure of \( S_{f,i,k} \). Note that each \( S_{f,i,k} \) is a compact set. For each \( i \), the \( \Phi_i(x) \) vector is defined as a set of positive, locally supported\(^1\) functions \( \Phi_i(x) \) for \( k = 1, \ldots, N \) such that each set \( S_{f,i,k} \) is connected with \( D = \bigcup_{k=1}^N S_{f,i,k} \) where \( N \) is a finite integer. This ensures that for any \( x \in D \), there exists at least one \( k \) such that \( \Phi_i(x) \neq 0 \). Therefore, \( \{S_{f,i,k}\}_{k=1}^N \) forms a finite cover for \( D \). Similarly, we define the support of the \( k \)-th basis function of \( g_i(x) \) as \( S_{g,i,k} \) with closure \( S_{g,i,k} \). The sets \( S_{g,i,k}, k = 1, \ldots, N \) also form a finite cover of region \( D \).

In this paper, we are not concerned with the selection of particular basis vectors \( \Phi_i \) or \( \Phi_g \). Any basis vectors which satisfy the above assumptions are qualified candidates for the regressor vectors. Splines, radial basis functions, certain wavelets, etc. satisfy these assumptions.

We define a set of parameters \( \theta_i^f \) that are optimal in the sense:

\[
\theta_i^f = \arg\min_{\theta} \left( \max_{x \in D} |f_i(x) - \theta^T \Phi_i(x)| \right)
\]

\(^1\)‘Locally supported’ means that \( \rho(S_{f,i,k}) < \mu \ll \rho(D) \), where for set \( A, \rho(A) = \max_{x,y \in A} (\|x - y\|) \).

Note that these optimal parameters are unknown. They are not used in the implemented control law, but are useful for the analysis that follows. Since \( D \) is compact and each \( f_i \) is continuous, the vector \( \theta_i^f \) exists and is well-defined. Define the parameter estimation error vector

\[
\hat{\theta}_i^f = \theta_i^f - \theta_i^f.
\]

Let

\[
\delta_i(x) = f_i(x) - (\theta_i^f)^T \Phi_i(x)
\]

represent the inherent or residual approximation error. Note that by the definition of \( \theta_i^f \) above, the maximum value of \( \delta_i(x) \) on \( D \) is bounded. This maximum value can be affected by the choice of the dimension and type of corresponding basis vector \( \Phi_i(x) \), but for a given choice of basis vector it cannot be decreased by the choice of the parameter vector \( \theta_i^f \). The upper bound on the magnitude of the residual approximation error only depends on the designer’s choice of approximator. The quantities \( \theta_i^f, \theta_i^g \) and \( \delta_i(x) \) are defined similarly.

With the above definitions, system equations (1)-(2) can be expressed as

\[
\dot{x}_i(t) = (\theta_i^f)^T \Phi_i(x) + \delta_i(x) + ((\theta_i^g)^T \Phi_i(x) + \delta_i(x)) x_{i+1}, 1 \leq i < n,
\]

\[
\dot{x}_n(t) = (\theta_i^f)^T \Phi_i(x) + \delta_i(x) + ((\theta_i^g)^T \Phi_i(x) + \delta_i(x)) u.
\]

### C. Bound Approximation

By the definition of the \( \delta_i \) and \( \delta_g \), the magnitude of these inherent approximation error functions are bounded on \( D \); however, the bound is not known. Our control approach will utilize an estimate of these upper bound functions. Therefore, we assume a form for the bounding functions with multiplicative parameters that will be estimated. To save computational effort, we reuse the same basis elements; however, the approach easily extends to the case of different basis elements.

By the above discussion, there exists a positive constant vector \( \Psi_i^f, i = 1, \ldots, n \), referred as the optimal bounding parameter, such that

\[
|\delta_i(x)| \leq (\Psi_i^f)^T \Phi_i(x), \quad \forall x \in D.
\]

The vector \( \Psi_i^f \) is not unique since any \( \Psi_i^f > \Psi_i^f \) satisfies this assumption. To avoid confusion, the optimal bounding parameter is defined to be the vector with the smallest \( 1 \)-norm such that the assumption is satisfied. A vector \( \Psi_i^f \) yielding a bound on \( |\delta_i| \) is defined similarly. Note that the optimal bounding parameter vectors \( \Psi_i^f \) and \( \Psi_i^g \) are unknown. They are used only for analytical purpose. The control law will use estimates \( \hat{\Psi}_i^f \) and \( \hat{\Psi}_i^g \) of the optimal bounding parameter vectors. Therefore, the approximated bounding functions are \( \hat{\Psi}_i^f \Phi_i(x) \) for \( |\delta_i| \) and \( \hat{\Psi}_i^g \Phi_i(x) \) for \( |\delta_i| \), where the vectors \( \hat{\Psi}_i^f \) and \( \hat{\Psi}_i^g \) will be estimated on-line. For the following analysis, we define bounding parameter estimation errors as

\[
\tilde{\Psi}_i^f = \Psi_i^f - \Psi_i^M, \quad \tilde{\Psi}_i^g = \Psi_i^g - \Psi_i^M.
\]
where each element of \(\Psi^M_{f_i}\) is defined as \(\Psi^M_{f_i,k} = \max\{\Psi^0_{f_i,k}, \Psi^0_{f_i,k}\} \), \(k = 1, \cdots, N\) with the vector \(\Psi^0_{f_i} = [\Psi^0_{f_i,1}, \cdots, \Psi^0_{f_i,N}]^T\) selected in the design stage. With these estimated upper bounds, we will select proper terms in the control signal or the intermediate state commands to properly handle the inherent approximation errors.

III. ADAPTIVE BACKSTEPPING-BASED DESIGN

Define the tracking error vector as \(\tilde{x} = [\tilde{x}_1, \cdots, \tilde{x}_n]^T\) where

\[
\tilde{x}_1 = x_1 - x_d
\]

\[
\tilde{x}_i = x_i - x_{ic} \text{ for } i = 2, \cdots, n
\]  

(4)  

(5)  

where the \(x_{ic}\) are defined below. The pseudocontrol signals \(\alpha_i\) of the backstepping procedure [6], [7] are defined as

\[
\alpha_1 = \frac{u_{a1}}{g_1 + \beta g_1}
\]

\[
\alpha_i = \frac{u_{ai}}{g_i + \beta g_i}
\]

(6)  

(7)  

where

\[
u_{a1} = -k_1 \tilde{x}_1 + \tilde{x}_d - \hat{f}_1 - \beta f_1 + u_{s1}
\]

\[u_{ai} = -k_i \tilde{x}_i + \tilde{x}_{ic} - \hat{f}_i - \beta f_i - (\hat{g}_i - 1 + \beta g_{i-1})\tilde{x}_{i-1} + u_{si}
\]

and \(u_{ai} \) for \(i = 2, \cdots, (n-1)\). The control gains, \(k_i, i = 1, \cdots, n-1\) are designer specified positive constants that will determine the decay rate for disturbances and initial condition errors. The \(u_{si}(t)\) terms are defined as

\[
u_{si}(t) = -r_i(t) \text{sign}(\tilde{x}_i)
\]

(8)  

where \(\tilde{x}_i\), which represent compensated tracking errors, will be defined below in eqn. (10). The \(u_{si}(t)\) terms are defined to return state \(x\) to the approximation region \(D\) and keep it there (i.e., to ensure that \(D\) is an invariant set). Here the gain \(r_i(t)\) is given by

\[
\frac{0}{b_{f_i} + \hat{b}_g |x_{i+1}|}, \text{ when } x \notin D
\]

(9)  

where \(b_{f_i}, \hat{b}_g\) are known upper bounds on \(|f_i(x)|\) and \(|g_i(x)|\), respectively. Note that if constants \(b_{f_i}\) and \(b_g\) are not known, then they could be estimated using the methods suggested in [13], [12]. We do not present such an adaptive bounding approach herein for \(x \notin D\) as it is not the main topic of this article.

The compensated tracking error signals \(\tilde{x}_i\) for \(i = 1, \cdots, n\) are defined as [3]

\[
\tilde{x}_i = \tilde{x}_i - \xi_i, \text{ for } i = 1, \cdots, n
\]

(10)  

where the \(\xi_i\) are defined below.

The signal \(x_{ic}\) required for eqn. (5) and its derivative \(\dot{x}_{ic}\) required for eqn. (6-7) are defined by the following the following procedure [3].

1) For \(i = 1: \)

\[
\dot{x}_1 = \hat{f}_1 + (\hat{g}_1 + \beta g_1) (\alpha_1 - \xi_2) - \tilde{x}_d - \hat{f}_1^T \Phi_{f_1}
\]

\[
- \beta g_1 x_{2c} + (\hat{g}_1 + \beta g_1) (x_{2c} - \hat{x}_{2c})
\]

\[
+ (g_1 x_2 - \hat{g}_1 x_{2c} + \delta f_1)
\]

\[
= -k_1 \tilde{x}_1 - \hat{f}_1^T \Phi_{f_1} - \beta f_1 + u_{s1} + \delta f_1
\]

\[
- \beta g_1 x_{2c} + (\hat{g}_1 + \beta g_1) (x_{2c} - \hat{x}_{2c})
\]

\[
+ (g_1 x_2 - \hat{g}_1 x_{2c}) - (\hat{g}_1 + \beta g_1) \xi_2.
\]

(11)  

The signals \(x_{ic}\) and \(\dot{x}_{ic}\) are defined as

\[
x_{ic} = -K_i (x_{ic} - x_{ic}^0)
\]

with \(K_i > k_i\) being a designer specified constant. Since the filter of (11) is being used as a means to compute \(x_{ic}\) and \(\dot{x}_{ic}\), without differentiation, the designer would typically select \(K_i > k_i\) so that \(x_{ic}\) accurately tracks \(x_{ic}^0\) over the bandwidth of \(x_{ic}^0\). Since (11) is a stable linear filter, \(x_{ic}\) and \(\dot{x}_{ic}\) will be bounded if the input \(x_{ic}^0\) is bounded.

b) Define

\[
\xi_{i-1} = -k_{i-1} \xi_{i-1} + (\hat{g}_{i-1} + \beta g_{i-1}) (x_{ic} - x_{ic}^0).
\]

This is a stable low pass filter. Its input is the product of \((\hat{g}_{i-1} + \beta g_{i-1})\) which we will prove to be bounded and \((x_{ic} - x_{ic}^0)\) which is small. For \(x_{ic}, x_{ic}^0 \in D\) we always have that \(|x_{ic} - x_{ic}^0| < 2\rho(D)\) where

\[
\rho(D) = \max_{x \in x_{ic} \in D} \|x_1 - x_2\|
\]

is the diameter of set \(D\). For any \(x\), each \(\xi_i\) is bounded by \(b\), i.e., \(|\xi_i| \leq b\), where

\[
b = 2\rho(D) \max_{i} \left[ \sup_{t} \left| (\hat{g}_{i-1} + \beta g_{i-1}) \right| \right]
\]

(12)  

\[
k = \min_{i} k_i.
\]

2) Define

\[
u = u_{ad} + u_{sn}
\]

(13)  

where \(u_{ad}\) and \(u_{sn}\) are defined as

\[
u_{ad} = \frac{u_{ad}}{g_n + \beta g_n},
\]

\[
u_{sn} = -k_n \tilde{x}_n + \tilde{x}_{nc} - \hat{f}_n - \beta f_n - (\hat{g}_n - 1 + \beta g_{n-1})\tilde{x}_{n-1} + u_{sn}
\]

and

\[
u_{sn} = -r_n(t) \text{sign}(\tilde{x}_n)
\]

(14)  

(15)  

where \(b_{fn}, \hat{b}_gn\) are defined as known upper bounds on \(|f_n(x)|\) and \(|g_n(x)|\) for \(x \notin D\), respectively. Similarly, we do not present a discussion herein for the case when \(b_{fn}\) and \(b_{gn}\) are unknown. For completeness, the signal \(\xi_n = 0\) and \(k_n\) is a designer specified positive constant.

A. Tracking Error Dynamics

This subsection uses the control approach defined above to derive the dynamics of the tracking error. This analysis can be divided into three cases.

1) For \(i = 1:\)

\[
\dot{x}_1 = f_1 + (\hat{g}_1 + \beta g_1)(\alpha_1 - \xi_2) - \tilde{x}_d - \hat{f}_1^T \Phi_{f_1}
\]

\[
- \beta g_1 x_{2c} + (\hat{g}_1 + \beta g_1) (x_{2c} - \hat{x}_{2c})
\]

\[
+ (g_1 x_2 - \hat{g}_1 x_{2c} + \delta f_1)
\]

\[
= -k_1 \tilde{x}_1 - \hat{f}_1^T \Phi_{f_1} - \beta f_1 + u_{s1} + \delta f_1
\]

\[
- \beta g_1 x_{2c} + (\hat{g}_1 + \beta g_1) (x_{2c} - \hat{x}_{2c})
\]

\[
+ (g_1 x_2 - \hat{g}_1 x_{2c}) - (\hat{g}_1 + \beta g_1) \xi_2.
\]

(16)
2) For $1 < i < n$:

$$
\dot{x}_i = \dot{f}_i + (\dot{g}_i + \beta_{g,i})(\alpha_i - \xi_i) - \bar{\theta}_{f,i}^T \Phi_{f,i} - \beta_{g,i} x_{i+1,c} + (\dot{g}_i + \beta_{g,i})(x_{i+1,c} - x_{i,c}) + (g_i x_{i+1} - \hat{g}_i x_{i+1,c}) + \delta f_i
$$

$$
= -k_i \dot{x}_i - (\bar{g}_{i-1} + \beta_{g,i-1}) \dot{x}_{i-1} - \bar{\theta}_{f,i}^T \Phi_{f,i} - \beta_{f,i} + u_s,i + \delta f_i - \beta_{g,i} x_{i+1,c} + (g_i x_{i+1} - \hat{g}_i x_{i+1,c}) - (\dot{g}_i + \beta_{g,i}) \xi_i + 1 + (\dot{g}_i + \beta_{g,i})(x_{i+1,c} - x_{i,c})
$$

$$
+ (\dot{g}_i + \beta_{g,i})(x_{i+1,c} - x_{i,c})
$$

(17)

3) For $i = n$:

$$
\dot{x}_n = f_n + g_n (u_{ad} + u_{sn}) - \dot{x}_{nc}
$$

$$
= f_n + (\dot{g}_n + \beta_{g,n}) u_{ad} - \dot{x}_{nc} - \bar{\theta}_{f,n}^T \Phi_{f,n} - \beta_{g,n} u_{ad} + (g_n - \hat{g}_n) u_{ad} + \delta f_n + g_n u_{sn}
$$

$$
= -k_n \dot{x}_n - (\bar{g}_{n-1} + \beta_{g,n-1}) \dot{x}_{n-1} - \beta_{f,n} - \bar{\theta}_{f,n}^T \Phi_{f,n} - \beta_{g,n} u_{ad} + (g_n - \hat{g}_n) u_{ad} + \delta f_n + g_n u_{sn}
$$

(18)

The equations of this section will be used in the following subsection to derive the dynamics of the compensated tracking errors defined in (10).

B. Compensated Tracking Error Dynamics

From Step 1b of the procedure described in Section III, the variables $\xi_i$, $i = 1, \cdots, n - 1$ are produced by filtering the unachieved portion of $x_{i+1,c}^0$. The variables $\dot{x}_i$ are referred as compensated tracking errors. These variables are obtained by removing the filtered unachieved portion of $x_{i+1,c}^0$ from the tracking error, as specified in eqn. (10). The dynamics of the compensated tracking errors are derived according to the three different cases in Section III-A.

1) For $i = 1$

$$
\dot{x}_1 = -k_1 \dot{x}_1 - \bar{\theta}_{f,1}^T \Phi_{f,1} - \beta_{f,1} - \beta_{g,1} x_{2,c} + (g_1 x_2 - \dot{g}_1 x_{2,c}) - (\dot{g}_1 + \beta_{g,1}) \xi_2 + \delta f_1 + u_{s,1}
$$

$$
= -k_1 \dot{x}_1 - \bar{\theta}_{f,1}^T \Phi_{f,1} - \beta_{f,1} - \beta_{g,1} x_{2,c}
$$

$$
+ (g_1 - \dot{g}_1) x_2 + (\dot{g}_1 + \beta_{g,1}) x_{2,c} + \delta f_1 + u_{s,1}
$$

$$
= -k_1 \dot{x}_1 - \bar{\theta}_{f,1}^T \Phi_{f,1} - \beta_{f,1} - \beta_{g,1} x_{2,c}
$$

$$
+ (g_1 + \beta_{g,1}) x_2 + \delta f_1 + \delta g_1, x_{i+1}
$$

(19)

2) Similarly, for $1 < i < n$

$$
\dot{x}_i = -k_i \dot{x}_i - (\bar{g}_{i-1} + \beta_{g,i-1}) \dot{x}_{i-1} + u_{s,i}
$$

$$
- \bar{\theta}_{f,i}^T \Phi_{f,i} - \beta_{f,i} - \beta_{g,i} x_{i+1,c} + (g_i + \beta_{g,i}) x_{i+1,c} + \delta f_i + \delta g_i, x_{i+1}
$$

$$
+ (\dot{g}_i + \beta_{g,i}) x_{i+1,c} + \delta f_i + \delta g_i, x_{i+1}
$$

(20)

3) For $i = n$, because $\dot{x}_n = \dot{x}_{n}$:

$$
\dot{x}_n = -k_n \dot{x}_n - (\bar{g}_{n-1} + \beta_{g,n-1}) \dot{x}_{n-1} - \beta_{f,n} - \bar{\theta}_{f,n}^T \Phi_{f,n} - \beta_{g,n} u_{ad} - \beta_{g,n} u_{ad} - \bar{\theta}_{f,n}^T \Phi_{f,n} + \delta f_n + g_n u_{sn}
$$

(21)

Given eqns. (19) - (21), we are now ready to analyze the stability of the specified control law.

IV. STABILITY AND PARAMETER ADAPTATION

We consider the following Lyapunov function candidate

$$
V = \sum_{i=1}^{n} V_i(\dot{x}_i, \bar{\theta}_f, \bar{\theta}_g, \bar{\Psi}_f, \bar{\Psi}_g)
$$

where

$$
V_i = \frac{1}{2} \left( \dot{x}_i^2 + \dot{\theta}_f^T \Gamma_f^{-1} \dot{\theta}_f + \dot{\theta}_g^T \Gamma_g^{-1} \dot{\theta}_g + \dot{\bar{\Psi}}_f^T \Phi_f \bar{\Psi}_f + \dot{\bar{\Psi}}_g^T \Phi_g \bar{\Psi}_g \right)
$$

with $\Gamma_f, \Gamma_g, \Phi_f, \Phi_g, \bar{\Psi}_f, \bar{\Psi}_g, i = 1, \cdots, n$ being defined as positive definite matrices representing the learning rates. The time derivative of the V is $\dot{V} = \sum_{i=1}^{n} \dot{V}_i$, and $\dot{V}_i$ along solutions of eqns. (19 - 21) are:

1) For $i = 1$,

$$
\dot{V}_1 = -k_1 \dot{x}_1^2 + \dot{\bar{\theta}}_f (\dot{\bar{\theta}}_f - \Gamma_f \Phi_f \bar{\theta}_f) + \dot{\bar{\theta}}_g (\dot{\bar{\theta}}_g - \Gamma_g \Phi_g \bar{\theta}_g)
$$

(23)

2) For $i = 2, \cdots, (n - 1)$,

$$
\dot{V}_i = -k_i \dot{x}_i^2 + \dot{\bar{\theta}}_f (\dot{\bar{\theta}}_f - \Gamma_f \Phi_f \bar{\theta}_f) + \dot{\bar{\theta}}_g (\dot{\bar{\theta}}_g - \Gamma_g \Phi_g \bar{\theta}_g)
$$

(24)

3) For $i = n$,

$$
\dot{V}_n = -k_n \dot{x}_n^2 + \dot{\bar{\theta}}_f (\dot{\bar{\theta}}_f - \Gamma_f \Phi_f \bar{\theta}_f) + \dot{\bar{\theta}}_g (\dot{\bar{\theta}}_g - \Gamma_g \Phi_g \bar{\theta}_g)
$$

(25)

In the above, for $i < n$:

$$
\Delta_i = \bar{x}_i (-\beta_{f,i} - \beta_{g,i} x_{i+1,c} + \delta f_i + \delta g_i, x_{i+1}) + \bar{\theta}_{f,i}^T \Phi_{f,i} + \bar{\theta}_{g,i}^T \Phi_{g,i}
$$

(26)

and for $i = n$:

$$
\Delta_n = \bar{x}_n (-\beta_{f,n} - \beta_{g,n} u_{ad} + \delta f_n + \delta g_n, u_{ad}) + \bar{\theta}_{f,n}^T \Phi_{f,n} + \bar{\theta}_{g,n}^T \Phi_{g,n}
$$

(27)

We choose the localized adaptive laws of $\theta_f$ and $\theta_g$ in $1, \cdots, n$ for $||\bar{x}|| > \sqrt{\mu \bar{x}_n}$ as

$$
\dot{\theta}_f, i = \Gamma_f \bar{x}_i \Phi_f, \dot{\theta}_g, i = \Gamma_g \bar{x}_i \Phi_g
$$

(28)

where a projection modification $\text{Proj} \{ \cdot \}$ is used to ensure that $\bar{g}_i, i = 1, \cdots, n$ are bounded away from zero. When $||\bar{x}|| \leq \sqrt{\mu \bar{x}_n}$, $\dot{\theta}_f, \dot{\theta}_g, \dot{\bar{\Psi}}_f, \dot{\bar{\Psi}}_g$ are defined in the discussion related to Theorem
1. Substituting (28) and (29) in (23) - (25), we obtain the derivative of $V$ defined in eqn. (22) as

$$
\dot{V} = - \sum_{i=1}^{n} k_i \bar{x}_i^2 + \sum_{i=1}^{n-1} (\bar{x}_i u_{s_i} + \Delta_i) + (g_n \bar{x}_n u_{s_n} + \Delta_n). \quad (30)
$$

Next, we will only consider the case when $x \in \mathcal{D}$ and perfect approximation is not possible. We are interested in developing bounds on the approximation error and using those bounds in the control law to achieve robustness to the approximation error. This goal is attained by defining smooth functions $\beta_f$, and $\beta_g$, $i = 1, \ldots, n$ as

$$
\beta_f = \Psi_f^T \Phi_f \tanh (\bar{x}_i / \epsilon) = \Psi_f^T \Omega_f, \quad (31)
$$

$$
\beta_g = \Psi_g^T \Omega_g, \quad (32)
$$

and

$$
\Omega_g = \begin{cases} 
\Phi_{g_n} \tanh (\frac{\bar{x}_i x_{n+1}}{\epsilon}), & \text{if } i < n \\
\Phi_{g_n} \tanh (\frac{x_{n+1}}{\epsilon}), & \text{if } i = n 
\end{cases} \quad (33)
$$

where $\epsilon > 0$ is a small design constant. Lemma 1 of [12] provides the inequality

$$
0 \leq |m| - m \cdot \tanh (m/\epsilon) \leq \eta \epsilon
$$

for any $\epsilon > 0$ and for any $m \in \mathbb{R}$, where $\eta$ is a constant that satisfies $\eta = e^{-(\eta+1)}$ (i.e. $\eta = 0.2785$). This Lemma is used below.

With this definition, starting from (26) and (27), we can reduce the expression for $\Delta_i$:

$$
\Delta_i \leq (\Psi_f^M)^T \Phi_f [\bar{x}_i] + (\Psi_g^M)^T \Phi_g [\bar{x}_i x_{i+1}] - (\Psi_f^M + \tilde{\Psi}_f) \Phi_f [\bar{x}_i x_{i+1}] - (\Psi_g^M + \tilde{\Psi}_g) \Phi_g [\bar{x}_i x_{i+1}] \tanh \left( \frac{\bar{x}_i x_{i+1}}{\epsilon} \right)
$$

$$
+ (\Psi^M_f + \tilde{\Psi}_f) \Phi_f [\bar{x}_i] \tanh \left( \frac{\bar{x}_i x_{i+1}}{\epsilon} \right) + (\Psi^M_g + \tilde{\Psi}_g) \Phi_g [\bar{x}_i] \tanh \left( \frac{\bar{x}_i x_{i+1}}{\epsilon} \right)
$$

$$
\leq \eta \left( (\Psi^M_f)^T \Phi_f + (\Psi^M_g)^T \Phi_g \right)
$$

$$
+ (\tilde{\Psi}_f) \Phi_f [\bar{x}_i] \tanh \left( \frac{\bar{x}_i x_{i+1}}{\epsilon} \right) + (\tilde{\Psi}_g) \Phi_g [\bar{x}_i] \tanh \left( \frac{\bar{x}_i x_{i+1}}{\epsilon} \right)
$$

$$
\leq \eta \epsilon(\Psi_f^M)^T \Phi_f + \eta \epsilon(\Psi_g^M)^T \Phi_g
$$

$$
+ |u_{ad}| \left| \frac{u_{ad}}{|u_{ad}} \right| (\Psi_f^M)^T \Phi_f + (\Psi_g^M)^T \Phi_g + \eta \epsilon(\Psi_f^M)^T \Phi_f + \eta \epsilon(\Psi_g^M)^T \Phi_g \quad (34)
$$

and similarly

$$
\Delta_n \leq \eta \epsilon(\Psi_f^M)^T \Phi_f + \eta \epsilon(\Psi_g^M)^T \Phi_g + |u_{ad}| \left| \frac{u_{ad}}{|u_{ad}} \right| \eta \epsilon(\Psi_f^M)^T \Phi_f + \eta \epsilon(\Psi_g^M)^T \Phi_g \quad (35)
$$

where the extension of Lemma 1 in [12], which provides the inequality$^2$

$$
|\bar{x}_u u_{ad}| - \bar{x}_u u_{ad} \cdot \tanh (\bar{x}_u u_{ad} / \epsilon) \leq \left| \bar{x}_u u_{ad} \right| \eta \epsilon,
$$

is used.

$^2$Note that $u_{ad}$ and $u_{ad}$ have the same sign since the denominator of the control equation is ensured to be bounded away from zero such that $\Theta_{g_n}^T \Phi_{g_n} + \beta_n > \theta_n > 0$.

Based on the inequalities (34–35), for $||\bar{x}|| > \sqrt{\frac{\epsilon + \mu}{\epsilon}}$ the localized adaptive laws of $\Psi_f$ and $\Psi_g$ with $\sigma$-modification are selected as

$$
\dot{\Psi}_i = \Gamma \Psi_i \left[ \bar{x}_i \tanh \left( \frac{\bar{x}_i}{\epsilon} \right) - \sigma_{\Psi, \Phi} \Phi_i \right], \quad (36)
$$

$$
\dot{\Phi}_i = \text{Proj} \{ \tau_{\Psi_i} \} \quad (37)
$$

where

$$
\tau_{\Psi_i} = \begin{cases} 
\Gamma \Psi_i \left[ \bar{x}_i x_{i+1} \tanh \left( \frac{\bar{x}_i x_{i+1}}{\epsilon} \right) - \sigma_{\Psi, \Phi} \Phi_i \right], & \text{if } i < n \\
\Gamma \Psi_i \left[ \bar{x}_n u_{ad} \tanh \left( \frac{x_{n+1}}{\epsilon} \right) - \sigma_{\Psi, \Phi} \Phi_i \right], & \text{if } i = n
\end{cases}
$$

the $|u_{ad}|$ is the square diagonal matrix with diagonal components equal to the vector $v$; $\sigma_{\Psi, \Phi} > 0$; and $\Psi_f^0$ and $\Psi_g^0$ are design parameters (vectors). When $||\bar{x}|| \leq \sqrt{\frac{\epsilon + \mu}{\epsilon}}$, $\dot{\Psi}_i = 0$ and $\dot{\Phi}_i = 0$.

Note that all $u_{ad}$ terms in (30) are zero for $x \in \mathcal{D}$. If we substitute (34–35) and (36–37) into (30), we attain

$$
\dot{V} \leq - \sum_{i=1}^{n} k_i \bar{x}_i^2 + \eta \epsilon \sum_{i=1}^{n-1} (\Psi_f^M)^T \Phi_f + \eta \epsilon \sum_{i=1}^{n-1} (\Psi_g^M)^T \Phi_g
$$

$$
+ |u_{ad}| \left| \frac{u_{ad}}{|u_{ad}} \right| \eta \epsilon (\Psi_f^M)^T \Phi_f + \eta \epsilon (\Psi_g^M)^T \Phi_g
$$

$$
- \sum_{i=1}^{n} \sigma_{\Psi, \Phi} \Phi_i R_i (\Psi_i - \Phi_i^0) + \sigma_{\Psi, \Phi} \Phi_i R_i (\Psi_i - \Phi_i^0)
$$

$$
\text{where } R_i = \text{diag}(\Phi_i) \text{ and } R_g = \text{diag}(\Phi_g) \text{. After applying the equation}
$$

$$
\dot{a}^T R (a - a^0) = \frac{1}{2} \dot{a}^T \dot{R} a + \frac{1}{2} (a - a^0)^T R (a - a^0)
$$

$$
- \frac{1}{2} (a - a^0)^T R (a - a^0)
$$

$$
to the two terms in the last summation, with the vector $a$ replaced by $\Psi_f$ and $\Psi_g$, $i = 1, \ldots, n$, respectively, we have

$$
\dot{V} \leq -c ||\bar{x}||^2 + d_0 + \rho \quad (38)
$$

where $c$, $d_0$ and $\rho$ are all positive constants given by

$$
c = \min_{i=1, \ldots, n} \{ k_i \} \quad (39)
$$

$$
d_0 = \eta \epsilon \sum_{i=1}^{n} (\Psi_f^M)^T \Phi_f + \eta \epsilon \sum_{i=1}^{n-1} (\Psi_g^M)^T \Phi_g
$$

$$
+ |u_{ad}| \left| \frac{u_{ad}}{|u_{ad}} \right| (\Psi_f^M)^T \Phi_f + (\Psi_g^M)^T \Phi_g \quad (40)
$$

$$
\rho = \frac{1}{2} \sum_{i=1}^{n} \left[ \sigma_{\Psi, \Phi} (\Psi_f^M - \Phi_i^0)^T R_i (\Psi_f^M - \Phi_i^0) + \sigma_{\Psi, \Phi} (\Psi_g^M - \Phi_g^0)^T R_g (\Psi_g^M - \Phi_g^0) \right] \quad (41)
$$
Let $\bar{\rho} > (d_0 + \rho)$ be a strict upper bound on $(d_0 + \rho)$. Select a small design constant $\mu > 0$. For $\|\bar{x}\| > \sqrt{\frac{\bar{\rho} + \mu}{c}}$, $\dot{V} < -\mu < 0$. When $\|\bar{x}\| \leq \sqrt{\frac{\bar{\rho} + \mu}{c}}$, a deadzone is included in the adaptive laws. Without such a deadzone, stability is not guaranteed and the parameters could drift.

Note that $m \tanh(m/e) \geq 0$; therefore, without leakage terms $\tilde{\Psi}_f$ and $\tilde{\Psi}_g$, would always be non-negative.

The stability properties are summarized in the following theorem:

**Theorem 1:** Assuming the upper bound $\bar{\rho} > d_0 + \rho > 0$ is known, for the system described by (1)-(2) with the adaptive feedback control law of eqns. (13-15) and the parameter adaptation laws of eqns. (28–29) and (36–37), we have the following stability properties:

1) For $x(0) \notin D$, $x(t)$ for $t > 0$ converges to region $D$ in finite time.

2) When $x \in D$ and $\delta_{f_i} = \delta_{g_i} = 0$:
   a) $\dot{x}_i \in L_2$;
   b) $\dot{x}_i \to 0$ as $t \to \infty$;
   c) $\tilde{\Psi}_f, \tilde{\Psi}_g, \tilde{\Psi}_f, \tilde{\Psi}_g \in L_\infty$.

3) When $x \in D$ and $\delta_{f_i} \neq 0$ or $\delta_{g_i} \neq 0$:
   a) $\tilde{x}_i, \tilde{\dot{x}}_f, \tilde{\dot{\Psi}}_g, \tilde{\dot{\Psi}}_g \in L_\infty$;
   b) $\dot{x}_i, \dot{\theta}_f, \dot{\theta}_g, \dot{\Psi}_f, \dot{\Psi}_g \in L_\infty$;
   c) $\tilde{x}_i, \tilde{\dot{\Psi}}_f, \tilde{\dot{\Psi}}_g, \tilde{\dot{\Psi}}_f, \tilde{\dot{\Psi}}_g \in L_\infty$;
   d) $\dot{x}$ is $\bar{\rho}$-small in the m.s.s. [4].
   e) The total time outside the deadzone is finite.
   f) $\|\bar{x}\|$ is ultimately bounded by $\|\bar{x}\| \leq \sqrt{\frac{\bar{\rho} + \mu}{c}}$, as $t \to \infty$.

**Pf.**

1) When $x \notin D$: We want to show that all initial conditions will return to and stay within region $D$.

Note that the $\Phi_f, \Phi_g, \dot{\theta}_f, \dot{\theta}_g, \dot{\beta}_f, \dot{\beta}_g$, terms are all zero, and $\delta_{f_i} = f_i(x)$ and $\delta_{g_i} = g_i(x) - g_i$ for $i = 1, \ldots, n$. Therefore, For $x \notin D$, we consider the Lyapunov function as

$$\tilde{V} = \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i^2.$$  \hspace{1cm} (42)

The derivative of $\tilde{V}$ can be easily shown to be similar to (30), where $\Delta_i$ terms defined in eqns. (26 - 27) are simplified to

$$\Delta_i = \begin{cases} \dot{x}_i (f_i + (g_i - g_i)x_{i+1}), & \text{for } 1 \leq i < n \\ \dot{x}_n (f_n + (g_n - g_n)u_{a_d}), & \text{for } i = n. \end{cases}$$

Applying the sliding control of (8) and (14), we obtain the derivative of $\tilde{V}$ defined in eqn. (42) as

$$\frac{d\tilde{V}}{dt} \leq -\sum_{i=1}^{n} k_i \dot{x}_i^2 + \sum_{i=1}^{n-1} (-r_i |\dot{x}_i| + |\Delta_i|)
+ (-g_n r_n |\dot{x}_n| + |\Delta_n|).$$  \hspace{1cm} (43)

Since the sliding gains of (9) and (15) yield, for $i < n$

$$r_i |\dot{x}_i| = (\dot{\theta}_f + \dot{\theta}_g|x_{i+1}|)|\dot{x}_i| \geq |\Delta_i|,$$

and, for $i = n$

$$g_n r_n |\dot{x}_n| = g_n (\bar{b}_f_n + \bar{b}_g_n)|u_{a_d}| |\dot{x}_n| \geq |\Delta_n|.$$

Then, we attain

$$\frac{d\tilde{V}}{dt} \leq -\sum_{i=1}^{n} k_i \dot{x}_i^2 - k\tilde{V}$$  \hspace{1cm} (44)

$$\tilde{V}(t) \leq e^{-kt} \tilde{V}(0),$$  \hspace{1cm} (45)

Then, for any $t$ larger than some finite time called $T_2$, $\tilde{V}(t) < \frac{\alpha^2}{k}$ which implies that $|\bar{x}(t)| < \frac{\alpha}{\sqrt{k}}$. In addition, for $x \notin D$, $|\xi_i| < \beta_i$ where $\beta_i = \frac{2\alpha D}{\sqrt{k}}$ by methods similar to those used to derive (12). Therefore, we can attain $|\xi(t)| < \frac{\beta}{2}$ by choosing $g_i$ sufficiently small for $x \notin D$. Therefore, for $t > T_2$,

$$|\bar{x}(t)| \leq |\bar{x}(0)| + |\xi(t)| < \gamma$$

which implies that $x$ returns to within $D$ in finite time. Once $x \in D$, the sliding mode term will not allow $x$ to leave $D$.

The reminder of this proof will only be concerned with the case of $x \in D$, where each sliding control term $u_{s_i}$ is zero. For $x \in D$, we will continue the analysis of $\tilde{V}$ for $V$ defined in (22).

2) When $x \in D$ and $\delta_{f_i} = \delta_{g_i} = 0$: Note that all $u_{s_i}$ terms in (30) are zero for $x \in D$. In addition, for the ideal case of perfect approximation, the $\beta_{f_i}, \beta_{g_i}, \delta_{f_i}$, and $\delta_{g_i}$ terms are identically zero, which yields directly $\Delta_i = 0$. Then, (30) is simplified as

$$\frac{dV}{dt} \leq -\sum_{i=1}^{n} k_i \dot{x}_i^2$$  \hspace{1cm} (46)

which is negative semi-definite. This implies that the variables $\tilde{x}_i, \tilde{\dot{x}}_f, \tilde{\dot{\Psi}}_g$, are each bounded. Since each term of $\tilde{x}_i$ is bounded, $\bar{x}$ can be directly shown to be bounded. Barbats’ lemma implies that each $\tilde{x}_i$ approaches zero as $t$ approaches infinity. Finally, integrating both sides of (46) yields

$$V(0) \geq \sum_{i=1}^{n} \int_{0}^{t} k_i \dot{x}_i^2 (\tau) d\tau,$$

which shows that each $\tilde{x}_i$ is in $L_2$.

3) When $x \in D$ and $\delta_{f_i} \neq 0$ or $\delta_{g_i} \neq 0$: Starting from the inequality (38), the derivative of $V$ for $|\bar{x}| > \sqrt{\frac{\bar{\rho} + \mu}{c}}$ is

$$\dot{V} \leq -c \|\bar{x}\|^2 + d_0 + \rho < -\mu < 0$$  \hspace{1cm} (47)

where $c, d_0$ and $\rho$ are given as (39–41), respectively. Therefore, if $|\bar{x}| > \sqrt{\frac{\bar{\rho} + \mu}{c}}$, then $V$ is decreasing.

If $|\bar{x}| \leq \sqrt{\frac{\bar{\rho} + \mu}{c}}$ then $\tilde{\dot{\theta}}_f, \tilde{\dot{\theta}}_g, \tilde{\dot{\Psi}}_f$, and $\tilde{\dot{\Psi}}_g$ are all constant and $|\bar{x}|$ is bounded. Thus, $V(t)$ is bounded by the maximum of $V(0)$ or max

$$\|\bar{x}\| = \sqrt{\frac{\bar{\rho} + \mu}{c}} \left(V(\bar{x}, \tilde{\dot{\theta}}_f(0), \tilde{\dot{\Psi}}_f(0), \tilde{\dot{\Psi}}_g(0))\right)$$

which shows that $\bar{x}_i, \tilde{\dot{x}}_f, \tilde{\dot{\Psi}}_f, \tilde{\dot{\Psi}}_g \in L_\infty$. The
The formulation of localized adaptive laws as defined in eqns. (36–37) localizes the effects of leakage terms to the vicinity of the present operating point, thus eliminating the problem with global forgetting. Localized forgetting also decreases the required amount of on-line computation, since all parameters associated with zero elements of basis vectors are left unchanged.

We have considered in this paper the robust adaptive control design for a wide class of $n$-th order uncertain nonlinear systems. A novel robust adaptive backstepping design procedure is proposed by incorporating the locally learned adaptive bounding functions on the residual approximation errors. This is an extension of the localized adaptive bounding technique proposed in [17] to higher order systems with $g_i(x) \neq 1$. Furthermore, the complexity of calculating time derivatives of intermediate state commands for the backstepping approach [6], [7] is addressed by the command filtering techniques proposed in [3]. We have proved that the overall adaptive scheme can guarantee the boundedness of both actual tracking errors and compensated tracking errors, by applying the Lyapunov stability analysis.

In addition, we successfully show that the localized adaptation algorithms with deadzone and parameter projection modification is effective to prevent the parameter drift and to guarantee the ultimate boundedness of the compensated tracking errors $\bar{x}$. Since we have shown that the m.s.s. bound on $\bar{x}$ is on the order of the residual function approximation errors, our future extension will focus on the adaptive enhancement of the structure of the approximator to achieve better tracking performance.

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REFERENCES


