Optimization methods for solving bang-bang control problems with state constraints and the verification of sufficient conditions

Helmut Maurer, Inga Altrogge, Nadine Goris
Institut für Numerische Mathematik und Angewandte Mathematik
Westfälische Wilhelms-Universität Münster
Einsteinstrasse 62, D-48149 Münster, Germany
Email: maurer@math.uni-muenster.de

II. OPTIMAL BANG-BANG CONTROL PROBLEMS WITH STATE CONSTRAINTS

We consider state-constrained optimal control problems with control appearing linearly. Let \( x(t) \in \mathbb{R}^n \) denote the state variable and \( u(t) \in \mathbb{R} \) the control variable at time \( t \in [0, t_f] \) where the final time \( t_f > 0 \) is either fixed or free. For simplicity, the control is assumed to be scalar. The following optimal control problem will be denoted by \((OC)\): determine a measurable control function \( u : [0, t_f] \to \mathbb{R} \) and a terminal time \( t_f > 0 \) such that the pair of functions \((x(\cdot), u(\cdot))\) minimizes the cost functional of Mayer type

\[
J(x, u, t_f) := g(x(t_f), t_f)
\]

subject to the constraints in the interval \([0, t_f]\),

\[
\dot{x}(t) = f(x(t), u(t)) = f_0(x(t)) + f_1(x(t))u(t),
\]

\[
x(0) = x_0,
\]

\[
\varphi(x(t_f), t_f) = 0,
\]

\[
u_{\text{min}} \leq u(t) \leq u_{\text{max}},
\]

\[
u_{\text{min}} < u_{\text{max}},
\]

and the scalar state inequality constraint

\[
S(x(t)) \leq 0 \quad \text{for} \quad 0 \leq t \leq t_f.
\]

The functions \( g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), \( f_0, f_1 : \mathbb{R}^n \to \mathbb{R}^n \), \( \varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^r \), \( 0 \leq r \leq n \), and \( S : \mathbb{R}^n \to \mathbb{R} \) are assumed to be twice continuously differentiable since we intend to derive and verify second order conditions. The state constraint is assumed to be of order one \([7], [15]\), i.e., the total time derivative of the function \( S(x(t)) \) contains the control explicitly,

\[
S^1(x, u) := S_x(x)f(x, u) = S_x(x)f_0(x) + S_x(x)f_1(x)u =: a(x) + b(x)u,
\]

where \( b(x) = S_x(x)f_1(x) \neq 0 \). Here and in the sequel, partial derivatives are denoted by subscripts.

An interval \([\tau_1, \tau_2] \subset [0, t_f]\) is called a boundary arc if \( S(x(t)) \equiv 0 \) holds for all \( t \in [\tau_1, \tau_2] \). If \( \tau_1 \) and \( \tau_2 \) are maximal with this property, then \( \tau_1 \) is called entry-time and \( \tau_2 \) is called exit-time of the boundary arc; \( \tau_1, \tau_2 \) are also called junction times. The following assumption is a standard regularity condition for a boundary arc \([7], [14], [16]\).

(A1) On a boundary arc the following condition holds:

\[
b(x(t)) \neq 0 \quad \forall \ t \in [\tau_1, \tau_2].
\]
In view of this assumption, we can compute the boundary control on a boundary arc from the equation $S^1(x,u) = a(x) + b(x)u = 0$ as the feedback expression

$$u_0(x) = -a(x)/b(x), \quad u(t) = u_0(x(t)). \quad (7)$$

The following assumption will be needed to determine the multiplier associated with the state constraint explicitly.

(A2) The boundary control lies in the interior of the control region:

$$u_{\text{min}} < u(t) = u_0(x(t)) < u_{\text{max}} \quad \forall \ t \in [\tau_1, \tau_2].$$

Assumptions (A1) and (A2) allow us to formulate first order necessary conditions of Pontryagin’s minimum principle in a computationally convenient form. We recall from [7], [16] that the Lagrange multiplier associated with the state constraint (5) is a measure that is represented by a function $\mu$ of bounded variation. Using (A1) and (A2) it has been shown in [15], [16], [14], [13] that the measure has a Radon–Nikodym derivative $\eta$. Hence, we may write the adjoint equation in a differential form. Suppose now that $\tilde{u} : [0,t_f] \to [u_{\text{min}}, u_{\text{max}}]$ is an optimal control with corresponding trajectory $\tilde{x}$ which satisfy assumptions (A1) and (A2) and for which the state constraint is not active at the initial and terminal time,

$$S(x(0)) < 0, \quad S(x(t_f)) < 0.$$ 

In the direct adjoining approach [7], [16], the augmented Pontryagin or Hamiltonian function is defined by

$$H(t, x, u, \lambda, \mu) = \lambda f(t, x, u) + \eta S(x) = \lambda f_0(t, x) + \lambda f_1(t, x)u + \eta S(x), \quad (8)$$

where the adjoint variable $\lambda \in IR^n$ is a row vector and $\eta$ is the multiplier associated with the state constraint. In the sequel, we will use the junction theorem in [15], Corollary 5.2 (ii), where it was shown that the adjoint variables are continuous at junction times provided that the state constraint is of first order and the control is discontinuous at junctions. Note that the discontinuity of the control follows from assumption (A2). Then there exist an absolutely continuous (a.c.) adjoint function $\lambda : [0,t_f] \to IR^n$, a piecewise a.c. multiplier function $\eta : [0,t_f] \to IR$ and a multiplier $\rho \in IR^r$ (row vector) such that the following conditions hold a.e. on $[0,t_f]$:

$$\dot{\lambda}(t) = -H_x(\tilde{x}(t), \tilde{u}(t), \lambda(t), \eta(t)), \quad (9)$$

$$\lambda(t_f) = l_x(\tilde{x}(t_f), t_f, \rho), \quad (10)$$

$$H(\tilde{x}(t), \tilde{u}(t), \lambda(t), \eta(t))|_{t=t_f} + l_x(\tilde{x}(t_f), t_f, \rho) = 0, \quad (11)$$

the minimum condition (12) yields

$$\eta(t) \geq 0, \quad \eta(t) = 0 \quad \text{if} \ S(x(t)) < 0, \quad (13)$$

where $l(x,t_f,\rho) := (g+\rho \varphi)(x(t_f), t_f)$ is the endpoint Lagrangian function. The factor at $u$ in the Hamiltonian is called the switching function

$$\sigma(x,\lambda) := \lambda f_1(x), \quad \sigma(t) = \sigma(x(t),\lambda(t)). \quad (14)$$

On interior arcs $S(x(t)) < 0$ the minimum condition (12) yields the control law

$$u(t) = \begin{cases} u_{\text{min}}, & \text{if} \ \sigma(t) > 0 \\ u_{\text{max}}, & \text{if} \ \sigma(t) < 0 \end{cases}. \quad (15)$$

The switching times of the controller are zeroes of the switching function. A singular arc occurs if the switching function $\sigma(t)$ vanishes on an open interval. In this paper, we do not consider singular arcs and make the following assumption

(A3) On interior arcs the control $u(t)$ is bang-bang and has only finitely many switching times.

For a boundary arc $[\tau_1, \tau_2]$ it was assumed in (A2) that the control takes values in the interior of the control set. Hence, the minimum condition (12) yields

$$\sigma(t) = \lambda(t)f_1(x(t)) = 0 \quad \forall \ t \in [\tau_1, \tau_2]. \quad (16)$$

This relation can be interpreted as the property that a boundary control behaves formally like a singular control, a fact that was exploited in [15] to obtain junction theorems. By differentiating (16) and using the adjoint equation (9) we find the following explicit representation of the multiplier $\eta(t)$; cf. [16], [13],

$$\eta(t) = \left[ \frac{\lambda(t)(f_1(x(t))f(x(t), u_0(x(t))) - \lambda(t)f_2(x(t), u_0(x(t))))f_1(x(t))}{b(x(t))}, \right]$$

where $u_b(x(t))$ is the boundary control (7).

III. THE INDUCED OPTIMIZATION PROBLEM AND SECOND ORDER SUFFICIENT CONDITIONS

Under assumptions (A1)–(A3), the optimal control problem can be transcribed into an optimization problem in the following way. We assume that the structure of the optimal control, i.e., the sequence of finitely many bang-bang and boundary arcs, is known. Let $t_i, i = 1, \ldots, s$, be the switching and junction times which are ordered as

$$0 =: t_0 < t_1 < \ldots < t_i < \ldots < t_s < t_{s+1} := t_f.$$

For simplicity, assume that there exists only a single boundary arc $[\tau_k, \tau_2] = [t_k, t_{k+1}]$ with an index $1 \leq k \leq s$. Then $[0,t_k)$ and $(t_{k+1}, t_f)$ are the interior arcs. By assumption, in every interval $I_j := [t_{j-1}, t_j]$ there exists a function $u^j(x)$ with the property that the optimal control is given by

$$u(t) = u^j(x(t)), \quad t_{j-1} \leq t \leq t_j, \quad (j = 1, \ldots, s, s+1). \quad (18)$$

The interval $I_{k+1}$ then represents the boundary arc. The function $u^j(x)$ is either the constant value of the bang-bang control on interior arcs or the boundary control $u^{k+1}(x) = u_b(x) = -a(x)/b(x)$.

Consider now the optimization variable

$$z := (t_1, \ldots, t_{s+1})^t \in IR^{s+1}, \quad t_{s+1} := t_f,$$

resp. $z := (t_1, \ldots, t_{s})^t \in IR^s$ for fixed final time $t_f$, where the asterisk denotes the transpose. Denote by $x(t; z)$ the absolutely continuous solution of the ODE system

$$\dot{x}(t) = f(x(t), u^j(x(t))) \quad \text{for} \quad t_{j-1} \leq t \leq t_j \quad (19)$$
with initial condition \( x(0) = x_0 \). Then the control problem \((OC)\) can be reformulated as the following induced optimization problem \((OP)\) with equality constraints:

\[
\text{(OP) \quad \text{Minimize} \quad G(z) := g(x(t_{s+1}; z), t_{s+1}) \\
\text{subject to} \quad \Phi(z) := \varphi(x(t_{s+1}; z), t_{s+1}) = 0, \quad S(z) := S(x(t_k; z)) = 0.}
\]

(20)

The last equation arises from the entry-condition for the boundary arc. We consider the Lagrangian for the induced optimization problem \((OP)\) in normal form,

\[
L(z, \rho, \beta) = G(z) + \rho \Phi(z) + \beta S(z) \tag{21}
\]

with multipliers \( \rho \in \mathbb{R}^r \) (row vector) and \( \beta \in \mathbb{R}^r \). First order necessary and second order sufficient conditions (SSC) for \((20)\) are well known in the literature. In the following theorem, we consider control problems with free final time which involve the optimization variable \( z \in \mathbb{R}^{r+1} \).

**Theorem 3.1: (SSC for the optimization problem \((OP)\))**

Let \( \bar{z} \) be feasible for the optimization problem \((20)\). Suppose there exist multipliers \( \rho \in \mathbb{R}^r \) and \( \beta \in \mathbb{R}^r \) such that the following three conditions hold:

(a) \( \text{rank} \left[ \Phi_z(\bar{z}) \mid S_z(\bar{z}) \right] = s + 1, \)

(b) \( L_z(\bar{z}, \rho, \beta) = 0, \)

(c) \( v^* L_{zz}(\bar{z}, \rho, \beta) v > 0 \) for all \( v \in \mathbb{R}^{r+1}, v \neq 0, \)

with \( \Phi_z(\bar{z}) v = 0, \ S_z(\bar{z}) v = 0. \)

Then \( \bar{z} \) is a strict local minimizer of the optimization problem \((OP)\).

Arguments similar to those in [17], [22] reveal that the first order conditions in part (a) and (b) of Theorem 3.1 are closely related to those in (9)–(11) involving the adjoint function \( \lambda(t) \). However, on the boundary, we obtain adjoint variables which correspond to the indirect adjoining approach described in [7], [16] where the function \( S^1_z(x, u) \) in \( (6) \) is adjoint in the Hamiltonian by a multiplier \( q^1 \) which is different from \( \eta \). The multiplier \( \beta \) in the Lagrangian \((21)\) yields the jump condition \( \lambda(\tau_{j+}) = \lambda(\tau_{j-}) - \beta S_z(x(\tau_j)) \) for the adjoint variable at the entry–time \( \tau_j \). Moreover, one can show the relation \( \beta = \int_{\tau_j}^{\tau_{j+}} \eta(t) dt > 0 \).

For bang-bang control problems without state inequality constraints, Agrachev, Stefani, Zezza [1] and Maurer, Osmolovskii [19], [20], [22] have shown that one further needs the so-called strict bang-bang property to obtain SSC for the bang–bang control problem. The following assumption gives an extension of the strict bang-bang property to control problems with state space constraints.

(A4) \( \text{(a) on interior arcs with switching times } t_i, \]

\( i = 1, \ldots, k - 1, k + 2, \ldots, s, \) it holds:

\( \sigma(t_i) = 0, \quad \partial \sigma(t_i)(u(t_i^-) - u(t_i^+)) > 0, \quad \sigma(t) \neq 0 \) for \( t \neq t_i. \)

(b) at the entry-time \( t_k \) and exit-time \( t_{k+1} \) of the boundary arc the following conditions hold:

\( \sigma(t_k^-)(u(t_k^-) - u(t_k^+)) > 0, \)

\( \sigma(t_{k+1})(u(t_{k+1}^-) - u(t_{k+1}^+)) > 0. \)

Finally, we need the property that the multiplier \( \eta(t) \) satisfies the strict complementarity condition.

(A5) \( \text{Strict complementarity: } \eta(t) > 0 \ \forall \ t \in [t_k, t_{k+1}]. \)

Note that assumptions (A4) and (A5) have also been used in [12], [13] to construct a local field of extremals near the boundary arc. Now we can state second order sufficient conditions for the state constrained control problem \((1)–(5)\) the proof will be published elsewhere.

**Theorem 3.2: (SSC for the state–constrained control problem \((OC)\))** Let \( \bar{u} \) be a feasible control for the control problem \((1)–(5)\) which has finitely many switching and junction times \( t_i, i = 1, \ldots, s \) and let \( \bar{x} \) be the corresponding trajectory. Suppose there exists an adjoint function \( \lambda : [0, t_f] \to \mathbb{R}^n \) and a multiplier \( \rho \in \mathbb{R}^r \) such that assumptions (A1)–(A5) hold where the multiplier function \( \eta : [0, t_f] \to \mathbb{R} \) is defined by \((17)\). Suppose further that the vector \( \bar{z} = (\bar{t}_1, \ldots, \bar{t}_s, \bar{t}_{s+1})^* \in \mathbb{R}^{s+1}, \bar{t}_{s+1} = t_f \) satisfies the SSC in Theorem 3.1. Then the control \( \bar{u} \) provides a strict strong minimum for the control problem \((OC)\).

IV. NUMERICAL METHODS FOR SOLVING THE INDUCED OPTIMIZATION PROBLEM

In this section, we shall extend the arc–parametrization method in [8], [17] to solve state-constrained control problems. Instead of directly optimizing the switching and junction times \( t_j, j = 1, \ldots, s, s + 1 \), one determines the arc durations \( \xi_j := t_j - t_{j-1}, \quad j = 1, \ldots, s, s + 1, \)

\( (22) \)

of bang–bang and boundary arcs. Therefore, the optimization variable \( z = (t_1, \ldots, t_s, t_{s+1})^* \), \( t_{s+1} := t_f \), is replaced by the optimization variable \( \xi := (\xi_1, \ldots, \xi_s, \xi_{s+1})^* \in \mathbb{R}^{s+1}, \quad \xi_j := t_j - t_{j-1}. \)

\( (23) \)

The variables \( z \) and \( \xi \) are related by a linear transformation involving the regular \((s + 1) \times (s + 1)\)-matrix \( R \),

\( \xi = R z, \quad z = R^{-1} \xi, \)

\( (24) \)

In the arc–parametrization method, the time interval \([t_{j-1}, t_j]\) is mapped to the fixed interval \( I_j := \left[ \frac{j - 1}{s + 1}, \frac{j}{s + 1} \right] \) by the linear transformation

\( t = a_j + b_j \tau, \quad \tau \in I_j = \left[ \frac{j - 1}{s + 1}, \frac{j}{s + 1} \right], \)

\( (25) \)

where \( a_j = t_j - 1 - (j - 1) \xi_j, \quad b_j = (s + 1) \xi_j \). Identifying \( x(\tau) \cong (x(a_j + b_j \tau) = x(t) \) in the relevant intervals, we obtain the ODE system

\( \dot{x}(\tau) = (s + 1) \xi_j f(x(\tau), u^l(x(\tau))) \quad \text{for } \tau \in I_j. \)

\( (26) \)

The solutions in the intervals \( I_j \) are concatenated to define the continuous solution \( x(t) = x(t; \xi) \) in the normalized
of cancer cells in the

subject to

$\tilde{\Phi}(\xi) := \varphi(x(1;\xi), t_f) = 0$, 
$\tilde{S}(\xi) := S(x(k/(s + 1);\xi)) = 0$.

Using the linear transformation (24) it can easily be seen that
the SSC for the optimization problems (OP) and (OP) are
equivalent; cf. similar arguments in [17].

To solve this optimization problem, we use a suitable
adaptation of the control package NUDOCCCS in B"uskens
[3], [5]. Then we can take advantage of the fact that this
routine also provides the Jacobian of the equality constraints
and the Hessian of the Lagrangian which are needed in the
check of the second order condition in Theorem 3.1.

V. NUMERICAL EXAMPLE: TWO-COMPARTMENT MODEL
IN CANCER CHEMOTHERAPY WITH A STATE CONSTRAINT

Ledzewicz and Sch"attler [11], [12] considered a two-
compartment model in cancer chemotherapy and established
the optimality using the methods outlined in [13]. Here,
we prove optimality by applying the numerical SSC test in
Theorem 3.2 which is conceptually different from the one
in [13]. The description of the control model is taken from [11]:
“The cell cycle is broken into two compartments of which
the first combines the first growth phase $G_1$ and the synthesis
phase $S$ while the second contains the second growth phase
$G_2$ and mitosis $M$. Let $x_i(t), i = 1, 2$, denote the number
of cancer cells in the $i$-compartment at time time $t$. The
control $u$ is the drug treatment which is measured by its
cell-killing effect. The control model is to minimize the cost
functional with fixed final time $t_f$

$J(x, u) = r_1x_1(t_f) + r_2x_2(t_f) + \int_0^{t_f} u(t)dt$

subject to

$\dot{x}_1 = -a_1x_1 + 2(1 - u)a_2x_2$,  
$\dot{x}_2 = a_1x_1 - a_2x_2$,  
$x_1(0) = x_{10}, x_2(0) = x_{20}$

$0 \leq u(t) \leq 1 \quad \forall t \in [0, t_f]$.

The cost functional (28) can be transformed to a functional
(1) of Mayer type by introducing the equation $x_3 = u$,
$x_3(0) = 0$, which yields

$J(x, u) = g(x(t_f)) = r_1x_1(t_f) + r_2x_2(t_f) + x_3(t_f)$.

In addition, we consider the state constraint of order one

$S(x(t)) := x_1(t) + x_2(t) - \alpha \leq 0, \quad 0 \leq t \leq t_f$,

which imposes an upper bound on the total number of tumor
cells in both compartments. The first total time derivative (6)
of $S(x)$ is given by

$S^1(x, u) = a_2x_2 - 2a_2x_2u$.

Obviously, assumption (A1) is satisfied since $b(x(t)) =
-2a_2x_2(t) \neq 0$ on $[0, t_f]$. The data in (28) and (29) are
taken from [11]; the initial values $x_{10}, x_{20}$ are extrapolated
from this paper:

$r_1 = 6.94, r_2 = 3.94, a_1 = 0.197, a_2 = 0.356$,

$x_1(0) = x_{10} = 0.86, x_2(0) = x_{20} = 0.55, t_f = 10$.

The parameter $\alpha$ in the state constraint (30) will be assigned
the value $\alpha = 1.7$ for which the state constraint becomes
active. The augmented Hamiltonian (8) is given by

$H = \lambda_1(-a_1x_1 + 2a_2x_2) + \lambda_2(a_1x_1 - a_2x_2) + \sigma u$

$+ \eta(x_1 + x_2 - \alpha)$,  

where $\sigma$ is the switching function

$\sigma = \sigma(x, \lambda) = 1 - 2a_2x_2\lambda_1$.  

The adjoint equation (9) and the transversality condition (10)
yield

$\dot{\lambda}_1 = a_1(\lambda_1 - \lambda_2) - \eta, \quad \lambda_1(t_f) = r_1$,  
$\dot{\lambda}_2 = a_2(2(u - 1)\lambda_1 + \lambda_2) - \eta, \quad \lambda_2(t_f) = r_2$.  

The boundary control $u_b(x)$ satisfies the equation

$S^1(x, u_b(x)) \equiv 0$ which gives

$u_b(x) \equiv 1/2$.

Hence, the boundary control lies in the interior of the control
set and satisfies assumption (A2). The multiplier $\eta$ for the
state constraint (30) is determined by equation (17):

$\eta(t) = a_1\lambda_1(t) \left( \frac{x_1(t)}{x_2(t)} + 1 \right) - a_2\lambda_1(t) - a_1\lambda_2(t)$.

To determine the structure of the optimal control we first
discretize the control problem with 500 gridpoints and apply
the program NUDOCCCS of B"uskens [3]. Figures 1 and 2
display the state, resp., adjoint variables, Figure 3 depicts
the optimal control and the switching function and Figure 4
gives the state constrained function $x_1 + x_2$.

The control has two bang-bang arcs and one boundary arc:

$u(t) = \begin{cases} 
0, & \text{for } t \in [0, t_1] \\
u_b(x(t)) = \frac{1}{2}, & \text{for } t \in [t_1, t_2] \\
1, & \text{for } t \in [t_2, t_f] 
\end{cases}$.

It can be seen from Figure 3, that the optimal control
satisfies assumptions (A3) and (A4) since, in particular, for
$k = 1$ in (A4) we have $\dot{\sigma}(t_1-) < 0$ and $\dot{\sigma}(t_2+) < 0$.
Moreover, Figure 5 shows that the multiplier $\eta$ satisfies the
strict complementarity condition (A5).

It remains to verify the SSC in Theorem 3.1 for the
optimization problem (27). The optimization variable is

$\xi = (\xi_1, \xi_2), \quad \xi_1 = t_1, \xi_2 = t_2 - t_1$.

Then the arc-length of the final time interval is given by

$t_f - \xi_1 - \xi_2, \quad t_f = 10$. 

926
Since no terminal state boundary conditions are prescribed, the only equality constraint is the entry–condition of the boundary arc,

\[ x_1(1/3; \xi) + x_2(1/3; \xi) = \alpha = 1.7. \]

The code NUDOCCCS gives the following results:

\[ t_1 = \xi_1 = 1.490713, \quad t_2 = \xi_1 + \xi_2 = 2.653005, \]
\[ \lambda_1(0) = 2.44417, \quad \lambda_2(0) = 2.82883, \]
\[ x_1(t_f) = 0.2635156, \quad x_2(t_f) = 0.2673589, \]
\[ J(x, u) = 10.81033. \tag{36} \]

The Hessian of the Lagrangian for (27) is computed as

\[ L_{\xi\xi} = \begin{pmatrix} 0.2253187 & 0.1280601 \\ 0.1280601 & 0.0992115 \end{pmatrix} \]

while the Jacobian of the equality constraint is given by

\[ \tilde{S}_{\xi} = (0.1979670, \ 0). \]

Obviously, the Hessian \( L_{\xi\xi} \) is positive definite and we have \( \text{rank}(\tilde{S}_{\xi}) = 1 \). Hence, we may conclude that the control (35) referring to the data (36) satisfies the SSC in Theorem 3.1 and provides a strict local minimum of the optimal control problem.

The results on SSC have an immediate application in sensitivity analysis of parametric bang-bang control problems with state constraints. The methods in [10], [17] can be extended to compute parametric sensitivity derivatives of switching and junction times, resp., arclengths of bang–bang and boundary arcs. For the chemotherapy problem under consideration we obtain the following sensitivity derivatives for the arclengths of the first bang-bang arc and the boundary...
arc:
\[
\begin{align*}
  d\xi_1/da_1 &= -1.513, \quad d\xi_2/da_1 = 11.99, \\
  d\xi_1/da_2 &= -3.350, \quad d\xi_2/da_2 = 0.5165, \\
  d\xi_1/dx_{10} &= -5.359, \quad d\xi_2/dx_{10} = 4.421, \\
  d\xi_1/dx_{20} &= -7.233, \quad d\xi_2/dx_{20} = 4.077.
\end{align*}
\]

In particular, note the high sensitivity of the arclength of the boundary arc w.r.t. a variation in the parameter \( a_1 \).

VI. CONCLUSION

We have presented second–order sufficient conditions (SSC) for bang–bang control problems which are subject to a first–order state constraint. The form of these SSC can be regarded as a generalization of those in [1], [19], [20], [22] for purely bang–bang controls. We have discussed numerical methods which efficiently solve the state–constrained bang–bang control problem and provide a test for SSC. The numerical methods were illustrated by an example in cancer chemotherapy. The proposed SSC have been successfully tested on further examples by Altrogge and Goris [2]: (1) a drug displacement problem with a toxicity constraint which was solved in [21]; (2) the control of an image converter with a constraint on the electric field [9]; (3) the control of a nuclear reactor [15]; (4) the cancer chemotherapy for a three–compartment model [23].

The methods in [4], [5], [10], [17] for computing parametric sensitivity derivatives of optimal solutions can be extended to bang–bang control problems with state constraints. In particular, as in [17] one obtains the sensitivity derivatives of switching and junction times which can be used to design real–time control algorithms for the online computation of optimal control and state trajectories under data perturbations; cf., e.g., [6].

REFERENCES