Feedback-Invariant Subspaces in Infinite-Dimensional Systems

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Abstract—We consider single-input single-output systems on a Hilbert space $X$, with infinitesimal generator $A$, bounded control element $b$, and bounded observation element $c$. Let $c^\perp$ be the subspace of $X$ perpendicular to $c$. We consider the problem of finding the largest feedback-invariant subspace of $c^\perp$. If $b$ is in $c^\perp$, and $c \notin D(A^*)$, a largest feedback-invariant subspace does not exist in general.

I. INTRODUCTION

A subspace $V$ is invariant for a linear system if for all initial conditions in $V$ there exists a control that keeps the state in $V$ for all times. If this is the case, the control can be a constant state feedback. Let $V^*$ be the largest feedback invariant subspace. The zeros of the original system are the eigenvalues of the controlled system restricted to $V^*$. Furthermore, a disturbance can be decoupled from the output if and only if it lies inside a feedback invariant subspace contained in the kernel of the observation operator [14].

In this paper we consider feedback invariance for single-input single-output infinite-dimensional systems with bounded control and observation. Let $X$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A$ be the infinitesimal generator of a $C_0$-semigroup $T(t)$ on $X$. Let $b$ and $c$ be elements of $X$. Let $U = Y = \mathbb{C}$ and $u(t) \in U$. We consider the following system in $X$:

$$\dot{x}(t) = Ax(t) + bu(t)$$

with the observation

$$y(t) = Cx(t) := \langle x(t), c \rangle.$$  \hspace{2cm} (1.1)

We sometimes refer to this system as $(A, b, c)$. The transfer function is $G(s)$ where $G(s) = \langle R(s, A)b, c \rangle$.

We denote the kernel of $C$ by

$$c^\perp := \{ x \in X \mid \langle x, c \rangle = 0 \}.$$  \hspace{2cm} (1.2)

When $b \notin c^\perp$, we show that the largest feedback-invariant subspace in $c^\perp$ exists, and is $c^\perp$ itself. We give an explicit representation of a feedback operator $K$ for which $c^\perp$ is $A + bK$-invariant. When $c \notin D(A^*)$, the operator $K$ is not bounded, so semigroup generation of $A + bK$ is not guaranteed.

If $\langle b, c \rangle = 0$ then the theory is quite different. A number of situations may occur, depending on the nature of $b$ and $c$. In particular, if $c \notin D(A^*)$, then in general no largest feedback-invariant subspace exists. This is in contrast to the finite-dimensional case, where a largest feedback invariant subspace always exists [14]. However, as in the finite-dimensional case, the spectrum of $A + bK$ is identical to the invariant zeros of the system.

This work builds on the results of Curtain and Zwart in the 1980’s, see [3], [16], [17], [18]. In [16], [17] there is a standing assumption that $(A, b)$ is such that $A + bK$ is a generator of a $C_0$-semigroup for any $A$-bounded $K$, which is a strong restriction on $b$. This paper also extends the results in [1], where it is assumed that $b \in D(A)$, $c \in D(A^*)$ and $\langle b, c \rangle \neq 0$. We remove the restrictions $b \in D(A)$ and $c \in D(A^*)$, and, most significantly, also examine the case where $\langle b, c \rangle = 0$.

We should note that even though in most infinite-dimensional systems analysis the assumption that $b$ and $c$ are in $X$ makes the analysis easier, the zeros for partial differential equations with boundary control and observation (which yields unbounded control and observation operators) is often more easily analyzed, see [11].

II. INVARIANCE CONCEPTS

For $\omega \in \mathbb{R}$, let

$$C_\omega = \{ z \in \mathbb{C} \mid \text{Re } z > \omega \}.$$  \hspace{2cm} (2.1)

Let $R(s, A) = (sI - A)^{-1}$, and let $\omega \in \mathbb{R}$ be such that $C_\omega$ is a subset of $\rho(A)$. For $\lambda_0 > \omega$, $R(\lambda_0, A)$ exists as a bounded operator from $X$ into $X$.

Definition 2.1: A subspace $Z$ of $X$ is feedback invariant if it is closed and there exists an $A$-bounded feedback $K$ such that $(A + bK)(Z \cap D(A)) \subset Z$.

The operator $K$ is not specified as unique in the above theorem. However, if $b \notin Z$, and there are two operators $K_1$ and $K_2$ that are both $(A, b)$-invariant on $Z$, then $b(K_1 x - K_2 x) \in Z$ and so $K_1 x = K_2 x$ for all $x \in Z$.

The following result shows that feedback invariance is equivalent to $(A, b)$-invariance, which is sometimes easier to work with.

Theorem 2.2: [17, Thm.II.26] A closed subspace $Z$ is feedback-invariant if and only if it is $(A, b)$-invariant, that is,

$$A(Z \cap D(A)) \subseteq Z + \text{span}\{b\}.$$  \hspace{2cm} (2.2)

Theorem 2.3: If $Z \subseteq c^\perp$ is a feedback-invariant subspace and $b \in Z$ then the system transfer function is identically zero.

Proof: Since $Z$ is feedback-invariant,

$$A(Z \cap D(A)) \subset Z + \text{span}\{b\} \subset Z.$$  \hspace{2cm} (2.3)

This implies that $Z$ is $A$-invariant. This implies that every $z \in Z$ can be written $z = (sI - A)\xi(s)$ where $\xi(s) \in D(A)$ and...
Z [17, Lem. 1.4], and \( s \in [r, \infty) \) for some \( r \in \mathbb{R} \). Since \( b \in Z \), \((sI - A)^{-1}b \in Z \) for all \( s \in [r, \infty) \). Since \( Z \subset c^\perp \), the system transfer function \( G(s) \) is zero for \( s \in [r, \infty) \). Since \( G \) is analytic on \( \rho(A) \), it must be identically zero on \( \rho(A) \). □

III. NICE CASES

If \( b \notin c^\perp \), the largest feedback-invariant subspace contained in \( c^\perp \) is \( c^\perp \).

**Theorem 3.1:** [9] Suppose \( \langle b, c \rangle \neq 0 \). Define

\[
Kx = -\frac{\langle Ax, c \rangle}{\langle b, c \rangle}, \quad D(K) = D(A),
\]

and define \((A + bK)x = Ax + bKx \) for \( x \in D(A + bK) = D(A) \). Then \((A + bK)(c^\perp \cap D(A)) \subset c^\perp \) and so the largest feedback-invariant subspace in \( c^\perp \) is \( c^\perp \) itself.

**Definition 3.2:** A closed subspace \( Z \) of \( X \) is **closed-loop invariant** if the closure of \( Z \cap D(A) \) in \( X \) is \( Z \) and there exists an \( A \)-bounded feedback \( K \) such that \((A + bK)(Z \cap D(A)) \subset Z \) and \( A + bK \) generates a \( C_0 \)-semigroup \( T_k \) on \( Z \).

The condition that \((A + bK)(Z \cap D(A)) \subset Z \) allows arbitrary elements of \( X \setminus D(A) \) to be appended to \( Z \). The additional condition that the closure of \( Z \cap D(A) \) is \( Z \) eliminates this ambiguity.

In general, \( A + bK \) does not generate a \( C_0 \)-semigroup. In this case \( c^\perp \) is not closed-loop invariant.

There are many results in the literature that give sufficient conditions for a relatively bounded perturbation of a generator of a \( C_0 \)-semigroup to be the generator of a \( C_0 \)-semigroup. For instance, if \( K \) is an admissible output element [12, Chap. 5], or if \( A \) generates an analytic semigroup [7, Chap. 9, sect. 2.2], then \( A + bK \) generates a \( C_0 \)-semigroup.

**Theorem 3.3:** [9] In addition to the assumptions of Theorem 3.1, assume that \( A + bK \) generates a \( C_0 \)-semigroup on \( X \). Then it generates a \( C_0 \)-semigroup on \( c^\perp \), hence \( c^\perp \) is closed-loop invariant under \( A + bK \).

If \( \langle b, c \rangle = 0 \), we can still find the largest feedback-invariant subspace in many cases.

We first give a definition of the relative degree of \((A, b, c)\), which is a generalization of the standard finite dimensional definition, see for example [5, pg. 99].

**Definition 3.4:** \((A, b, c)\) is of relative degree \( n \in \mathbb{Z}^+ \) if

1) the function \((s^nG(s))^{-1}\) is in \( H^\infty(\mathbb{C}) \) for some \( \gamma \in \mathbb{R} \);
2) \( \lim_{s \to -\infty} s^j G(s) = 0 \) for \( j = 1, 2, \ldots, (n-1) \).

In finite dimensions condition (1) in Definition 3.4 is equivalent to

\[
\lim_{s \to -\infty, s \in \mathbb{R}} s^n G(s) \neq 0.
\]

The above definition of relative degree seems to be the most general definition for infinite dimensional systems that guarantees some (limited) regularity of closed loop solutions, see [9].

Define

\[
Z_n = c^\perp \cap (A^*c)^\perp \cap \cdots (A^{n}c)^\perp.
\]

The existence of a largest feedback invariant subspace depends on whether \( c \in D(A^{n}) \), where \( n + 1 \) is the relative degree of the system.

**Theorem 3.5:** [9] Suppose \( n \in \mathbb{Z}^+ \) is such that

\[
c \in D(A^n), \quad b \in Z_{n-1}
\]

and

\[
\langle b, A^n c \rangle \neq 0.
\]

Then the largest feedback-invariant subspace \( Z \) in \( c^\perp \) is \( Z_n \).

We can use this to prove the following:

**Theorem 3.6:** Suppose \( n \in \mathbb{Z}^+ \cup \{0\} \) is such that \((A, b, c)\) is of relative degree \( n + 1 \) and \( c \in D(A^n) \). Then the largest feedback-invariant subspace \( Z \) in \( c^\perp \) is \( Z_n \).

Closed-loop invariance of \( Z_n \) exists under conditions similar to those for the case \( \langle b, c \rangle = 0 \). That is, if \( Z_n \) is feedback-invariant under the operator \( A + bK_n \), and \( A + bK_n \) generates a \( C_0 \)-semigroup on the original space \( X \), then \( Z_n \) is also closed-loop invariant [9].

IV. NOT SO NICE CASE

The following example illustrates that if \( \langle b, c \rangle = 0 \) and \( c \notin D(A^*) \) a largest feedback-invariant subspace as defined in Definition 2.1 might not exist.

**Example IV.1.** The following example of a controlled delay equation first appeared in Pandolfi [10]:

\[
\begin{align*}
\dot{x}_1(t) & = x_2(t) - x_2(t-1) \\
\dot{x}_2(t) & = u(t) \\
y(t) & = x_1(t).
\end{align*}
\]  

(4.6)

The transfer function for this system is

\[
G(s) = \frac{1 - e^{-s}}{s^2}. 
\]

(4.7)

The system of equations (4.6) can be written in a standard state-space form (1.1, 1.2), see [4]. Choose the state-space

\[
X = \mathbb{R}^2 \times L_2(-1,0) \times L_2(-1,0).
\]

A state-space realization on \( X \) is

\[
b = (0 \quad 1 \quad 0 \quad 0 \quad 0), \quad c = (1 \quad 0 \quad 0 \quad 0 \quad 0).
\]

Define \( D(A) \) to be \([r_1, r_2, \phi_1, \phi_2]^T \in X \) such that

\[
\phi_1(0) = r_1, \phi_2(0) = r_2, \phi_1 \in H^1(-1,0), \phi_2 \in H^1(-1,0).
\]

For \([r_1, r_2, \phi_1, \phi_2]^T \in D(A) \),

\[
A(r_1, r_2, \phi_1, \phi_2) = \begin{pmatrix}
\phi_2(t) - \phi_2(t-1) \\
0 \\
\phi_1 \\
\phi_2
\end{pmatrix}.
\]

In this example \( \langle b, c \rangle = 0 \) and \( c \notin D(A^*) \). From the transfer function (4.7) we can see that the system has relative degree 2.

Pandolfi [10] showed that the largest feedback-invariant subspace \( Z \subset c^\perp \), if it exists, is not a delay system. We
now show that this system does not have a largest feedback-invariant subspace in \( c^1 \). Define

\[
e_k = \begin{bmatrix} 0 \\ 1 \\ \exp(2\pi i kt) \end{bmatrix} \in D(A) \cap c^1.
\]

For each \( k \), the subspace \( \text{span}\{e_k\} \) is \( (A, b) \)-invariant and hence feedback-invariant (Thm. 2.2). Define

\[
V_n = \text{span}_{-n \leq k \leq n} e_k.
\]

Each subspace \( V_n \) is feedback-invariant. Define also the union of all finite linear combinations of \( e_k \),

\[
V = \bigcup V_n.
\]

By well-known properties of the exponentials \( \{ e^{2\pi i kt} \}_{k=1}^\infty \) in \( L^2(0, 1) \), the closure of \( \{ \exp(2\pi i kt) \} \) is \( L^2(0, 1) \). Consider a sequence of elements in \( V, [0, 1, 0, z_0] \) where \( z_0(0) = 1 \) and \( \lim_{n \to \infty} z_n = 0 \). This sequence converges to \( [0, 1, 0, 0] \) and so we see that the closure of \( V \) in \( X \) is \( \tilde{V} = 0 \times R \times 0 \times L^2(-1, 0) \). If there is a largest feedback-invariant subspace \( Z \) in \( c^1 \), then \( Z \supset \tilde{V} \). The important point now is that although \( b \notin V, b \in V \). Since \( b \) cannot be contained in any feedback invariant subspace (Theorem 2.3), \( V \) is not feedback-invariant. Hence no largest feedback-invariant subspace exists for this system. \( \square \)

Assume \( \langle b, c \rangle = 0 \). Theorem 2.2 implies that any element \( x \in D(A) \) of an \( (A, b) \)-invariant subspace of \( c^1 \) is contained in the set

\[
Z = \{ z \in c^1 \cap D(A) \mid \langle Az, c \rangle = 0 \}.
\]

The closure of \( Z \) is a natural candidate for the largest feedback-invariant subspace of \( c^1 \). When \( c \in D(A^*) \), the closure of \( Z \) is \( Z_1 = c^1 \cap (A^*)^c \). If \( \langle b, A^*c \rangle \neq 0 \), this is the largest feedback-invariant subspace in \( c^1 \) (Thm. 3.6). The situation when \( c \notin D(A^*) \) is quite different.

**Theorem 4.1:** If \( c \notin D(A^*) \), the set \( Z \) is dense in \( c^1 \). Furthermore, \( Z \neq c^1 \cap D(A) \).

**Proof:** This will be proven by showing that if \( Z \) is not dense in \( c^1 \), then \( c \notin D(A^*) \). Let \( A = \rho(A) \) and \( A_\lambda = A - \lambda I \), so \( D(A_\lambda) = D(A) \). \( D(A) \) is a Hilbert space with the graph norm, and the graph norm is equivalent to

\[
\| x \|_1 := \| A_\lambda x \|.
\]

The corresponding inner product on \( D(A) \) is

\[
\langle x, y \rangle_1 := \langle A_\lambda x, A_\lambda y \rangle.
\]

Define \( e = (A_\lambda^*)^{-1}c \in X \). For \( x \in D(A) \), the condition \( \langle c, x \rangle = 0 \) is written

\[
0 = \langle c, x \rangle = \langle A_\lambda x, A_\lambda A_\lambda^{-1}c \rangle = \langle x, A_\lambda^{-1}c \rangle.
\]

For \( x \in c^1 \cap D(A_\lambda) \), the condition \( \langle Ax, c \rangle = 0 \) is equivalent to \( \langle A_\lambda x, c \rangle = 0 \). Hence for such \( x \) we have

\[
0 = \langle A_\lambda x, c \rangle = \langle A_\lambda x, A_\lambda A_\lambda^{-1}c \rangle = \langle x, A_\lambda^{-1}c \rangle.
\]

We can write \( Z \) as

\[
\{ x \in D(A) \mid \langle x, A_\lambda^{-1}c \rangle = 0 \}
\]

We now introduce the notation

\[
(y)_1 := \{ x \in D(A) \mid \langle x, y \rangle = 0 \}.
\]

Using this notation,

\[
Z = (A_\lambda^{-1}e)_1 \cap (A_\lambda^{-1}c)_1.
\]

Now suppose that \( Z \) is not dense in \( c^1 \). Then there exists \( v \in c^1 \) such that \( \langle x, v \rangle = 0 \) for all \( x \in Z \). Define \( w = (A_\lambda^{-1}e)_1 \). As in (4.11), for \( x \in D(A) \), the condition \( \langle x, v \rangle = 0 \) is equivalent to

\[
\langle x, A_\lambda^{-1}w \rangle = 0.
\]

Hence we see that

\[
Z = (A_\lambda^{-1}e)_1 \cap (A_\lambda^{-1}w)_1.
\]

Let \( R \) be the orthogonal projection from \( D(A) \) onto \( (A_\lambda^{-1}e)_1 \) (using the inner product \( \langle \cdot, \cdot \rangle_1 \)). Then

\[
Z = (A_\lambda^{-1}e)_1 \cap (RA_\lambda^{-1}c)_1
\]

and

\[
(A_\lambda^{-1}e)_1 \cap (A_\lambda^{-1}w)_1 = (A_\lambda^{-1}e)_1 \cap (RA_\lambda^{-1}w)_1.
\]

Hence (4.14) becomes

\[
(A_\lambda^{-1}e)_1 \cap (RA_\lambda^{-1}c)_1 \subseteq (A_\lambda^{-1}e)_1 \cap (RA_\lambda^{-1}w)_1.
\]

This implies that there is a scalar \( \gamma \) such that

\[
RA_\lambda^{-1}c = \gamma RA_\lambda^{-1}w.
\]

We obtain that

\[
A_\lambda^{-1}c = \alpha A_\lambda^{-1}w + \beta A_\lambda^{-1}e.
\]

Applying \( A_\lambda \) to both sides of this equation,

\[
c = \alpha w + \beta e.
\]

Since \( w = (A_\lambda^*)^{-1}v \) and \( c = (A_\lambda^*)^{-1}c \), we see that \( c \in D(A_\lambda^*) = D(A^*) \). Thus, if \( Z \) is not dense in \( c^1 \) in \( c^1 \) then \( c \notin D(A^*) \).

Now assume that \( Z = c^1 \cap D(A) \). Then \( (A_\lambda^{-1}e)_1 \cap (A_\lambda^{-1}w)_1 = (A_\lambda^{-1}e)_1 \cap (RA_\lambda^{-1}e)_1 \), so, as above, \( c = \beta e \) would imply that \( c \notin D(A^*) \).

**Corollary 4.2:** Suppose that \( q \in X \) and \( c \notin D(A^*) \). Then \( q^1 \cap Z \) is dense in \( q^1 \cap c^1 \). Furthermore, \( q^1 \cap Z \neq q^1 \cap c^1 \cap D(A) \).

**Proof:** If \( q = \lambda c \) for some scalar \( \lambda \), then \( q^1 \cap Z = Z \) and \( q^1 \cap c^1 = c^1 \), and the result follows immediately from Theorem 4.1.

Assume now that \( q \) is not parallel to \( c \). Let \( P \) be the orthogonal projection of \( X \) onto \( c^1 \), and \( \tilde{q} = Pq \), so \( \tilde{q} \neq 0 \). Let \( X = \tilde{q}^1 \), and let \( Q \) be the orthogonal projection of \( X \) onto \( \tilde{q}^1 \). By construction, \( c = Qe \in \tilde{X} \). Let

\[
\tilde{A} = QA|_{\tilde{X}}, \ D(\tilde{A}) = D(A) \cap \tilde{X},
\]

\[
\tilde{Z} = \{ x \in D(\tilde{A}) \mid \langle x, c \rangle = 0 \text{ and } \langle A\tilde{x}, c \rangle = 0 \}.
\]

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We wish to show that \( c \not\in D(\tilde{A}^*) \). Note that for \( x \in \tilde{X} \),
\[
\langle \tilde{A}x, c \rangle = \langle \tilde{Q}Ax, c \rangle = \langle Ax, Qc \rangle = \langle Ax, c \rangle.
\]
(4.16)

Therefore \( c \not\in D(\tilde{A}^*) \) if the functional \( x \to \langle Ax, c \rangle \) is
unbounded on \( \tilde{X} \). To show this let \( b_0 \in D(\tilde{A}) \cap \tilde{X} \) and
let \( Q_0 \) be the (possibly not orthogonal) projection onto \( \tilde{X} \)
given by
\[
Q_0 x = x - \frac{\langle x, \tilde{q} \rangle}{\langle q_0, \tilde{q} \rangle} q_0.
\]

Then \( \langle Ax, c \rangle \) is unbounded on \( \tilde{X} \) if \( \langle A Q_0 x, c \rangle \) is unbounded
on \( X \). Since
\[
\langle A Q_0 x, c \rangle = \langle Ax, c \rangle - \frac{\langle x, \tilde{q} \rangle}{\langle q_0, \tilde{q} \rangle} \langle A q_0, c \rangle.
\]

The second term on the right is clearly bounded on \( X \), and
the first term on the right is unbounded on \( X \) since \( c \not\in D(\tilde{A}^*) \), so \( \langle A Q_0 x, c \rangle \) is not a bounded operator on \( X \), hence \( c \not\in D(\tilde{A}^*) \).

Now we can apply Theorem 4.1 to \( \tilde{X} \), \( \tilde{A} \), \( c \), and \( \tilde{Z} \) and conclude
that \( \tilde{X} \cap \tilde{Z} \) is dense in \( \tilde{X} \cap c^\perp \) and \( \tilde{X} \cap \tilde{Z} \neq \tilde{X} \cap c^\perp \cap D(\tilde{A}) \).

For \( x \in c^\perp \), \((x, Pq) = \langle x, q \rangle \) and so
\[
\tilde{X} \cap c^\perp = \{ x \in X \mid \langle x, c \rangle = 0, \langle x, Pq \rangle = 0 \}
\]
\[
\qquad = \{ x \in X \mid \langle x, c \rangle = 0, \langle x, q \rangle = 0 \}
\]
\[
\qquad = q^\perp \cap c^\perp.
\]

Similarly,
\[
\tilde{X} \cap \tilde{Z} = \{ x \in D(\tilde{A}) \mid \langle x, c \rangle = 0, \langle x, q \rangle = 0, \langle \tilde{A}x, c \rangle = 0 \}.
\]
(4.17)

This can be written
\[
\tilde{X} \cap \tilde{Z} = \{ x \in D(\tilde{A}) \mid \langle x, c \rangle = 0, \langle x, q \rangle = 0 \}
\]
\[
\qquad = q^\perp \cap Z.
\]

Thus we have shown that \( q^\perp \cap Z \) is dense in \( q^\perp \cap c^\perp \), and
that the two spaces are not equal. \( \Box \)

If \( \langle b, c \rangle = 0 \), \( c \in D(\tilde{A}^*) \), and \( \langle b, A^*c \rangle \neq 0 \),
the largest invariant subspace in \( c^\perp \) is \( Z_1 = c^\perp \cap (A^*c)^\perp \).

Defining \( \alpha = \frac{1}{\langle b, A^*c \rangle} \),
\[
A + bK = A + ab(Ax, A^*c), \quad \text{with}
\]
\[
D(\tilde{A} + bK) = \{ z \in c^\perp \cap D(\tilde{A}) \mid \langle Az, c \rangle = 0 \},
\]
is \( Z_1 \)-invariant. In many cases, this operator generates a \( C_0 \)-
semigroup on \( Z_1 \). It is tempting to hope, that even if \( c \not\in D(\tilde{A}^*) \),
the operator (with some value of \( \alpha \))
\[
A + bK = A + ab(\tilde{Q}Ax, c),
\]
\[
D(\tilde{A} + bK) = \{ z \in c^\perp \cap D(\tilde{A}^2) \mid \langle Az, c \rangle = 0 \}
\]
is a generator, or has an extension which is a generator. However, we see from the next result that this operator is not closable, so that no extension of it is a generator of a \( C_0 \)-semigroup.

**Theorem 4.3:** Suppose \( b \in X \) and \( c \not\in D(\tilde{A}^*) \). Then the operator
\[
A_F x = Ax + b \langle A^2 x, c \rangle,
\]
\[
D(A_F) = \{ x \in c^\perp \cap D(\tilde{A}^2) \mid \langle Ax, c \rangle = 0 \}
\]
is not closable.

**Proof:** Let \( \lambda \in \rho(A) \) and \( A_\lambda = A - \lambda I \), as above.
From Corollary 4.2 we see that \( ((A_\lambda^{-1})^*c)^\perp \cap Z \) is dense in
\( ((A_\lambda^{-1})^*c)^\perp \cap c^\perp \). Let
\[
Tx := \langle A_\lambda x, c \rangle, \quad D(T) = ((A_\lambda^{-1})^*c)^\perp \cap c^\perp \cap D(A).
\]

We will now show that \( T \) is not closable. From Corollary 4.2,
\( ((A_\lambda^{-1})^*c)^\perp \cap Z \neq D(T) \). Thus we can choose \( f \in D(T) \)
such that \( f \not\in ((A_\lambda^{-1})^*c)^\perp \cap Z \), and there exists \( \{ f_n \} \subset ((A_\lambda^{-1})^*c)^\perp \cap Z \) such that \( \lim f_n = f \). From the definition of \( Z \), \( T f_n = 0 \) for all \( n \). Let \( x_n = f - f_n \), so
\[
\lim x_n = 0, \quad \text{and} \quad \lim Tx_n = Tf \neq 0,
\]
(4.18)

which shows that \( T \) is not closable [15, Section II.6, Proposition 2]. It then follows that \( I + bT \) with domain \( D(T) \) is not closable.

Now note that \( y \in D(A_F) \) if and only if \( A_\lambda y \in D(T) \), and
that for \( y \in D(A_F) \)
\[
A_F y = (I + bT)A_\lambda y + \lambda y,
\]
so \( A_F \) is closable if and only if \( (I + bT)A_\lambda \) is closable.

Using the sequence \( \{ x_n \} \subset D(T) \) defined above, define \( y_n = A_\lambda^{-1}x_n \). Note that \( \{ y_n \} \subset D(A_F) \) and
\[
\lim y_n = 0 \quad \text{and} \quad \lim(I + bT)A_\lambda y_n = bTf \neq 0.
\]

Hence \( (I + bT)A_\lambda \) is not closable, so \( A_F \) is not closable. \( \Box \)

**Definition 4.4:** The invariant zeros of (1.1), (1.2) are the
set of all \( \lambda \) such that
\[
\begin{pmatrix}
\lambda I - A & b \\
C & 0
\end{pmatrix}
\begin{pmatrix}
x \\
u
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
(4.19)

has a solution for \( u \in U \) and non-zero \( x \in D(A) \).

One of the important properties of a largest invariant
subspace, is the following well-known result. A proof for
infinite-dimensional system can be found in, for instance, [9].

**Theorem 4.5:** Assume a largest feedback-invariant
subspace \( Z \) of \( (A, b, c) \) exists and \( G(s) \) is not identically zero, and
let \( K \) be an operator such that \( A + bK \) is \( Z \)-invariant.

Then the eigenvalues of \( (A + bK)|_Z \) are the invariant zeros of the
system.

We now show that, for a large class of relative degree 2
systems we can find a feedback \( K \) and a subspace of \( c^\perp \)
that is \( (A + bK) \)-invariant. In general, such a \( A + bK \)
is not closable on the original Hilbert space, hence does not
generate a \( C_0 \)-semigroup in the original norm. However, the
spectrum of \( A + bK \) does yield the invariant zeros. In order
to define this space we need to extend \( \langle A, c \rangle \) to a larger set
than \( D(A) \). Define
\[
C_A x = \lim_{s \to -\infty, s \in \mathbb{R}} \langle sA R(s, A)x, c \rangle
\]
(4.20)

with domain
\[
D(C_A) = \{ x \in X \mid \lim_{s \to -\infty, s \in \mathbb{R}} \langle sA R(s, A)x, c \rangle \text{ exists} \}.
\]

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(This is the same as $(CA)_L$ where the $L$-extension is given by [13, Defn. 5.6].) It is straightforward to verify that $D(C_A) \supseteq D(A)$. If $x \in D(A)$, then $C_A(x) = \langle Ax, c \rangle$. Also, if $c \in D(A^*)$, then $D(C_A) = X$ and $C_A x = \langle x, A^*c \rangle$.

Proposition 4.6: Assume that $(A, b, c)$ has relative degree at least 2. Then $\lim_{s \to \infty} s^2G(s)$ exists for real $s$ if and only if $b \in D(C_A)$. In this case,

$$\lim_{s \to \infty} s^2G(s) = C_A b.$$  (4.21)

Proof: First note that since the relative degree of the systems is at least 2, $\lim_{s \to \infty} sG(s) = 0$. But,

$$\lim_{s \to \infty} sG(s) = \lim_{s \to \infty} \langle s(sI - A)^{-1}b, c \rangle = \langle b, c \rangle$$

and so $\langle b, c \rangle = 0$. Since

$$s^2G(s) = \langle s(sI - A)(sI - A)^{-1}b, c \rangle + \langle sA(sI - A)^{-1}b, c \rangle,$$

we obtain

$$\lim_{s \to \infty} s^2G(s) = \lim_{s \to \infty} s(b, c) + \lim_{s \to \infty} sA(sI - A)^{-1}b, c)$$

$$= \lim_{s \to \infty} \langle sA(sI - A)^{-1}b, c \rangle.$$

The result follows. $\square$

Using the operator $C_A$, the space $Z$ defined above in (4.8) can be extended to

$$Z_A = \{x \in c^{|} \cap D(C_A)| C_A x = 0 \}.$$

If $c \in D(A^*)$, then $Z_A = Z_1$.

The following theorem is now straightforward, so we omit the proof.

Theorem 4.7: Assume that a system $(A, b, c)$ has relative degree 2 and $\lim_{s \to \infty} s^2G(s)$ exists. Define on $c^{|}$

$$A_Kx = Ax + bKx,$$  (4.22)

where

$$Kx = -\frac{C_A(Ax)}{C_A b}$$  (4.23)

with domain

$$D(A_K) = \{x \in D(A) \cap c^{|}| A \in D(C_A), C_A x = 0 \}.$$

The space $Z_A$ is invariant under $A_K$.

The operator $K$ in this theorem is in general not $A$-bounded. If $c \in D(A^*)$, then $K$ is the same $A$-bounded operator defined above. For the general case, we need the extension of $(A, c)$ to $C_A$ in order to define $K$.

Theorem 4.8: Assume that the system $(A, b, c)$ has relative degree 2 and $\lim_{s \to \infty} s^2G(s)$ exists. The invariant zeros of $(A, b, c)$ are the eigenvalues of $A_K$, where $A_K$ is as defined in (4.22, 4.23).

Proof: First assume that $\lambda$ is an eigenvalue of $A_K$ with eigenvector $v$. Note that $\psi \in D(A) \cap c^{|}$, so set $x = v$ and $u = -Kv$ in (4.19) to obtain that $\lambda$ is an invariant zero of the original system.

Now assume that $\lambda$ is an invariant zero. That is, there exists $u \in \mathbb{R}$ and $v \neq 0$ such that $v \in c^{|} \cap D(A)$ and

$$\lambda v - Av + bu = 0.$$  

We need to first show that $v \in D(A_K)$. First, note that

$$Av = \lambda v - bu.$$  

Since $\lim_{s \to \infty} s^2G(s)$ exists, $b \in D(C_A)$ and since $D(A) \subseteq D(C_A)$. Also,

$$C_A v = \langle Av, c \rangle = \lambda \langle v, c \rangle + u(b, c) = 0 + 0.$$  

Thus, $v \in D(A_K)$. It follows that

$$\begin{bmatrix} \lambda I - A & b \\ c & 0 \end{bmatrix} \begin{bmatrix} v \\ Kv + u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

Since $b \notin Z_A, Kv + u = 0$ and $\lambda$ is an eigenvalue of $A_K$ on $c^{|}$ with the given domain. $\square$

The following result follows immediately from Theorem 4.3.

Corollary 4.9: Suppose $(A, b, c)$ has relative degree 2 and $\lim_{s \to \infty} s^2G(s)$ exists. If $c \notin D(A^*)$ then the operator $A_K$ with domain $D(A_K)$ defined in (4.22) is not closable.

It is shown in the next example that, in general, it is not possible to restrict $D(A_K)$ to $D(A^2)$ and obtain the invariant zeros.

Example IV.1 continued. Recall that this controlled delay system has no largest feedback-invariant subspace. A straightforward calculation shows that the invariant zeros of this control system are $i2n\pi r$, where $n$ is any integer. We now verify that these are the eigenvalues of $A_K$ on $c^{|}$.

We can calculate $C_A$ from its definition to be

$$C_A x = r_2 - \lim_{s \to \infty} s e^{-s} \int_{-1}^{0} e^{-s\tau} \phi_2(\tau)d\tau.$$  

Denote the limiting value of

$$\lim_{s \to \infty} s e^{-s} \int_{-1}^{0} e^{-s\tau} \psi(\tau)d\tau$$

by $E_{-1}\psi$, when this limit exists. (If the value of $\psi$ at $-1$ exists, $E_{-1}\psi = \psi(-1)$.) Then

$$D(C_A) = \{|r_1, r_2, \phi_1, \phi_2|^T \in X; E_{-1} \phi_2 \text{ defined}\}$$

$$\cup \{|r_1, r_2, \phi_1, \phi_2|^T \in X; \phi_2 \in H_1(-1, 0)\}.$$  

We have $C_A b = 1$ and $A_K = A + bK$, where

$$Kx = -C_A(Ax) = E_{-1} \phi_2,$$  (4.24)

with $D(A_K)$

$$\{(0, r_2, \phi_1, \phi_2); \phi_1(0) = 0, \phi_2(0) = \phi_2(-1) = r_2, \phi_1 \in H_1(-1,0), \phi_2 \in H_1(-1,0), E_{-1} \phi_2 \text{ defined}\}.$$  

When $A_K x = \lambda x, x \in D(A_K)$, we obtain

$$0 = 0$$

$$E_{-1} \phi_2 = \lambda r_2$$

$$\phi_1 = \lambda \phi_1$$

$$\phi_2 = \lambda \phi_2.$$  

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This system of equations has a non-trivial solution in $D(A_K)$ for $\lambda = i2n\pi$ with

$$x = \begin{bmatrix} 0 \\ r_2 \\ 0 \\ r_2 e^{i2\pi n t} \end{bmatrix}. $$

Thus, the invariant zeros of this system are $i2n\pi$. These are exactly the invariant zeros. Suppose we restrict the domain $D(A_K)$ to the more obvious

$$D(A_K) = \{ x \in D(A) \cap c^+ | Ax \in D(A), \langle Ax, c \rangle = 0 \}. $$

This yields that $A_K$ is invariant on $Z$ as defined in (4.8). For this example, $D(A_K)$ is

$$\{(0, r_2, \phi_1, \phi_2); \phi_1(0) = 0, \phi_2(0) = \phi_2(-1) = r_2, \phi_1 \in H_2(-1), 0), \phi_2 \in H_2(-1, 1), \phi_1(0) = 0, \phi_2(0) = 0 \}. $$

However, with this choice of domain, $A_K$ does not have any eigenvalues. □

The feedback (4.24) matches that obtained in [18] by direct calculation on the delay differential equation. However, not only do we now have a general definition of the appropriate feedback, we have an rigorous definition of its domain.

Example IV.2 We give here a system $(A, b, c)$ for which there is no largest feedback invariant subspace of $c^+$. Let $X$ be the Hilbert space $\ell^2$, with index set $N$. Let $h = [1, 1, 1, \ldots], \bar{0} = [0, 0, 0, 0, \ldots]^T$ and $D = \text{diag}\{\lambda_2, \lambda_3, \lambda_4, \ldots\}$, where $\lambda_j = -j$ for $j = 2, 3, \ldots$. Define

$$A = \begin{bmatrix} -1 & h \\ 0 & D \end{bmatrix}, \quad c = [1, 0, 0, 0, \ldots]^T, $$

and, for any fixed integer $N > 2$,

$$b = [0, b_2, b_3 \ldots b_N, 0, 0, \ldots]^T, \quad \sum_{j=2}^{N} b_j \neq 0. $$

It is easy to verify that $\langle b, c \rangle = 0$ and $c \notin D(A^*)$. Also, since $b \in D(A)$, $C_A b = \langle Ab, c \rangle = \sum_{j=2}^{N} b_j \neq 0$. For positive integers $n > N$, define the subspace of $X$

$$V_n = \{[0, x_2, \ldots, x_n, 0, \ldots]^T; x_j = 0 \text{ if } j > n, \sum_{k=2}^{n} x_k = 0 \}; $$

For $x \in V_n$, define

$$K_n x = \frac{1}{C_A b} \sum_{j=2}^{n} jx_j. $$

It is easy to verify that $V_n$ is $A + bK_n$-invariant. Define $V = \cup_{n \in N} V_n$.

Any largest feedback-invariant subspace must contain $V$. It is clear that $V$ is dense in $Z = \{[x_j]_{j \in N} \in D(A) | x_1 = 0, \sum_{j \in N} x_j = 0 \}$. Since $Z$ can also be written as (4.8), Theorem (4.1) implies that $V$ is dense in $c^+$. However, $b \in c^-$ and so, from Theorem 2.3 the closure of $V$ is not feedback invariant. Hence, no largest feedback-invariant subspace exists. □

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REFERENCES


