The Duality Relation between Maximal Output Admissible Set
and Reachable Set

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Abstract—The concept of positive invariance is central
to several problems in control theory, such as constrained
control, disturbance rejection and robustness analysis. This
paper considers two specific types of positively invariant sets,
known as the maximal output admissible set and reachable set,
determined for linear continuous-time system. In this paper, the
inner and outer approximations of both the maximal output
admissible set and reachable set are established by using the
forward Euler approximated system and the modified zero
order hold system. The main purpose is to show that there exists
the duality relation between the maximal output admissible
set and reachable set. An application of these two positively
invariant sets to characterize the $L^p$-induced norm of Hankel
operator is discussed.

I. INTRODUCTION

A subset of state space is said to be positively invariant
if it has the property that, if it contains the system state at
some time, then it will contain the state trajectory originating
from such state also in the future for any admissible control
function. Because of this significant property, the existence
and characterization of positively invariant sets are the funda-
mental tools in many control analysis and synthesis problems
[1], such as constrained control [2], [3], disturbance rejection
[4], robustness analysis [5], model predictive control and
switching control scheme [6] etc.

In particular, the positively invariant sets known as the
maximal output admissible set and reachable set are well-
known and extensively applied in various aspects of synthesis
and analysis for linear discrete-time systems. The maximal
output admissible set is usually defined as the largest set
of the initial state which makes resultant output trajectory
remain inside given polyhedral region consistently [2], [3]
which can be interpreted as a special case of our consider-
ations. On the other hand, the reachable set is the set of all
reachable states from the origin under the effect of unknown
but bounded exogenous inputs [7], [8].

To characterize such given constraints on state and control
variables, the $\ell^\infty$ norm was typically employed in a number
of the previous papers [2], [3], [4], [5], [6], [7], [8], [9]. In
contrast with the other existing results, this paper considers
the maximal output admissible set and reachable set for
linear continuous-time systems and equips the constrains on
input and output signals with $L^p$ norm where $p \in [1, \infty].$
Accordingly, there are two crucial issues arising from the
transitions respectively from discrete-time to continuous-time
and from $\ell^\infty$ norm to $L^p$ norm. The reason which makes
the construction of the positively invariant sets complicated even
for simple systems is that there exists constraints applied to
the system variables continuously. The characterization of the
positively invariant sets for linear continuous-time system
naturally requires the intersection of an infinite number of
half spaces. Except in the case of $L^2$ norm, it is impossible
to construct the polyhedral invariant sets. Therefore, the
problem becomes how to approximate the sets appropriately.

Our previous results presented in [10] show that the
inclusion between the positively invariant sets of continuous-
time system and its forward Euler approximated discrete-
time system holds. In addition, the inclusion monotonically
holds in the case of the forward Euler approximated systems,
discretized by different sampling periods. The maximal out-
put admissible set for linear continuous-time system always
contains the maximal output admissible set of the corre-
sponding forward Euler approximated system as its subset.
The improved approximations are consistently obtainable
while the smaller sampling periods are applied. On the other
hand, the reachable set for a linear continuous-time system is
the subset of the reachable set of the corresponding forward
Euler approximated discrete-time system. The approximation
improves as long as the sampling period decreases.

The contributions of this paper are divided into three parts.
First, this paper examines the duality relation between the
maximal output admissible set and reachable set. Because the
$(L^p, L^p)$ norm constraints those act on the control variables
are convex, the Hahn-Banach theorem [11] can be imple-
mented to derive the duality results which are necessary
in deriving other significant properties. Second, the outer
approximation of the maximal output admissible set and the
inner approximation of the reachable set for continuous-time
systems are established by utilizing the modified zero order
hold systems. Third, a characterization of the $L^p$-induced
norm of Hankel operator based on the positively invariant
sets are proposed.

The remainder of this paper is organized as follows:
We define the positively invariant sets in consideration and
formulate the problem in Section 2. In Section 3, we review
the previous results presented in [10] and address the main
contributions of this paper in detail. Section 4 shows how
the results can be applied to the characterization of the $L^p$-
induced norm of Hankel operator. Section 5 is devoted to the
numerical examples of the results proposed in Section 3.

Notations: Let $I^-$ and $I^+$ denote the set of negative and
positive integers. The set of positive real numbers and real
numbers are denoted by $\mathbb{R}^+$ and $\mathbb{R}$, respectively. Let $\|f\|_{L^p}$
denote the $L^p$ norm of function $f \in L^p$ and $\|g\|_{\ell^\infty}$
denote the $\ell^\infty$ norm of function $g \in \ell^\infty$. Let $A$ be a subset in $\mathbb{R}^n$,
then int$(A)$, cl$(A)$ and conv$(A)$ denote interior, closure and
convex hull of set $A$. Let $\alpha A = \{\alpha a : a \in A\}$, where $\alpha \in \mathbb{R}$.
II. Definitions and Problem Formulation

The linear continuous-time system under consideration is expressed by the following state space equation:

\[ \Sigma^c: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ z(t) = Cx(t). \end{cases} \]

The vector signals are defined as follows: \( w \in \mathbb{R} \) denotes the exogenous input, \( x \in \mathbb{R}^n \) denotes the system state and \( z \in \mathbb{R} \) denotes the controlled output.

We make the following assumptions for the remainder of the paper that \( \Sigma^c \) is asymptotically stable. The pair \((A, B)\) is controllable and the pair \((C, A)\) is observable. For representing the system state \( x \) under the effect of an exogenous input \( w \) and unforced output of system \( \Sigma^c \), let us introduce operators \( \mathcal{B}: L^p \rightarrow \mathbb{R}^n \), \( \mathcal{C}: \mathbb{R}^n \rightarrow L^p \) as follows:

\[ \mathcal{B}w := \int_0^\infty e^{At}Bw(t)dt, \quad \mathcal{C}x(t) := Cx(t). \]

Remind that the dual system of the linear continuous-time system \( \Sigma^c \) can be represented as follows:

\[ \Sigma^{c o}_r: \begin{cases} \dot{x}(t) = A^*x(t) + C^tw(t), \\ z(k) = B^*x(k). \end{cases} \]

The state under an exogenous input \( w \) and autonomous output of system \( \Sigma^{c o}_r \) are expressed by adjoint operators of \( \mathcal{C} \) and \( \mathcal{B} \), respectively.

Consider the forward Euler approximated system of system \( \Sigma^c \) with sampling period \( \tau > 0 \):

\[ \Sigma^e_r: \begin{cases} x(k + 1) = (I + \tau A)x(k) + (\tau B)w(k), \\ z(k) = Cx(k). \end{cases} \]

We further assume that the forward Euler approximated system is asymptotically stable i.e. \( |\lambda_i(I + \tau A)| < 1 \), for all \( i \in \{1, \cdots, n\} \). This implies that the sampling period must be chosen within the following range:

\[ 0 < \tau < \min_{i \in \{1, \cdots, n\}} \frac{-2R|\lambda_i(A)|}{|\lambda_i(A)|^2} =: \theta. \]

To express the state and output of system \( \Sigma^e_r \), define the operators \( \mathcal{B}^e_r: \ell^p \rightarrow \mathbb{R}^n \), \( \mathcal{C}^e_r: \mathbb{R}^n \rightarrow \ell^p \), \( p \in [1, \infty] \) as

\[ \mathcal{B}^e_r w := \sum_{k=0}^{\infty} (I + \tau A)^k(Bw(k)), \quad \mathcal{C}^e_r x(k) := C(I + \tau A)^kx. \]

Now let us consider the modified zero order hold system of \( \Sigma^e \) with sampling period \( h > 0 \) expressed by the followings:

\[ \Sigma^e_h: \begin{cases} x(k + 1) = e^{Ah}x(k) + E_hw(k), \\ z(k) = Ch^{-1}E_hx(k). \end{cases} \]

where, \( E(h) := \int_0^h e^{At}dt \), \( \mathcal{B}^e_h: \ell^p \rightarrow \mathbb{R}^n \), \( \mathcal{C}^e_h: \mathbb{R}^n \rightarrow \ell^p \), \( p \in [1, \infty] \) as follows:

\[ \mathcal{B}^e_h w := \sum_{k=0}^{\infty} e^{Ah}E_hw(k), \quad \mathcal{C}^e_h x(k) := Ch^{-1}E_he^{Ah}x. \]

Consider the dual systems of the linear discrete-time systems \( \Sigma^{e}_r \) and \( \Sigma^{e}_h \) determined as follows:

\[ \Sigma^{e}_r: \begin{cases} x(k + 1) = (I + \tau A)x(k) + (\tau C^t)w(k), \\ z(k) = B^*x(k). \end{cases} \]

\[ \Sigma^{e}_h: \begin{cases} x(k + 1) = e^{Ah}x(k) + E_h^tC^tw(k), \\ z(k) = B^*h^{-1}E_h^tE_hx(k). \end{cases} \]

In fact, the above two systems are not the true dual systems of \( \Sigma^c \) and \( \Sigma^d \), because the positions of the sampling parameters \( \tau \), \( h^{-1} \) are unaffected. However, for the purposes of this paper we consider these systems to be the dual systems. Note that it is possible to describe the state under exogenous input \( w \) and unforced output of systems \( \Sigma^{e}_r \) and \( \Sigma^{e}_h \) by adjoint operators of \( \mathcal{C}^e_r \), \( \mathcal{B}^e_r \), \( \mathcal{C}^e_h \) and \( \mathcal{B}^e_h \), respectively.

Let us consider the zero order hold of linear continuous-time system \( \Sigma^c \) with sampling period \( h > 0 \):

\[ \Sigma^h: \begin{cases} x(k + 1) = e^{Ah}x(k) + E_hw(k), \\ z(k) = Cx(k). \end{cases} \]

It is obvious that it is possible to express the state of system \( \Sigma^h \) corresponding to an exogenous input \( w \) by utilizing the operator \( \mathcal{B}^h \) determined in (5). Let us define the linear operators \( \mathcal{B}^h: \ell^p \rightarrow \mathbb{R}^n \) as \( \mathcal{B}^h w := \mathcal{B}^e_h w \) and \( \mathcal{C}^h: \mathbb{R}^n \rightarrow \ell^p \), \( p \in [1, \infty] \) as \( \mathcal{C}_c^h x(k) := Ce^{Ah}x. \)

First, we recall the following definitions:

**Definition 1** Let us consider the specified set of functions \( W \) and the system:

\[ \Sigma: \dot{x}(t) = A(x(t)) + B(u(t)), \quad x(k + 1) = A(k)x(k) + Bw(k), \quad \text{where}, \ w \in W. \]

The set \( S \) is a positively invariant set of \( \Sigma \) if \( x(0) \in S \) then \( x(t) = e^{A \mu}x(0) + \int_0^t e^{A \mu_1}Bw(t)dt \in S \) for all \( w \in W \) and all \( t \in R^+ \). In the case of discrete-time system, \( x(k) = A^k_1x(0) + \sum_{j=0}^{k-1}A^{k-j}_1Bw(j) \in S \) for all \( w \in W \) and all \( k \in I^+ \) respectively.

Now let us consider the two types of positively invariant set known as the maximal output admissible set and reachable set.

**A. Maximal Output Admissible Set**

Consider linear systems \( \Sigma^c \), \( \Sigma^e_r \), \( \Sigma^d \) and \( \Sigma^h \) with \( B = 0 \). The set of initial state such that the corresponding output satisfies given \( (L^p, \ell^p) \) norm constraints is to be characterized by the following:

**Definition 2** The output admissible set of system \( \Sigma^c \) is the set of initial states of the linear continuous-time system \( \Sigma^c \) such that its corresponding output \( z \) satisfies the control specification \( ||z||_{L^p} \leq 1 \), for fixed \( p \in [1, \infty] \).

The maximal output admissible set is the union of all output admissible sets. In other words, the maximal output admissible set is the largest output admissible set and can be expressed employing previously defined operators as follows:

**Definition 3** The maximal output admissible set for linear continuous-time system \( \Sigma^c \) is defined as

\[ \mathcal{M}_{L^p}(\Sigma^c) := \{ x \in \mathbb{R}^n : \|\mathcal{C}x\|_{L^p} \leq 1 \}. \]
The maximal output admissible set for discrete-time systems $\Sigma^e$, $\Sigma^e_h$ and $\Sigma^d$ are respectively determined by

$$\mathcal{M}_L(\Sigma^e) := \{ x \in \mathbb{R}^n : \| \tau^e \|_{LP} \leq 1 \} \ ,$$
$$\mathcal{M}_L(\Sigma^e_h) := \{ x \in \mathbb{R}^n : h^e \| \Sigma^e_h \|_{LP} \leq 1 \} \ ,$$
$$\mathcal{M}_L(\Sigma^d) := \{ x \in \mathbb{R}^n : h^d \| \Sigma^d \|_{LP} \leq 1 \} \ .$$

For linear discrete-time systems, it is worth noting that the constraints are configured by the $LP$ norm of the output multiplied by $\tau^e$ or $h^e$ which are nothing but the $LP$-induced norm of the hol operator.

**Theorem 1** The following duality relations hold.

$$(\mathcal{R}_L(\Sigma^e))^o = \mathcal{M}_L(\Sigma^e) \ , \ p \in [1, \infty],$$
$$(\mathcal{M}_L(\Sigma^e))^o = \mathcal{R}_L(\Sigma^e) \ , \ p \in [1, \infty],$$
$$(\mathcal{M}_L(\Sigma^e))^o = \text{cl}(\mathcal{R}_L(\Sigma^e)) \ .$$

where $1/p + 1/q = 1$.

**Proof:** Let us consider $x \in \mathcal{M}_L(\Sigma^e) = \{ x \in \mathbb{R}^n : \| B^e A^e(x) \|_{LP} \leq 1 \} \ and \ y \in \mathcal{R}_L(\Sigma^e).$ We have that:

$$\langle x, y \rangle = \langle x, \mathcal{R}_L(\Sigma^e) \rangle \leq \| x \|_{LP} \| B^e A^e(x) \|_{LP} \leq 1.$$

Hence, $\mathcal{M}_L(\Sigma^e) \subset (\mathcal{R}_L(\Sigma^e))^o$. Conversely, suppose that $x \in (\mathcal{R}_L(\Sigma^e))^o \ \mathcal{M}_L(\Sigma^e)$. Since $\mathcal{M}_L(\Sigma^e)$ is closed and convex and $\mathcal{R}_L(\Sigma^e)$ contains the origin as its interior, by Hahn-Banach theorem, there exists $d > 0$ and $y \in \mathcal{R}_L(\Sigma^e)$ such that $\langle x, y \rangle > d > \sup_{m \in \mathcal{M}_L(\Sigma^e)} \langle m, y \rangle$. However,

$$\sup_{m \in \mathcal{M}_L(\Sigma^e)} \langle m, y \rangle = \sup_{m \in \mathcal{M}_L(\Sigma^e)} \langle m, \mathcal{R}_L(\Sigma^e) \rangle \leq 1 \ .$$

Therefore, there exists $y \in \mathcal{R}_L(\Sigma^e)$ such that $\langle x, y \rangle > 1$, contradicting the fact that $x \in (\mathcal{R}_L(\Sigma^e))^o$. This implies $(\mathcal{R}_L(\Sigma^e))^o \subset \mathcal{M}_L(\Sigma^e)$. Since for any $p \in [1, \infty]$, $\mathcal{R}_L(\Sigma^e)$ is closed, convex and $0 \notin \mathcal{R}_L(\Sigma^e)$, then $(\mathcal{R}_L(\Sigma^e))^o = \mathcal{R}_L(\Sigma^e) = (\mathcal{M}_L(\Sigma^e))^o$. After some rearrangements, We can conclude that $(\mathcal{M}_L(\Sigma^e))^o = \mathcal{R}_L(\Sigma^e), \ p \in [1, \infty]$. Even though $\mathcal{R}_L(\Sigma^e)$ is not closed, it is straightforward to verify that $(\mathcal{M}_L(\Sigma^e))^o = \text{cl}(\mathcal{R}_L(\Sigma^e))$.

**Remark 3** In the case of linear discrete-time system, the duality relations are summarized as follows:

$$(\mathcal{M}_L(\Sigma^e))^o = \mathcal{R}_L(\Sigma^e), \ (\mathcal{R}_L(\Sigma^e))^o = \mathcal{M}_L(\Sigma^e),$$
$$(\mathcal{M}_L(\Sigma^e))^o = \mathcal{R}_L(\Sigma^d), \ (\mathcal{R}_L(\Sigma^e))^o = \mathcal{M}_L(\Sigma^d).$$

The above results can be verified in the same fashion as described in the proof of Theorem 1. Additionally, it is unnecessary to manipulate the case of the $L^1$ norm distinctly because reachable sets $\mathcal{R}_L(\Sigma^e)$ and $\mathcal{R}_L(\Sigma^d)$ are closed inherently.

The inner and outer approximation among the maximal output admissible sets and reachable set will be explicitly exhibited in Theorem 2. In addition, the inclusions monotonically hold in cases of discrete-time system, discretized by different sampling periods. The improved inner and outer approximations of the sets are consistently obtainable while the smaller sampling period is applied.

**Theorem 2** Let $p \in [1, \infty], \ 0 < \tau_2 \leq \tau_1 < \theta, \ h > 0$ and $N \in \mathbb{Z}^+$. Then the inclusions

$$\mathcal{M}_L(\Sigma^e) \subset \mathcal{M}_L(\Sigma^e), \in \mathcal{M}_L(\Sigma^e),$$
$$\mathcal{M}_L(\Sigma^e) \subset \mathcal{M}_L(\Sigma^d), \in \mathcal{M}_L(\Sigma^d),$$
$$\mathcal{R}_L(\Sigma^e) \subset \mathcal{R}_L(\Sigma^e) \subset \mathcal{R}_L(\Sigma^e),$$
$$\mathcal{R}_L(\Sigma^e) \subset \mathcal{R}_L(\Sigma^d) \subset \mathcal{R}_L(\Sigma^d).$$
hold. $\mathcal{M}_P(\Sigma_{\tau_1}^c)$ is an output admissible set for $\Sigma_{\tau_2}$ and $\Sigma^c$. Moreover,
\[
\text{cl}\left(\bigcup_{\tau_0>0} \mathcal{M}_P(\Sigma_{\tau_0}^c)\right) = \bigcap_{h>0} \mathcal{M}_P(\Sigma_{h}^d) = \mathcal{M}_{L,P}(\Sigma^c).
\]
are satisfied. On the other hand, the reachable set $\mathcal{R}_P(\Sigma_{\tau_1}^c)$ is an invariant set for systems $\Sigma_{\tau_2}$ and $\Sigma^c$. Furthermore,
\[
\text{cl}\left(\bigcup_{h>0} \mathcal{R}_P(\Sigma_{h}^d)\right) = \bigcap_{\tau_0>0} \mathcal{R}_P(\Sigma_{\tau_0}^c) = \mathcal{R}_{L,P}(\Sigma^c), \quad p \in (1, \infty],
\]
are satisfied.

Proof: As previously assumed in this paper, let sampling periods satisfy $0 < \tau_2 \leq \tau_1 < \theta$. For convenience, let us define the ratio of $\tau_2$ and $\tau_1$ as $r := \tau_2/\tau_1 \in (0, 1]$. Since $I + \tau_2 A$ can be expressed as
\[
I + \tau_2 A = (1-r)I + r(I + \tau_1 A). \quad (16)
\]
When $r = 1$, it is obvious that $\mathcal{M}_P(\Sigma_{\tau_1}^c) \subset \mathcal{M}_P(\Sigma_{\tau_2}^c)$ holds. It was shown in [10] that for fixed $p \in [1, \infty)$ and for any $x \in \mathbb{R}^n$ and $r \in (0, 1)$ the next inequality holds.
\[
\sum_{k=0}^{\infty} \left| C(I + \tau_2 A)^k x \right|_p^{\tau_2} \leq \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left| C(I + \tau_1 A)^j x \right|_p^{r \tau_1} = \sum_{k=0}^{\infty} \left| C(I + \tau_1 A)^k x \right|_p^{\tau_1}.
\]
Thus, we can conclude that $\tau_2^{1/p}\|\mathcal{G}_{\tau_2}^p x\|_p \leq \tau_1^{1/p} \|\mathcal{G}_{\tau_1}^p x\|_p$ holds. In the case of $p = \infty$, it is also easy to show that $\|\mathcal{G}_{\tau_2}^\infty x\|_{\infty} \leq \|\mathcal{G}_{\tau_1}^\infty x\|_{\infty}$. Because we can take sampling period $\tau > \tau_0$ arbitrarily small, for $p \in [1, \infty]$, the relations $\|\mathcal{G}_p x\|_{\infty} \leq \tau_1^{1/p} \|\mathcal{G}_p x\|_p$ are inherited. Next, the outer approximation of $\mathcal{M}_{L,P}(\Sigma^c)$ will be considered. For fixed $p \in [1, \infty]$ and any $N \in I^+$, the next inequality is satisfied.
\[
(Nh) \sum_{k=0}^{\infty} \left| C(Nh)^{-1} \left( \sum_{j=0}^{N} e^{J_{A\theta}h} \sum_{j=0}^{\infty} e^{J_{A\theta}h} \right) e^{kA\theta} h x \right|_p^p \leq h \sum_{k=0}^{\infty} \left| Ch^{-1} \left( \sum_{j=0}^{\infty} e^{J_{A\theta}h} \right) e^{kA\theta} h x \right|_p^p.
\]
Hence, $h^{1/p}\|\mathcal{G}_{h}^\infty x\|_p \leq (h/\theta)^{1/p} \|\mathcal{G}_{\theta h}^\infty x\|_p$ is verified. It is possible to show that $\|\mathcal{G}_{h}^\infty x\|_{\infty} \leq \|\mathcal{G}_{\theta h}^\infty x\|_{\infty}$ is satisfied. As a result, for any $p \in [1, \infty]$ the inclusions (14) hold. Due to (16), if a state $\xi \in \mathcal{M}_P(\Sigma_{\tau_2}^c)$, then for any sampling period $\tau_2$ such that $0 < \tau_2 \leq \tau_1 < \theta$, $(I + \tau_2 A) \xi \in \mathcal{M}_P(\Sigma_{\tau_2}^c)$ holds. This implies $\mathcal{M}_P(\Sigma_{\tau_2}^c)$ is the positively invariant set for systems $\Sigma_{\tau_2}$ and $\Sigma^c$. Now we will show that if the sampling period decreases, the approximated set converges to the maximal output admissible set of continuous-time system, i.e.
\[
\text{cl}\left(\bigcup_{\tau_0>0} \mathcal{M}_P(\Sigma_{\tau_0}^c)\right) = \bigcap_{h>0} \mathcal{M}_P(\Sigma_{h}^d) = \mathcal{M}_{L,P}(\Sigma^c).
\]
of Hankel operator is defined as
\[ \|H\|_p := \sup_{w \in L^p} \frac{\|Hw\|_{L^p}}{\|w\|_{L^p}}, \quad (18) \]

Next, consider the linear discrete-time-invariant systems \( \Sigma_1, \Sigma_2 \) Hankel operator with relation to the above systems \( H^L_z : \ell^p \to \ell^p, H^h_z : \ell^p \to \ell^p \) are defined as follows:
\[ H^L_z w(k) := C((I + \tau A)^k) \sum_{j=-\infty}^{0} (I + \tau A)^{-j}(\tau B)w(j), \]
\[ H^h_z w(k) := Ch^{-1}E(h)e^{kAh} \sum_{j=-\infty}^{0} e^{-jAh}E(h)Bw(j), \]

where \( \ell^p \) and \( \ell^p \) are shorthand notations for \( \ell^p(\mathbb{N}^+ \cup \{0\}) \) and \( \ell^p(\mathbb{N}^+ \cup \{0\}) \), respectively. Similarly, their \( \ell^p \)-induced norms are defined as follows:
\[ \|H^L_z\|_p := \sup_{w \in \ell^p} \frac{\|H^L_z w\|_{\ell^p}}{\|w\|_{\ell^p}}, \quad \|H^h_z\|_p := \sup_{w \in \ell^p} \frac{\|H^h_z w\|_{\ell^p}}{\|w\|_{\ell^p}}. \]

**Theorem 3** The following equations hold.
\[ \|H\|_p = \inf\{\alpha : \mathcal{R}_L(\Sigma^d) \subset \alpha \mathcal{M}_L(\Sigma^d)\}, \quad (19) \]
\[ \|H^h\|_p = \inf\{\alpha : \mathcal{R}_h(\Sigma^d) \subset \alpha \mathcal{M}_h(\Sigma^d)\}, \quad (20) \]
\[ \|H^L\|_p = \inf\{\alpha : \mathcal{R}_L(\Sigma^d) \subset \alpha \mathcal{M}_L(\Sigma^d)\}. \quad (21) \]

**Proof:** By the definition, it is obvious that \( \|H\|_p \leq \alpha \) for all \( \alpha > 0 \) such that \( \mathcal{R}_L(\Sigma^d) \subset \alpha \mathcal{M}_L(\Sigma^d) \). In the converse direction, for all \( \epsilon > 0 \), there exists \( w \in L^p \) such that \( \|Hw\|_{L^p}/\|w\|_{L^p} \geq \inf\{\alpha : \mathcal{R}_L(\Sigma^d) \subset \alpha \mathcal{M}_L(\Sigma^d)\} - \epsilon \), since \( \mathcal{R}_L(\Sigma^d) \) is a reachable set. The linear discrete-time cases can be proved in the same manner. \( \blacksquare \)

**Remark 5** The following equations can be derived from Theorem 1 and Theorem 3.
\[ \|H\|_p = \inf\{\alpha : \mathcal{M}_L(\Sigma^d) \supset \mathcal{R}_L(\Sigma^d)\}; \quad (22) \]
\[ \|H^h\|_p = \sup\{\alpha : \mathcal{M}_h(\Sigma^d) \supset \mathcal{R}_h(\Sigma^d)\}; \quad (23) \]
\[ \|H^L\|_p = \sup\{\alpha : \mathcal{M}_L(\Sigma^d) \supset \mathcal{R}_L(\Sigma^d)\}. \quad (24) \]

**V. Numerical Examples**
Consider the following data to illustrate the inclusions between the maximal output admissible sets. The computational results are given in Fig.1, Fig.2 and Fig.3.
\[ A = \begin{bmatrix} -1.3602 & 0.1253 \\ -1.6656 & -0.6390 \end{bmatrix}, \quad C = [1 \ 3]. \quad (22) \]

In Fig.1, Hankel singular value was utilized to construct the polyhedral set that achieves the \( \ell^1 \) norm constraint [12]. In this set, when we construct a positively invariant set for system \( \Sigma^e \) by a standard algorithm proposed in [3], the inner approximation of sets \( \mathcal{M}_L(\Sigma^d), \mathcal{M}_h(\Sigma^d) \) which is an output admissible set for systems \( \Sigma^e \) and \( \Sigma^c \) can be constructed. Even though the outer approximation is quite difficult to construct directly, according to Theorem 2 and Remark 3, the outer approximation can be obtained by calculating the inner approximation of \( \mathcal{R}_L(\Sigma^d) \) using the algorithm proposed in [9]. Then we can compute the polar of \( \mathcal{R}_L(\Sigma^d) \) to derive the outer approximation of \( \mathcal{M}_L(\Sigma^d) \).

It is well-known that we can employ the observability gramian of the corresponding systems to construct \( \mathcal{M}_L(\Sigma^d), \mathcal{M}_h(\Sigma^d) \) and \( \mathcal{M}_L(\Sigma^d) \) shown in Fig.2.

Fig.3 is obtained by applying the algorithm introduced in [3]. The detail that should be emphasized is that there are two methods to construct the outer approximation from Theorem 2 and Remark 4. Experimentally, \( \mathcal{M}_L(\Sigma^d) \) converges to \( \mathcal{M}_L(\Sigma^d) \) more rapidly than \( \mathcal{M}_L(\Sigma^d) \).

The following parameters are given for verifying the inclusions between the reachable sets presented in Section 3. The results are shown in Fig.4, Fig.5, and Fig.6.
\[ A = \begin{bmatrix} -0.3721 & -0.0479 \\ 0.3878 & -0.5598 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3710 \\ 0.2749 \end{bmatrix}. \quad (23) \]

In general, it is quite hard to construct the reachable set with relation to \( (L^1, \ell^1) \) norm directly. According to Theorem 1, Theorem 2, Remark 3 and Remark 4, we can still derive the reachable set from computing the polar sets of the sets \( \mathcal{M}_L(\Sigma^d) := \left\{ x \in \mathbb{R}^n : \|B^TE\mathcal{A}(\mathcal{X})x\|_{\ell^1} \leq 1 \right\}; \mathcal{M}_L(\Sigma^d), \mathcal{M}_L(\Sigma^d) \) and \( \mathcal{M}_L(\Sigma^d) \). Hence, the inner approximations \( \mathcal{R}_L(\Sigma^d), \mathcal{M}_L(\Sigma^d) \) of \( \mathcal{R}_L(\Sigma^d) \) and the outer approximation \( \mathcal{R}_L(\Sigma^d) \) of \( \mathcal{R}_L(\Sigma^d) \) are computable as in Fig.4.

The controllability gramian was employed to exhibit \( \mathcal{R}_L(\Sigma^d), \mathcal{R}_L(\Sigma^d) \) and \( \mathcal{R}_L(\Sigma^d) \) in Fig.5.

By applying the algorithm proposed in [7], we can obtain the inner approximation of \( \mathcal{R}_L(\Sigma^d) \) which is also the inner approximation of \( \mathcal{R}_L(\Sigma^d) \), and the outer approximation of \( \mathcal{R}_L(\Sigma^d) \) which is the outer approximation of \( \mathcal{R}_L(\Sigma^d) \) as shown in Fig.6. It is worth noting that the outer approximation \( \mathcal{R}_L(\Sigma^d) \) which is an invariant set for linear continuous-time system \( \Sigma^c \) can be also effectively constructed by calculating the polar set of \( \mathcal{M}_L(\Sigma^d) \).

The legends of the figures are given in the following format. The first element denotes the system in consideration. The second parameter determines the sampling period used in discrete-time systems. The third argument, if it exists, prescribes whether the inner (i) or the outer approximation (o) of the sets characterized by the first two parameters is represented.
VI. CONCLUSIONS

This paper has considered the maximal admissible set and reachable set for linear continuous-time system. We show that the inner and outer approximations are obtained by employing the forward Euler approximated discrete-time system and the modified zero order hold system. The inclusions between the positively invariant sets of the continuous-time system and the positively invariant sets of the discrete-time systems hold. The approximated sets approach to the positively invariant set of continuous-time system monotonically within arbitrary accuracy. The duality relations between the maximal output admissible set and reachable set for each system are verified. This duality results provide us an alternative computational method for the construction of the positively invariant sets. Furthermore, the duality relations are necessary in deriving other significant results. Finally, it was prescribed that the $L^p$-induced norm of Hankel operator is the smallest scaling parameter such that the reachable set is contained in the scaled maximal output admissible set.

REFERENCES