Stability and $L_2$—Gain Analysis of Systems with Time-Varying Delays: Input-Output Approach

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Abstract—Stability and $L_2$ ($l_2$)-gain analysis of linear (continuous-time and discrete-time) systems with uncertain bounded time-varying delays possessing non-zero nominal values and norm-bounded uncertainties. The delay derivatives (in the continuous-time) are not assumed to be less than $q < 1$. An input-output approach is applied by introducing a new input-output model, which leads to effective frequency domain and time domain criteria. The new method essentially improves the existing results for delays with derivative not greater than 1, which were treated in the past as fast-varying delays (without any constraints on the delay derivative). New BRLs are derived for systems with state and objective vector delays. Numerical examples illustrate the efficiency of the new method.

Keywords: time-varying delay, small-gain theorem, BRL, Lyapunov-Krasovskii functional, uncertainties.

I. INTRODUCTION

The stability and control of systems with uncertain time-delay is a subject of recurring interest. Most of the works are based on application of different types of Lyapunov Krasovskii Functionals (LKFs) (see e.g. [1], [2],[6], [9]-[11]). The continuous-time systems with fast varying delays were for the first time treated via descriptor type LKF [2], where the derivative of the LKF along the trajectories of the system depends on the state and the delay derivative.

Robust stability has been analyzed also via IO approach, which reduces the stability analysis of the uncertain system to the stability analysis of the class of systems with the same nominal part but with additional inputs and outputs. This approach was introduced for constant delays in [7]. The stability conditions for constant delays by LKFs were recovered by this approach in [12]. The method of [12] has been generalized to the case of slowly-varying delays (i.e. to delays with the derivative less than $q < 1$) in [6]. All the above works on IO approach consider the continuous-time case.

Frequency domain stability criteria for continuous-time and discrete-time systems with fast-varying delays have been derived in [8] in terms of transfer functions. In [4] a frequency domain stability criterion for continuous-time systems with fast-varying delays has been found via direct application of the Laplace transform.

The present paper is inspired by [6], where the IO approach was developed for the case where the delay derivative is smaller than $q < 1$. We develop here the IO approach to the continuous- and discrete-time systems with norm-bounded uncertainties and with fast varying delays from known segments. We introduce a new IO model with an output, which explicitly depends on $\dot{x}(t)$ ($x(k+1)-x(k)$). This corresponds to the term with $\dot{x}(t)$ in the derivative of descriptor LKF [2]. For the first time, we apply IO approach to $L_2(l_2)$-gain analysis. As a result, new BRLs with the delayed objective vector are obtained both, in the frequency and in the time domain. The time domain results are based on the application of the descriptor type LKF combined with the free weighting matrices technique of [11].

II. STABILITY AND BRL IN THE FREQUENCY DOMAIN

A. Robust stability: continuous-time systems

We consider the following linear system with uncertain coefficients and uncertain time-varying delays $\tau_i(t)$ (i=1,2):

$$\dot{x}(t) = (A_0+H\Delta E_0)x(t) + \sum_{i=1}^{2}(A_i+H\Delta E_i)x(t-\tau_i(t)), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $A_i$, $E_i$, $i = 0,1,2$ and $H$ are constant matrices of appropriate dimensions and
\( \Delta(t) \) is a time-varying uncertain \( n \times n \) matrix that satisfies
\[
\Delta^T(t) \Delta(t) \leq I_n. \tag{2}
\]

The uncertain delays \( \tau_i(t) \) are piecewise-continuous functions of the form
\[
\tau_i(t) = h_i + \eta_i(t), \quad i = 1, 2, \quad |\eta_i(t)| \leq \mu_i \leq h_i \tag{3}
\]
with the known upper bounds \( \mu_1 \) and \( \mu_2 \).

To obtain a less conservative result in the case of sign-varying \( \eta_i(t) \), we assume additionally that \( t - \tau_i(t) \) is a non-decreasing function. The latter assumption means that \( \tau_1(t) \) is differentiable almost for all \( t \geq 0 \) and \( \dot{\tau}_1(t) = \dot{\eta}_1(t) \leq 1 \) almost for all \( t \geq 0 \). Note that this derivative constraint is less strong than \( \dot{\tau}_1 \leq q < 1 \) of [6]. Moreover, \( \tau_1(t) \) may include such delays that \( t - \tau_1(t) \) is piecewise constant. This kind of delay appears [5] in sampled-data control \( u(t) = K x(t_k), \quad t \in [t_k, t_{k+1}) \) with the sampling times \( 0 = t_0 \leq t_1 \leq \ldots \) satisfying \( t_{k+1} - t_k \leq 2 \mu_1 \), where \( x(t_k) \) can be represented as \( x(t_k) = x(t - \mu_1 - \eta_1(t)) \) with \( \eta_1(t) = t - \mu_1 - t_k, \quad t \in [t_k, t_{k+1}) \).

We assume
\textbf{A1} The nominal system
\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + A_2 x(t - h_2), \tag{4}
\]
is asymptotically stable.

The results are easily generalized to the case of any finite number of the delays.

We represent (1) in the form:
\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{2} A_i x(t - h_i) - \sum_{i=1}^{2} A_i \int_{-h_i}^{0} E_i x(t - h_i - \eta_i) \, dt + H u_3(t),
\]
where
\[
\dot{y}_1(t) = \sqrt{\mu_1} \dot{x}(t), \quad \dot{y}_2(t) = \sqrt{2 \mu_2} \dot{x}(t),
\]
\[
y_3(t) = E_0 x(t) + \sum_{i=1}^{2} E_i x(t - h_i) + \sum_{i=1}^{2} \sqrt{\mu_i} E_i u_i(t).
\]

Following the idea of [7], [12], [6] to embed the perturbed system (5) into a class of systems with additional inputs and outputs, the stability of which guarantees the stability of (5), we introduce the following forward system:
\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{2} A_i x(t - h_i) + \sum_{i=1}^{2} \sqrt{\mu_i} A_i u_i(t) + H u_3(t),
\]
\[
y_1(t) = \sqrt{\mu_1} \dot{x}(t), \quad y_2(t) = \sqrt{2 \mu_2} \dot{x}(t),
\]
\[
y_3(t) = E_0 x(t) + \sum_{i=1}^{2} E_i x(t - h_i) + \sum_{i=1}^{2} \sqrt{\mu_i} E_i u_i(t),
\]
with feedback
\[
u_1(t) = -\frac{1}{\sqrt{\mu_1}} \int_{-h_1}^{0} y_1(t + s) \, ds,
\]
\[
u_2(t) = -\frac{1}{\sqrt{2 \mu_2}} \int_{-h_2}^{0} y_2(t + s) \, ds,\quad u_3(t) = \Delta y_3(t). \tag{7}
\]

Note that \( y_1(t) \) and \( y_2(t) \) differ from the output of [7], [12], [6], and correspond to the term with \( \dot{x}(t) \) in \( \dot{V} \) by descriptor approach [2].

Let \( u^F = [u_1^F \ u_2^F \ u_3^F], \ y^F = [y_1^F \ y_2^F \ y_3^F] \). Then the forward system (6) can be written as \( y = G u \) with transfer function matrix
\[
G(s) = \begin{bmatrix} \sqrt{\mu_1} s I & \sqrt{2 \mu_2} s I \\ \sqrt{\mu_1} E_1 & 0 & E_0 + \sum_{i=1}^{2} E_i e^{-h_i s} \end{bmatrix} (s I - A_0 - \sum_{i=1}^{2} A_i e^{-h_i s})^{-1}
\begin{bmatrix} \sqrt{\mu_1} A_1 & \sqrt{\mu_2} A_2 H \\ \sqrt{\mu_1} E_1 & 0 \end{bmatrix}.
\]

Assume that \( y_i(t) = 0, \quad \forall t \leq 0, \quad i = 1, 2, 3 \) the following result is proved.

\textbf{Lemma 2.1:} The following holds:
\[
\|u_i\|_{L_2} \leq \|y_i\|_{L_2}, \quad i = 1, 2, 3. \tag{9}
\]

\textbf{Proof.} For \( i = 1 \) we have by Cauchy-Schwartz inequality for all \( t \geq 0 \)
\[
\mu_1^2 \|u_1(t)\|^2 = \|f_{t-h_1}^t y_1(t + s) \, ds\|^2 \leq \eta_1(t) f_{t-h_1}^t \|y_1(s)\|^2 \, ds. \tag{10}
\]

Denote \( s = p(t) = t - h_1 - \eta_1(t) \). Since \( p(t) \) is a non-decreasing function, the set of segments \( t \in [t_1, t_2) \), where \( p(t) \) is constant, is countable, while out of these segments \( p(t) \) is increasing. Hence, for almost all \( s \) the inverse \( t = p^{-1}(s) = q(s) \) is well-defined and satisfies \( |q(s) - s - h_1| \leq \mu_1 \).

Then, integrating (10) in \( t \) from 0 to \( \infty \), changing the order of integration and taking into account that \( y_1(t) = 0 \), \( t \leq 0 \), we find that
\[
\mu_1^2 \|u_1\|_{L_2}^2 \leq \int_0^\infty \eta_1(t) f_{t-h_1}^t \|y_1(s)\|^2 \, ds \, dt = \int_0^\infty \int_{t-h_1}^{t+\eta_1(t)} \eta_1(q(s)) \|y_1(s)\|^2 \, ds \, dt \leq \mu_1^2 \|y_1\|_{L_2}^2.
\]

For \( i = 2 \) the proof is similar. For \( i = 3 \) (9) is immediate.

From Lemma 2.1 it follows by the small gain theorem (see e.g. [6]) that the system (1) is IO stable (and thus asymptotically stable, since the nominal system is time-invariant) if
\[
\|G\|_{\infty} < 1. \tag{11}
\]

\textbf{Theorem 2.1:} Consider (1) with delays given by (3), where \( \eta_i(t), \quad i = 0, 1 \) are piece-wise-continuous functions and \( \dot{\eta}_i(t) \leq 1 \) for almost all \( t \geq 0 \). Under A1 the system is asymptotically stable if (11) holds, where \( G \) is given by (8).

\textbf{Remark 2.1:} A stronger result is obtained by scaling \( G \):
\[
G_N(s) = \text{diag}(X_1, X_2, \rho I_n) G(s) \text{diag}(X_1^{-1}, X_2^{-1}, \rho^{-1} I_n),
\]

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where $X_i, i = 1, 2$ are non-singular $n \times n$ matrices and $\rho \neq 0$ is a scalar. $G$ is not scaled by $\sqrt{\rho_i}$ so as substituting of $\mu_i = 0$ leads to $G_X$, which corresponds to the case of known constant delays $\tau_i \equiv h_i$ and norm-bounded uncertainties.

Hence, under A1 (1) is asymptotically stable for all delays satisfying (3) if there exist $X_1$ and $\rho$ such that $\|G_X\|_{\infty} < 1$.

**B. BRL: continuous-time systems**

We consider the following linear system with uncertain coefficients and uncertain time-varying delays $\tau_i(t) (i=1,2)$ as above:

$$\dot{x}(t) = (A_0 + H \Delta E_0)x(t) + \sum_{i=1}^2 (A_i + H \Delta E_i)x(t - \tau_i(t)) + (B_1 + H \Delta E_3)w(t),$$

$$z(t) = C_0x(t) + \sum_{i=1}^2 C_i x(t - \tau_i(t)), \quad x(s) = 0, \ s \leq 0,$$

(12)

where $w(t) \in \mathbb{R}^q$ is an arbitrary disturbance vector in $L_2[0, \infty)$ and $z(t) \in \mathbb{R}^p$ is the objective vector. Given $\gamma > 0$, we search a condition which guarantees that $L_2$-gain of (5) is less than $\gamma$, i.e. that the following inequality holds:

$$\|z\|_{L_2}^2 < \gamma^2 \|w\|_{L_2}^2, \quad \forall 0 \neq w \in L_2.$$  

(13)

Consider an auxiliary system

$$\dot{x}(t) = (A_0 + H \Delta E_0)x(t) + \sum_{i=1}^2 (A_i + H \Delta E_i)x(t - \tau_i(t)) + \gamma^{-1}(B_1 + H \Delta E_3)\bar{w}(t),$$

$$z(t) = C_0x(t) + \sum_{i=1}^2 C_i x(t - \tau_i(t)), \quad x(t) = 0, \ t \leq 0,$$

(14)

It is clear that

$$\|z\|_{L_2}^2 < \|\bar{w}\|_{L_2}^2, \quad \forall 0 \neq \bar{w} \in L_2$$

(15)

for (14) is equivalent to (13) for (5).

To derive 'scaled conditions' consider the following forward system

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^2 A_i x(t - h_i) + \sum_{i=1}^2 \sqrt{\rho_i} A_i X_i^{-1} u_i(t) + \rho^{-1} H u_3(t) + \gamma^{-1} B_1 \bar{w}(t),$$

$$y_1(t) = \sqrt{\rho_1} X_1 \bar{x}(t), \quad y_2(t) = \sqrt{2 \rho_2} X_2 \bar{x}(t),$$

$$y_3(t) = \rho E_0 x(t) + \sum_{i=1}^2 E_i x(t - h_i),$$

$$\sum_{i=1}^2 \sqrt{\rho_i} E_i X_i^{-1} u_i(t) + \gamma^{-1} E_3 \bar{w}(t),$$

$$z(t) = C_0x(t) + \sum_{i=1}^2 C_i x(t - \tau_i(t)) + \sum_{i=1}^2 \sqrt{\rho_i} C_i X_i^{-1} u_i(t),$$

(16a-d)

with the feedback of (7).

The forward system (16) can be written as

$$\begin{bmatrix} y \\ z \end{bmatrix} = G_r \begin{bmatrix} u \\ \bar{w} \end{bmatrix}, \quad u^T = [u_1^T \ u_2^T \ u_3^T], \quad y^T = [y_1^T \ y_2^T \ y_3^T]$$

(17)

with transfer matrix

$$G_r(s) = \begin{bmatrix} \sqrt{\rho_1} s X_1 \\ \sqrt{2 \rho_2} s X_2 \\ \rho E_0 + \sum_{i=1}^2 \rho E_i e^{-h_i s} C_0 + \sum_{i=1}^2 \sqrt{\rho_i} C_i e^{-h_i s} \end{bmatrix} (sI - A_0)$$

$$- \sum_{i=1}^2 A_i (e^{-h_i s})^{-1} \begin{bmatrix} \sqrt{\rho_1} A_1 X_1^{-1} \\ \sqrt{2 \rho_2} A_2 X_2^{-1} \\ \rho \sqrt{\rho_1} E_1 X_1^{-1} \end{bmatrix} \begin{bmatrix} \frac{H \ E_0}{\rho} \\ \frac{H \ E_3}{\rho} \\ 0 \ \bar{E}_3 \end{bmatrix}.$$  

(18)

**Theorem 2.2:** Assume A1. Given $\gamma > 0$, (12) is internally stable and has $L_2$-gain less than $\gamma$ for all delays satisfying (3), if there exist non-singular $n \times n$-matrices $X_1, X_2$ and a scalar $\rho \neq 0$ such that

$$\|G_r\|_{\infty} < 1.$$  

(19)

**Proof:** Eq. (17) and (19) imply that

$$\|y\|_{L_2} + \|z\|_{L_2} < \|u\|_{L_2} + \|\bar{w}\|_{L_2}.$$  

The latter inequality together with $\|u\|_{L_2} \leq \|y\|_{L_2}$ yield (15) and (13).

**C. Extension to the discrete-time delay systems**

We consider the following linear discrete system with uncertain coefficients and uncertain time-varying delays $\tau_i(k) (i=1,2)$:

$$x(k + 1) = (A_0 + H \Delta E_0)x(k) + \sum_{i=1}^2 (A_i + H \Delta E_i)x(k - \tau_i(k)) + (B_1 + H \Delta E_3)w(k),$$

$$z(k) = C_0x(k) + \sum_{i=1}^2 C_i x(k - \tau_i(k)), \quad x(0) = 0, \ l \leq 0,$$

(20)

where $x(k) \in \mathbb{R}^n$ is the system state, $A_1, A_2, H, E_i$ and $B_1$ are constant matrices of appropriate dimensions and $\Delta(k)$ is a time-varying uncertain matrix that satisfies

$$\Delta^T(k) \Delta(k) \leq I.$$  

(21)

The uncertain delays $\tau_i(k)$ are supposed to have the following form:

$$\tau_i(k) = h_i + \eta_i(k), \ i = 1, 2,$$

$$-h_i \leq -\mu_- \leq \eta_i(k) \leq \mu_+ + h_i, \quad |\mu_- - \mu_+| \leq 1$$

(22)

with the known bounds $\mu_+ \geq 0$ and $\mu_- \geq 0$. Note that similarly to the continuous-time case we choose $h_i$ in the 'middle' of the delay interval. Denote $\mu_i = \max\{|\mu_{i-} - \mu_{i+}|, i = 1, 2.$

We assume additionally that $k - \tau_1(k)$ is increasing function, i.e. that $\tau_1$ satisfies the following constraint: $\tau_1(k+$
1) – τ₁(κ) ≤ 0. Note that the constraint on τ₁ is more restrictive, than in the continuous-time case, where t – τ₁ is supposed to be non-decreasing.

We assume

**A1d** The nominal system

\[
x(k + 1) = A_0 x(k) + A_1 x(k – h_1) + A_2 x(k – h_2),
\]

is asymptotically stable.

Given γ > 0, we are looking for a condition which guarantees that (20) is internally stable (i.e. asymptotically stable for w = 0) and has l₂-gain less than γ, i.e. that the following inequality holds:

\[
\|z\|_{l_2}^2 < \gamma^2 \|w\|_{l_2}^2, \quad \forall 0 \neq w \in l_2.
\]  

We represent (20) in the form:

\[
x(k + 1) = A_0 x(k) + \sum_{i=1}^{2} A_i x(k – h_i) - \sum_{i=1}^{2} A_i \sum_{j=k-h_i-n}^{k-h_i-2} \Delta x(j) + H \Delta E_0 x(k) + \sum_{i=1}^{2} E_i x(k – h_i) \|w(k)\|_\infty, \]

\[
y_1(k) = \sqrt{\mu_1} x_1(k + 1 – x(k)),
\]

\[
y_2(k) = \sqrt{\mu_2} E_0 x(k) + \sum_{i=1}^{2} E_i x(k – h_i) + \sum_{i=1}^{2} \frac{\gamma}{\sqrt{\mu_3}} E_3 w(k),
\]

\[
z(k) = C_0 x(k) + \sum_{i=1}^{2} C_i x(k – h_i) - \sum_{i=1}^{2} \sqrt{\mu_1} C_i x_i^{-1} u_i(k) + \gamma \sum_{i=1}^{2} E_3 w(k),
\]

with the feedback matrix

\[
G_d(z) = \begin{bmatrix}
\sqrt{\mu_1} (z – 1) X_1 \\
\sqrt{\mu_2} + \rho E_0 + \sum_{i=1}^{2} \rho E_i z^{-h_i} C_0 + \sum_{i=1}^{2} \rho E_i z^{-h_i} \\
\sum_{i=1}^{2} \rho E_i z^{-h_i}^{-1} [\sqrt{\mu_1} A_1 X_1^{-1}\sqrt{\mu_2} A_2 X_2^{-1} – \frac{H}{\rho} \frac{\rho E_3}{\gamma}] + \rho \sqrt{\mu_1} E_1 X_1^{-1} + \rho \sqrt{\mu_2} E_2 X_2^{-1} 0 \frac{\rho E_3}{\gamma} \sqrt{\mu_2} C_1 X_1^{-1} 0 0 \end{bmatrix}
\]

Theorem 2.3: Assume A1d. Given γ > 0, (20) is internally stable and has l₂-gain less than γ for all delays satisfying (22) and τ₁(k + 1) – τ₁(k) ≤ 0, if there exist non-singular n × n-matrices X_i, i = 1, 2 and a scalar ρ ≠ 0 such that

\[
\|G_d\|_\infty < 1.
\]

Proof is similar to the continuous-time case.

III. STABILITY AND BRL IN THE TIME DOMAIN

In the continuous-time case, let \( V_n \) be Lyapunov-Krasovskii Functional (LKF), which guarantees the stability of the nominal system (4). The following condition along (16)

\[
\mathcal{W} \triangleq V_n(t) + ||y(t)||^2 + ||z(t)||^2 – ||u(t)||^2 – ||\bar{w}(t)||^2 < -\varepsilon (||x(t)||^2 + ||u(t)||^2 + ||\bar{w}(t)||^2), \quad \varepsilon > 0
\]

guarantees that the \( H_\infty \)-norm of (16) is less than 1. Therefore, (31) is a sufficient condition for the feasibility of the frequency domain condition (19) of Theorem 2.2.

In the discrete-time case the corresponding condition along (26) has a form

\[
\mathcal{W}_d \triangleq V_n(k + 1) – V_n(k) + ||y(k)||^2 + ||z(k)||^2 – ||u(k)||^2 – ||\bar{w}(k)||^2 < -\varepsilon (||x(k)||^2 + ||u(k)||^2 + ||\bar{w}(k)||^2).
\]

Note that in this section we will derive conditions, which will be sufficient for the feasibility of the frequency domain conditions of Theorems 2.2 and 2.3. We choose the descriptor type \( V_n [2] \).

A. Discrete-time results

We combine the discrete-time descriptor LKF (see e.g. [1]):

\[
V_n(k) = x^T(k) P_1 x(k) + \sum_{i=1}^{2} \sum_{j=k-h_i}^{k-1} \sum_{m=-h_i}^{k-1} \sum_{j=k+h_i}^{k+m} y(j)^T R_i y(j) + \sum_{i=1}^{2} \sum_{j=k-h_i}^{k-1} x(j)^T S_i x(j), \quad \bar{y}(k) = x(k + 1) – x(k),
\]

\[
P_1 > 0, \quad R > 0, \quad S > 0.
\]
with the free weighting matrices technique of [11].

**Lemma 3.1:** The nominal system (23) is asymptotically stable if there exist $n \times n$ matrices $0 < P_1, P_2, P_3, S_i > 0, Y_1, Y_2, Z_{i1}, Z_{i2}, Z_{i3}, R_i > 0, W_1, W_2, W_3, T_i$ such that the following LMI s are feasible:

$$
\Gamma_n = \begin{bmatrix}
\Psi_n & P^T \begin{bmatrix}
0 & A_1 \\
-1 & T_1 \\
0 & 0
\end{bmatrix} - Y_T^T + \begin{bmatrix} T_1 \\
0
\end{bmatrix} + h_1 W_1 \\
* & -S_1 - T_1 - T_1^T + h_1 W_13 \\
* & * \\
0 & -S_2 - T_2 - T_2^T + h_2 W_23 \\
\end{bmatrix} < 0
\end{bmatrix}
$$

$$
\Xi_i = \begin{bmatrix}
Z_i & W_i & Y_i^T \\
* & W_i & T_i^T \\
* & * & R_i
\end{bmatrix} \geq 0, \quad i = 1, 2.
$$

(34a,b)

where

$$
P = \begin{bmatrix} P_1 & 0 & 0 \\
P_2 & P_3 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad Y_i = [Y_{i1} \ Y_{i2}], \quad W_i = \begin{bmatrix} W_{i1} \\
W_{i2}
\end{bmatrix},
$$

$$
P = \begin{bmatrix}
0 & I \\
A_0 - I & -I
\end{bmatrix} + \begin{bmatrix} A_T^0 - I \\
I & -I
\end{bmatrix} P + \sum_{i=1}^2 h_i Z_i
$$

and

$$
Y_i = \psi \begin{bmatrix} Y_{i1} \\
Y_{i2}
\end{bmatrix} + \sum_{i=1}^2 h_i Z_i
$$

Thus, along the trajectories of (26) we have

$$
V_n(k + 1) - V_n(k) \leq [\xi_T^T(k) x(k - h_2)] \Gamma_n \begin{bmatrix} \xi_T^T(k) \\
x(k - h_2)
\end{bmatrix} + 2 \tilde{\psi}(k) P^T \begin{bmatrix}
0 & 0 \\
0 & \sum_{i=1}^2 \sqrt{h_i} A_i X_i^{-1} u_i(k) + \frac{1}{\rho} u_3(t) + \frac{B_i}{\gamma} \tilde{w}(k)
\end{bmatrix}
$$

and

$$
\mathcal{W}_d \leq \eta^T(k) \Gamma \eta(k) + ||y(k)||^2 + ||z(k)||^2
$$

where

$$
\zeta^T(k) = [\xi_T^T(k) x(k - h_2)^T \ u_3^T(k) \ u_5^T(k) \ u_6^T(k) \ \tilde{w}(k)]
$$

and

$$
\Gamma = \begin{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\end{bmatrix}
$$

By applying Schur complements to the terms $||y(k)||^2 + ||z(k)||^2$ and multiplying the resulting matrix by $\text{diag}\{I_n, \mu_1 X_1, \mu_2 X_2, \rho I, \gamma I, I_2, \rho I, I_p\}$ and its transpose from the right and from the left we conclude that (32) is satisfied if

$$
\begin{bmatrix}
0 & 0 & \mu_1 R_{1a} & (\mu_2 - \mu_2^2) R_{2a} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_1^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_2^2 \\
-\mu_1 R_{1a} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & (\mu_2 - \mu_2^2) R_{2a} & 0 & 0 & 0 & 0 \\
* & * & * & -r I_n & 0 & 0 & 0 \\
* & * & * & * & -I_p \\
\end{bmatrix} < 0,
$$

(36)

We thus obtained the following.

**Theorem 3.1:** Given $\gamma > 0$, (20) is internally stable and has $L_2$-gain less than $\gamma$ for all delays satisfying (22), if there exist $n \times n$ matrices $0 < P_1, P_2, P_3, S_i > 0, Y_1, Y_2, Z_{i1}, Z_{i2}, Z_{i3}, R_i, W_1, W_2, W_3, T_i, R_{ia}, i = 1, 2$ and a scalar $\gamma > 0$ such that LMIs (34b) and (36) are feasible.

**B. Continuous-time results**

By combining the descriptor model transformation and the corresponding LKF [2]

$$
V_n = x^T(t) P_1 x(t) + \sum_{i=1}^2 \int_{-h_i}^0 \int_{-t+\theta}^{t+\theta} \tilde{w}(s) R_i \tilde{w}(s) ds d\theta + \int_{-t}^0 \tilde{w}^T(s) S_i \tilde{w}(s) ds, \quad R_i > 0, S_i > 0,
$$

with the technique of [11] and the arguments as above we obtain:

**Theorem 3.2:** Given $\gamma > 0$, (12) is internally stable and has $L_2$-gain less than $\gamma$ for all delays satisfying (22), if there exist $n \times n$ matrices $0 < P_1, P_2, P_3, S_i > 0, Y_1, Y_2, Z_{i1}, Z_{i2}, Z_{i3}, R_i, W_1, W_2, W_3, T_i, R_{ia}, i = 1, 2$ and a scalar $p > 0$ such that LMIs (34b) and (36) are feasible,
where $\mu_{2-} + \mu_{2+}$ should be changed by $2\mu_2$ and $\Psi_n$ should be substituted by $\Psi_{nc}$ given by

$$
\Psi_{nc} = PT \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^T \\ I & -I \end{bmatrix} P + \sum_{i=1}^{\infty} h_i Z_i + \left[ \sum_{i=1}^{\infty} S_i \right]_0 + \sum_{i=1}^{\infty} R_i + \sum_{i=1}^{\infty} Y_i + \sum_{i=1}^{\infty} Y_i^T.
$$

C. Examples

In order to verify the conditions of Theorems 2.1 and 2.2 for the continuous case, or the one of Theorem 2.3 for discrete-time, a constant nonsingular matrix $D$ of a specific diagonal block structure is sought that satisfies, for say $G$ of Theorem 2.1, the following inequality:

$$
D^{-T}G^T(-j\omega)D G(j\omega) D^{-1} < I, \quad \forall \omega \in [0, \infty).
$$

By Remark 2.1, the matrix $D$ should possess the structure $\text{diag}(X_1, X_2, \rho I)$.

Denoting $Q = DT D$, the latter inequality becomes:

$$
G^T(j\omega) Q G(j\omega) < Q, \quad Q > 0, \quad \forall \omega \in [0, \infty) \quad (37a,b)
$$

which can be solved for $Q$ of the above structure for preselected values of $\omega$ in $[0, \infty)$.

Example 1 [11]: (continuous-time). Consider (1) with

$$
A_0 = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix}, \quad A_2 = E_2 = 0, \quad H = I, \quad E_1 = 0.2I, \quad i = 0, 1,
$$

In this example, for $\tau_1 \leq 0.9$, the following stability interval was obtained in [11]: $\tau_1(t) \in [0, 0.242]$. The LMI s of Theorem 3.2 are feasible for all fast varying delays, where $h_1 = \mu_1 = 0.146$. The LMI s of Theorem 3.2, where $W$ and $T$ with corresponding indices are taken to be zero, are feasible for smaller values: $h_1 = \mu_1 = 0.133$. Hence, the system is stable for all fast varying delays from a larger interval: $\tau_1(t) \in [0, 0.292]$. For delays $\tau_1 \leq 1$ the corresponding interval is $[0, 0.388]$. By applying the frequency domain result of Theorem 2.2, it is found that the system is asymptotically stable for delays from slightly wider intervals: $[0, 0.298]$ in the fast varying case and $[0, 0.4]$ if $\dot{\tau}_1 \leq 1$.

In the case of constant delay $\tau_1 = h_1$ by Theorem 3.2 without free weighting matrices $W$ and $T$ with indices we find $h_1 \leq 0.68$, while the above matrices improve the result till $h_1 \leq 0.84$. In the case of known system matrices ($H = 0$) the free weighting matrices do not improve the result. Thus in the fast varying case we have $h_1 = \mu_1 = 0.34$.

IV. Conclusions

Stability and $L_2$ ($l_2$)-gain analysis of linear retarded type continuous-time and discrete-time systems with uncertain time-varying delays from given segments and norm-bounded uncertainties is studied. IO approach is applied to stability of systems with fast varying delays (without any constraints on the delay derivative) and is extended, for the first time, to BRL. A new IO model is introduced. New BRLs are derived in the case of delayed state and objective vector, which allows to solve delayed state-feedback $H_\infty$ control problem. Both, frequency domain and time domain criteria are derived. Equivalent LMI s may be derived in the fast-varying delay case by direct application of LKF [3], where the same nominal LKF is added by additional terms which compensate the uncertainties. However, when the delay $\tau$ is not too fast ($\dot{\tau}(t) \leq 1$ for almost all $t \geq 0$), the results of [3] are improved.

REFERENCES