Output Feedback Variable Structure Control of Uncertain Linear Systems

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Abstract—The output feedback variable structure control problem is considered for a class of single input single output uncertain linear plants with arbitrary relative degree and with parameter perturbations, unmodeled dynamics and bounded disturbance simultaneously. A novel passivity-based control scheme composed of linear feedback and variable structure control is proposed. If exact differentiations of system output provided, this scheme results in asymptotic stability; if a class of singular perturbation based linear differentiators adopted, it leads to an arbitrarily small regulation/tracking error.

I. INTRODUCTION

Variable structure control (VSC) has become one of the most popular methods to deal with linear or nonlinear uncertain plants for its remarkable robust properties ([1], [2]). Particularly, the research on static or dynamic output feedback variable structure control (OFVSC) has attracted considerable attentions in the recent decade.

As is well known, relative degree is a crucial factor in a VSC system. For a linear plant with relative degree one, static OFVSC is possible ([3]–[5]), and a general result was achieved in [2]. In contrast, OFVSC of higher relative degree plants is a much more challenging problem, and several complex methods have been introduced. Firstly, the concept high order sliding mode (HOSM) may play an important role ([6]–[10]). With the so-called robust exact differentiators ([8], [11]), a number of 2-order sliding mode controllers ([12]) and universal HOSM controllers ([7], [13]) have been proposed to deal with nonlinear plants. Secondly, high gain observer (HGO) is another useful tool for output feedback control. For instance, [14], [15] and [16] discussed how to synthesis a VSC system with linear HGOs to realize an arbitrarily small tracking error. Thirdly, variable structure model reference adaptive control (VS-MRAC) schemes are applicable to linear models with unknown parameters and unmodeled dynamics ([17]–[19]). Finally, in a single-relay control system where the relative degree of the continuous part is greater than one, oscillations will inevitably occur. Fortunately, if the continuous system can be divided into a slowly-varying sub-system and a rapidly-varying one, the closed loop behavior can be subtly studied by singular perturbation analysis ([20], [21]).

In this paper, we study a class of single input single output uncertain linear systems. Higher relative degree, bounded external disturbance at the input channel, parameter perturbations and unmodeled dynamics are all considered. We will adopt VSC strategy combined with dynamic output feedback to realize robust stabilization and disturbance attenuation, where the passivity-based linear feedback is introduced to stabilize (passify) the nominal model set and the variable structure control to deal with other uncertainties and disturbances. The proposed scheme guarantees local asymptotic stability if exact differentiations of system output provided, or guarantees an arbitrarily small regulation/tracking error if the exact differentiators are replaced by a class of singular perturbation based linear differentiators.

We summarize the innovations of this paper as follows. First, the passivity idea is introduced to the synthesis of VSC systems, and the robust passification problem is considered and solved for the first time (Lemma 2). Second, in comparison with some existing control methods (e.g. [7], [13]–[16] and [17]–[19]), the proposed one seems simpler and more suitable for the stabilization of a general class of well-modeled uncertain linear plants (where various kinds of uncertainties are permitted). Finally, this scheme can be viewed as a new analysis framework of a class of relay control systems as well (Lemma 4).

II. PROBLEM STATEMENT

Consider a class of single input single output uncertain linear systems as shown in Fig. 1. The plant is described by

\[ y(t) = G(s)[u(t) + d(t)], \]

where the transfer function is

\[ G(s) = G_p(s)[1 + \Delta(s)] = \frac{n_p(s)}{d_p(s)}[1 + \Delta(s)]. \]

The nominal model \(G_p(s)\), unmodeled dynamics \(\Delta(s)\) and external disturbance \(d(t)\) satisfy the following assumptions.

Assumption A: The numerator \(n_p(s) = n_0 s^l + \cdots + n_1 s + n_0\) (of a known degree \(l\)) and the denominator \(d_p(s) = s^q + d_{q-1}s^{q-1} + \cdots + d_1 s + d_0\) (of a known degree \(q\) are both interval polynomials. System relative degree \(r = q - l \geq 2\).

\[ 1 \]In the representation \(y(t) = G(s)u(t)\), \(y(t)\) and \(u(t)\) are the output and input signals of linear filter \(G(s)\) respectively.

\[ 2 \]Since the variable structure control problem of systems with \(r = 1\) has been well-solved, we will not consider such a simple case. In fact, our conclusions can be shown compatible with the existing results in this case.
Uncertain coefficients $n_i \in [\bar{n}_i, \bar{n}_i]$ ($i = 1, 2, \cdots, l$), $d_i \in [\bar{d}_i, \bar{d}_i]$ ($i = 1, \cdots, q - 1$); especially, $\bar{n}_i > 0$. Besides, every possible $n_p(s)$ should be Hurwitz. The set of $n_p(s)$ and $d_p(s)$ are denoted as $\mathcal{F}_n$ and $\mathcal{F}_d$ respectively, and the set of nominal plant $G_p(s)$ is denoted as $\mathcal{G}$, that is:

$$\mathcal{G} := \left\{ G_p(s) \mid G_p(s) = \frac{n_p(s)}{d_p(s)}, \ n_p \in \mathcal{F}_n, \ d_p \in \mathcal{F}_d \right\}$$

**Assumption B:** The unmodeled dynamics $\Delta(s) \in \mathcal{RH}_\infty$ satisfies $\|\Delta(s)\|_1 \leq \iota < 1$. The set of $\Delta(s)$ is denoted as $\mathcal{D}$.

**Assumption C:** The external disturbance $d(t)$ is bounded as $\|d(t)\| \leq \nu(y(t)) + \kappa_0$, where $\kappa_0$ is a known constant and $\nu(\cdot)$ is a known continuous non-negative function, $\nu(0) = 0$.

**Remark 1:**
- Equation (1) describes how the unknown external disturbance $d(t)$ affects the plant, and (2) describes both the unmodeled dynamics and the interval plant model. It is not difficult to transform the transfer function $G_p(s)$ into a state-space representation $\Sigma(A, b, c)$ of a fixed order $q$, while this is impossible for $G(s)$ because of the presence of unmodeled dynamics $\Delta(s)$.

**Assumption A** is quite similar to the assumptions of [22] or the assumptions of Theorem 13.1 (simultaneous strong stabilization) in [23]. It is required that every $n_p(s) \in \mathcal{F}_n$ is Hurwitz. This condition can be verified by Kharitonov’s Theorem (see e.g. [23]).

Since $\|\Delta(s)\|_1 < 1$, the relative degree of $G(s)$ is the same as the relative degree of $G_p(s)$.

It should be pointed out that all the coefficients of $G_p(s)$ are defined in bounded closed intervals (compact sets).

As a result the set $\mathcal{G}$ is compact; moreover, the range of system norm $\|\cdot\|_\mu : \mathcal{G} \rightarrow \mathbb{R}^+$ is a bounded closed interval in $\mathbb{R}^+$, infimum and supremum exist.

In this note, we will first solve the regulation problem. That is, to find a control law under which the system output $y(t)$ will converge to zero or to a small neighborhood of zero. After that, we will revise the proposed control law so that $y(t)$ can track a class of bounded reference signals $y_r(t)$.

### III. Exact Differentiation Based OFVSC Control

As implied in [2], strictly positive real (SPR) transfer function may play an important role in variable structure control systems.

**Definition 1 (SPR, [24]):** A transfer function $\phi(s)$ for a linear single input single output system, with relative degree $m = 1$, is SPR if and only if

(a) $\phi(s)$ is analytic in $\mathbb{R}^+$;

(b) $\Re\{\phi(j\omega)\} > 0$, $\forall \omega \in (-\infty, \infty)$;

(c) $\lim_{\omega \rightarrow \infty} \omega^2 \Re\{\phi(j\omega)\} > 0$.

**Remark 2:** If the relative degree $m = 0$, the condition (c) should be modified as: $\Re\{\phi(\infty)\} > 0$.

**Lemma 1:** Consider a closed loop system as shown in Fig. 2, where the strictly proper linear plant $G(s) \in \mathcal{SPR}$, and the time-variant relay satisfies

$$u(t) = -\bar{K}(t) \text{sgn}(y(t)),$$

where $0 < \kappa \leq \bar{K}(t) \leq \kappa + D$. Then for any initial state $x_0$, $\lim_{t \rightarrow -\infty} x(t) \rightarrow 0$ and $y(t)$ converges to zero in finite time. Moreover, for any fixed constant $\kappa > 0$, there exists a neighborhood $B(\delta)$ of the equilibrium point such that $\forall x_0 \in B(\delta)$, $y(t)$ converges monotonously.

**Proof:** Assume $\Sigma(A, b, c)$ is a minimum realization of $G(s)$. Define $V(x) = x^T P x$ as a Lyapunov function of the closed loop system, where the positive definite matrix $P$ satisfies the Kalman-Yakubovich-Popov conditions

$$PA + A^T P = -Q < 0, \quad Pb = c^T. \quad (4)$$

Then in the non-sliding mode dynamics,

$$\dot{V}(x) = -x^T Q x - 2\bar{K}(t)\|y(t)\| < 0. \quad (5)$$

Since $G(s)$ is minimum phase, the sliding mode dynamics (if exists) is also globally asymptotically stable. Hence the whole VSC system is globally asymptotically stable. After a finite-time transient period, $\|x(t)\| < (cbk - \varepsilon)/\|cA\|$ (where $\varepsilon < cbk$ is a given positive constant). Then

$$\dot{y}(t) \text{sgn}(y(t)) = cA x \text{sgn}(y(t)) - cb\bar{K} < -\varepsilon < 0, \quad (6)$$

which means the existence of sliding mode in Filippov sense and finite-time convergence of $y(t)$. Furthermore, inequality (6) holds in a neighborhood of equilibrium point if $\kappa$ fixes. This implies the monotonous convergence of $y(t)$.

**Lemma 1** shows that SPR property is welcome under the VSC framework. Inspired by this result, we will first robustly passify (means rendering a system SPR) the nominal model set $\mathcal{G}$ by a linear dynamic feedback controller $C_1(s)$, a cascade filter $C_2(s)$ and a Hurwitz polynomial $h(s)$ of degree $(r - 1)$. Then according to Lemma 1, an additional variable structure control scheme may deal with other uncertainties and disturbances. Fig. 3 shows the entire control scheme.
Theorem 3 (exact differentiation based OFVSC scheme): Consider the VSC system shown in Fig. 3, where the plant to be controlled satisfies Assumptions A, B and C. Assume robust exact differentiators can provide 1-order to \((r - 1)\)-order differentiations of signal \(y(t)\), and two linear filters \(C_1(s)\), \(C_2(s)\) and a Hurwitz polynomial \(h(s)\) satisfy Lemma 1. Then in a neighborhood of the equilibrium point, the following control law

\[
\begin{align*}
    u(t) &= u_{usc}(t) - C_1(y(t)) \\
    u_{usc}(t) &= -K_{usc} \text{sgn}(\hat{y}(t)) \\
    \hat{y}(t) &= C_2(h(s)y(t))
\end{align*}
\]

(9)  

(where \(K_{usc} > \kappa_0(1 + \epsilon)/(1 - \epsilon)\) is a positive constant) guarantees asymptotic stability of the closed loop system and the convergence of \(y(t)\).

**Remark 3:** Since a transfer functions \(s^k\) represents \(k\)-order exact differentiation, in (10), \(h(s)y(t) = \sum_{k=0}^{r-1} h_k y^{(k)}(t)\) represents a linear combination of the output signal \(y(t)\) and its 1-order to \((r - 1)\)-order differentiations. The construction of robust exact differentiators can be found in [11] and [8].

In Lemma 1, only the nominal set \(G\) is considered and passed, despite whether the real plant (2) has been stabilized or not. In another word, single linear dynamic feedback \(C_1(s)\) does not guarantee stability.

**Theorem 3** leads to local asymptotic stability. The larger \(K_{usc}\), the larger stability region. In a special case that \(D\) is null (no unmodeled dynamics), \(\hat{d}(t) = d(t)\). Let

\[
K_{usc}(t) = \nu(y(t)) + \kappa_0 + \epsilon \quad \epsilon > 0
\]

instead of (9), inequality (13) will hold for all \(t \geq 0\). This results in global asymptotic stability.

**IV. LINEAR DIFFERENTIATION BASED OFVSC CONTROL**

In comparison with the nonlinear differentiator which is exact in Filippov sense, a class of singular perturbation based linear differentiators is more practically used for its simpler structure and less calculation burdens (see e.g. [16]). In this section, we try to answer such a question: with the same control scheme in section III, can we directly replace the nonlinear exact differentiators by linear differentiators? And what is the behavior of the new closed loop system?

Define a Hurwitz polynomial

\[
f(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_{r-1} s^{r-1} \quad (a_0 = 1)
\]

whose roots are all negative real, then \(F(s) := \frac{1}{f(s)}\) is a low pass filter with \(F(0) = 1\). Define

\[
f_\mu(s) = f(\mu s) \quad F_\mu(s) = \frac{1}{f_\mu(s)}
\]

and in another word, \(\frac{1}{f_\mu(s)}u(t)\) may track \(u(t)\), \(\frac{1}{F_\mu(s)}u(t)\) may track \(u'(t)\), \(\frac{2}{F_\mu(s)}u(t)\) may track \(u''(t)\), \(\cdots\) \(\frac{r-1}{F_\mu(s)}u(t)\) may track \(u^{(r-1)}(t)\) respectively. Finally, regarding \(\frac{1}{F_\mu(s)}u(t)\) as a tracker of \(u^{(r-1)}(t)\), we obtain a group of linear differentiators. Now the linear combination of exact differentiations \(h(s)\) may be replaced by \(h(s)\), which is a stable, minimum phase and proper linear filter.

**Lemma 4:** Assume the closed loop system shown in Fig. 5 is composed of a linear plant \(G(s) \in \text{SPR}\), a fast filter \(F_\mu(s)\) defined by (16), and a time-variant relay satisfies (3). Consider a neighborhood of the equilibrium point, for any given error bound \(\varepsilon\), there exists a sufficiently small constant \(\mu\) and a finite time \(T_0\), such that \(\forall t > T_0\), \(|y(t)| < \varepsilon\).

**Fig. 5:** Closed loop system composed of a SPR linear unit, a fast filter and a time-variant relay.
Proof: See Appendix.

**Theorem 5 (linear differentiation based OFVC law):** Under the same condition of Theorem 3, where the exact differentiation polynomial \( h(s) \) is directly replaced by a linear filter \( F_p(s)h(s) \) (\( F_p(s) \) is defined by (16) with all poles negative real). In a neighborhood of the origin, for any given constant \( \varepsilon > 0 \), there exists a sufficiently small constant \( \mu \), such that the control law (8), (9) (where \( K_{vsc} > \kappa_0 \frac{1}{\mu} \)) and (17)

\[
\hat{y}(t) = C_2(s)[F_p(s)h(s)]y(t)
\]

guarantees \(|y(t)| < \varepsilon\) after a finite-time transient process.

**Proof:** As have been done in the proof of Theorem 3, we first modify the disturbance \( d(t) \) by \( \hat{d}(t) \) so that the unmodeled dynamics can be removed. It can be seen that (11) does not vary and (13) still holds in a neighborhood of origin. Dragging the fast filter \( F_p(s) \) to the front of \( G_p(s) \), we get an equivalent block diagram (Fig. 6) of the closed loop system described by Theorem 5.

![Fig. 6. Equivalent control scheme of Theorem 5.](image)

Since \( \frac{G_p}{1 + C_1C_2h}C_2h \in \text{SPR} \), according to Lemma 5, if \( \mu \) is sufficiently small, \( \hat{y}(t) \) will converge to an arbitrarily small neighborhood of zero. Notice that

\[
y(t) = \frac{f_p(s)}{h(s)} \frac{1}{C_2(s)} \hat{y}(t) = \left( \sum_{k=0}^{\infty} \frac{\mu^k a_k s^k}{h(s)} \right) \frac{1}{C_2(s)} \hat{y}(t),
\]

\( \hat{y}(t) \) can be arbitrarily small means \( y(t) \) can be arbitrarily small as well. Then the theorem conclusion is immediate. □

**Remark 4:**
- As singular perturbation based linear filter introduced, there is only one switching function appears in the proposed control law in Theorem 5.
- The peaking phenomenon aroused by singular perturbation based linear differentiators does not matter, because the auxiliary signal \( \hat{y}(t) \) is not an energy signal and only the sign of it is concerned. But, on the other hand, linear differentiators are sensitive to measurement noise\(^3\), which may be the main defect of this method.
- As \( \mu \to 0 \) is sufficiently small, the larger \( K_{vsc} \), the larger neighborhood of the origin that can be regulated. Generally speaking, \( K_{vsc} \) should be determined according to the system initial states or the neighborhood which is concerned, and \( \mu \) should be determined according to \( K_{vsc} \) and the desired error bound \( \varepsilon \).
- This control scheme is different from the proposed ones in [14], [15] and [16], because (a) we introduce signal differentiators rather than state observers; (b) we do not construct any sliding mode dynamics in Filippov sense; in fact, no sliding mode exists at all. Although the closed loop behaviors are similar, our scheme seems simpler and more practical to deal with uncertain linear plants.

**V. OUTPUT TRACKING PROBLEM**

Assume that the reference signal \( y_r(t) \) satisfies:

**Assumption D:** Reference signal \( y_r(t) = G_R(s)r_0(t) \) is generated by an exo-signal \( r_0(t) \) and a reference model \( G_R(s) \), where \( G_R(s) = n_R(s)/d_R(s) \) is stable, minimum phase and with relative degree \( r \) at least, and \( r_0(t) \) is uniformly bounded, i.e. \( \sup |r_0(t)| \leq \theta \).

\[
d_R(t) = y_r(s) - G_p(s)\hat{y}(t) = \frac{G_R(s)}{G_p(s)[1 + \Delta(s)]}r_0(t)
\]

and denote

\[
\sup_{G_p(s) \in \mathcal{G}} \left\| \frac{G_R(s)}{G_p(s)} \right\|_1 \sup_{G_p(s) \in \mathcal{G}} \left\| \frac{n_R(s)d_R(s)}{d_R(s)n_p(s)} \right\|_1 := \zeta,
\]

then

\[
\sup |d_R(t)| \leq \frac{\zeta}{1 - t^{\beta}} \hat{\psi} := \psi
\]

is uniformly bounded. Thus the tracking problem has been converted into an equivalent regulation problem.

**VI. SIMULATION EXAMPLE**

Consider an uncertain linear plant, where the nominal set and unmodeled dynamics are restricted as

\[
\mathcal{G} = \left\{ \frac{[1,1.2]}{s^3 + [1,3]s^2 + [-3,0]s + [1,1.5]} \right\}
\]

and \( \mathcal{D} = \{||\Delta(s)||_1 < 0.2\}; \) disturbance \( |d(t)| \leq 2.5 \). The system output is expected to track a bounded reference signal \( y_r(t) \), which is the solution of Van der Pol equation \( \dot{y}_r + (y_r^2 - 1)y_r + y_r = 0 \) with initial states \( y_r(0) = 1 \) and \( \dot{y}_r(0) = 1 \). (thus max\( y_r(t) < 2 \).) In simulation, we take \( G_p(s) = 1.05/(s^3 + 3s^2 - 0.5s + 1.2) \) as the “real” model and \( \Delta(s) = 0.15\frac{r^{\alpha+1}(s-\alpha)}{r^{\alpha+1}(s-\alpha+1)} \) as the “real” unmodeled dynamics; and \( d(t) \) is composed of a periodic pulse signal (amplitude \( [0,2] \), period 0.1 and pulse-rate 30%) and a stochastic signal uniformly distributed in \([ -0.5, 0.5]\).

It can be verified that \( C_1(s) = \frac{150(s+1)^2}{(s+5)(s+10)} \), \( C_2(s) = \frac{s+3}{s+10} \) and \( h(s) = (s + 1)^2 \) will guarantee \( \frac{C_1}{1 + C_1C_2} \in \text{SPR} \)

\(^3\)In comparison, the exact differentiators are, to some extent, robust ([11]).
Theorem 5 (see Appendix A.) The controller is designed according to sliding motion exists at all.

Simulation result is shown in Fig. 8. Besides of the tracking ability and convergence, it can be seen (a) the auxiliary signal \( \tilde{e}(t) \) oscillates around zero with high switching frequency; (b) the oscillations seem similar to the "chattering" phenomenon in standard VSC systems, although no ideal sliding motion exists at all.

VII. CONCLUSIONS

We have proposed a novel output feedback variable structure control scheme based on the passivity idea. In the case that exact differentiations provided, it guarantees asymptotic stability; in the case that singular perturbation based linear differentiators adopted, it guarantees an arbitrarily small regulation/tracking error.

We believe this scheme is practically significant that: (a) it can deal with a wide class of uncertain linear plants; (b) it seems rather simple and the control performance is acceptable; and (c) the coefficients of the proposed controller are physically significative and can be tuned up easily. We also believe the scheme is theoretically significant that it can be viewed as a new analysis framework of a class of relay control systems without sliding mode (Lemma 4).

APPENDIX

A. Sketch of the proof of Lemma 2.

\( \forall G_p \in \mathcal{G} \), (coefficients can be turned-up “off-line” easily, see Appendix A.) The controller is designed according to Theorem 5, where \( F_\mu(s) = (\mu s + 1)^{-2} \). The large coefficient \( K_{esc} = 15 \) and the sufficient small coefficient \( \mu = 0.05 \) are determined “on-line”, according to the “real” model.

Simulation result is shown in Fig. 8. Besides of the tracking ability and convergence, it can be seen (a) the auxiliary signal \( \tilde{e}(t) \) oscillates around zero with high switching frequency; (b) the oscillations seem similar to the “chattering” phenomenon in standard VSC systems, although no ideal sliding motion exists at all.

Finally, define \( d_2(s) := (s + K_2)d_1(s) = s(s + K_2)d(s) \) and so on, until a sufficiently large constant \( K_r \), (depends on \( K_1, K_2, \ldots, K_{r-1} \)) makes the polynomial \( n_r(s) := d_{r-1}(s) + K_{r-1}n_{r-1}(s) \) Hurwitz; and moreover, \( \sup_{\omega \geq 0} \phi \left\{ \bar{n}_{r-1}(j\omega) \right\} \) can be arbitrarily small.

It can be verified that we finally obtain a proper, stable and minimum phase controller \( C_1(s) \):

\[
C_1(s) = \frac{K_1 K_2 \cdots K_r (s + 1)^{-1}}{(s + K_2)(s + K_3) \cdots (s + K_r)}
\]

that stabilizes \( n_p(s)/d_p(s) \). Define

\[
C_{20}(s) = (s + K_1 \bar{n}_1) h(s) = (s + 1)^{-1}
\]

the feedback and cascade transfer function will be

\[
\bar{G}(s) := \frac{G_p(s)}{1 + G_p(s)C_1(s)} = \frac{C_{20}(s)h(s)}{n_p(s)(s + 1)^{-1}(s + K_1 \bar{n}_1) \prod_{i=2}^{r} (s + K_i)}
\]

which is with relative degree 0 and satisfies

\[
\sup_{\omega \geq 0} \phi \left\{ \bar{G}(j\omega) \right\} \leq \sup_{\omega \geq 0} \left| \phi \left\{ \frac{j\omega + K_1 \bar{n}_1}{j\omega + K_2 \bar{n}_1} \right\} \right| + \sum_{i=2}^{r} \sup_{\omega \geq 0} \left| \phi \left\{ \frac{n_{i-1}(j\omega)(j\omega + K_i)}{n_i(j\omega)} \right\} \right| + \sup_{\omega \geq 0} \left| \phi \left\{ \frac{n_p(j\omega)(j\omega + 1)^{-1}(j\omega + K_1 \bar{n}_1)}{n_1(j\omega)} \right\} \right|
\]

Notice that the first item in the righthand side of (22) is less than \( \frac{\pi}{2} \) and the following two items can be arbitrarily small. This means: properly choosing coefficients \( K_1, K_2, \ldots, K_r \), we have \( |\phi \left\{ \bar{G}(j\omega) \right\}| < \pi/2 \), and thus \( \bar{G}(s) \in \text{SPR} \). Since the set of \( \bar{G}(s) \) (denoted as \( \mathcal{G} \)) is compact, this inequality is strict. It can be proved that there exists a sufficiently large constant \( \beta \), such that \( \forall \bar{G}(s) \in \mathcal{G}, \bar{G}(s) := \bar{G}(s)/(s + \beta) \) is SPR.

Finally, define a proper, stable and minimum phase transfer function

\[
C_2(s) = \frac{C_{20}(s)}{s + \beta} = \frac{s + K_1 \bar{n}_1}{s + \beta},
\]

we complete the proof of Lemma 2 constructively, where the transfer functions \( C_1(s), C_2(s) \) and \( h(s) \) are defined in (19), (20) and (23). It should be pointed out that it is easy to adjust these coefficients \( (K_1, \ldots, K_r, \beta) \), because they are merely expected to be sufficiently large.
B. Sketch of the proof of Lemma 4.

First, it can be proved that system output $y(t)$ oscillates; that is, if $y(0) \neq 0$, there exists a constant $T < \infty$ such that $y(T) = 0$. When the sign of $y(t)$ changes, the sign of $u(t)$ will change immediately; and after most $O(\mu)$ time$^5$, the sign of $\dot{u}(t)$ will be the same as the sign of $u(t)$ and retain $|\dot{u}(t)| > \rho \epsilon (0 < \rho < 1)$, until $y(t)$ returns to zero and another switching occurs.

Denote $t_k$ as the switching time series of $y(t)$, and denote $t_k \in (t_k, t_{k+1})$ (if exists) such that $\forall t \in (t_k, t_{k+1})$, $\dot{u}(t) > 0$ (if $u(t) > 0$) or $\dot{u}(t) < -\rho \epsilon$ (if $u(t) < 0$). Assume $S(x) = x^T P x$ is the storage function of $G(s)$, where the positive definite matrix $P$ satisfies conditions (4). Since $G(s) \in {\cal S}P$, $\forall t > 0$, $S(x_t) \leq S(x_0) + \int_0^t \dot{u}(t) y(t) d\tau$ Consider the time interval $[t_k, t_{k+1}]$,

$$S_{k+1} = S_k + \int_{t_k}^{t_{k+1}} \dot{u}(t) y(t) d\tau + \int_{t_k}^{t_{k+1}} \dot{u}(t) y(t) d\tau < S_k + \beta(t_k - t_k) - \rho \epsilon \int_{t_k}^{t_{k+1}} |y(t)| d\tau$$

$$:= S_k + D_k^{(1)} - D_k^{(2)}$$

where $S_k$ is the abbreviation of $S(x(t_k))$, and $\beta = \sup |\dot{u}(t)| \times \sup |y(t)|$ is a boundary constant (independent of $\epsilon$ and $\mu$). Intuitively, since $(t_k - t_k) < \mu(\mu)$, the term $D_k^{(1)}$ can be arbitrarily small as $\mu \to 0$; while the term $D_k^{(2)}$ cannot be “too small” because $|y(t)|$ is bounded independent of $\mu$. Mathematically speaking,

$$\int_{t_k}^{t_{k+1}} |y(t)| d\tau \geq \frac{1}{2\alpha} \left( \max_{t \in [t_k, t_{k+1}]} |y(t)| \right)^2$$

where $\alpha = \sup |y(t)|$ is independent of $\mu$.

Assume $(t_k - t_k) \leq \lambda \mu$, then

$$D_k^{(1)} + \lambda \beta \mu \geq D_k^{(2)}$$

will hold sooner or later after a finite-time transient process (otherwise the “energy” series $\{ S_k \}$ will decrease at a speed greater than $\lambda \beta \mu$ to negative.) From then on, at each interval $[t_k, (t_k', t_{k+1})]$, the series $\{ S_k \}$ can be divided into 3 cases:

1. $S_k - S_{k+1} \leq \lambda \mu$;
2. $S_k > S_{k+1} + \lambda \beta \mu$ but $S_k \leq S_k'$;
3. $S_k > S_{k+1} + \lambda \beta \mu$ and $S_k > S_k' > S_{k+1}$;

The three cases are complete (if time $t_{k}'$ exists) and case (1) is finite-time recurrent.

It can be proved that every time interval $[t_k, t_{k+1}]$ belongs to (c1) or (c2) satisfies

$$\max_{t \in [t_k, t_{k+1}]} |y(t)| < O(\sqrt{\mu}).$$

(26)

While case (c3) means “energy” decreasing monotonously and (26) can also be obtained.

Finally, if $\dot{u}(t) < \rho \epsilon$ holds in some whole time interval $[t_k, t_{k+1}]$, we also have $\max_{t \in [t_k, t_{k+1}]} |y(t)| < O(\mu)$. The analysis above indicates that as long as $\mu \to 0$, $y(t)$ will converge to an arbitrarily small neighborhood of zero and will not escape any longer.

REFERENCES