On a Robust Bounded Control Design of the Combined Wheel Slip for an Autonomous 4WS4WD Ground Vehicle

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Abstract

This study presents a robust bounded control that prevents wheel skidding for an autonomous 4WS4WD vehicle under uncertain tire/road condition and aerodynamic drag. Constraint on combined wheel slip is incorporated as a priori in design to prevent the skidding. A robust low-and-high gain technique is used to suppress the path-tracking error and to enhance the utilization of the limited wheel slip. Simulation shows that, under the uncertain tire/road condition and aerodynamic drag, the proposed control scheme can effectively limit the combined wheel slip and achieve the goal of path tracking. Moreover, the control inputs of wheel torque and wheel steering are coordinated well during the path tracking.

Keywords: Bounded control, autonomous control, automotive control, combined wheel slip constraint, singular perturbation theory.

1. Introduction

In this paper, we study an issue of guiding an autonomous vehicle to approach and track a reference path without wheel skidding, when uncertainties of tire/road condition and aerodynamic drag are taken into account. Wheel skidding incurs abnormal wear of tire and, in cases, can cause vehicles out of control. The wheel skidding is related to the magnitude of combined wheel slip [1]-[2]. When the magnitude of combined wheel slip exceeds the threshold associated with road condition, tire/road friction saturates and wheel skidding occurs. Conversely, when the magnitude of combined wheel slip is retained within the threshold, the skidding does not take place.

The combined wheel slip consists of two coupled components: namely, the longitudinal and lateral wheel slips. However for simplicity, most vehicle control designs separate these two components to independently develop the relevant controllers. And only few designs, later for integration, combine these independently developed controllers with modifications using the concept of friction circle (see [3]-[7] and references therein). As a result, the overall stability of the new combined system is not easily guaranteed and the coordination between wheel torque and wheel steering could not be well in response to the combined control.

Due to the locally fast and exponential convergence of the wheel subsystem to its quasi-steady state, Peng, et al [8] incorporate the combined wheel slip as a priori in the control design, and regard the quasi-steady state of the combined wheel slip as the constraint target to develop the anti-skidding controller. By the approach, the combined wheel slip can be effectively limited below a pre-specified constraint during the tracking control. However in their study, the influence of aerodynamic drag is not considered. It is known that aerodynamic drag is the main disturbance that reduces vehicle speed from its set point. Thus, it is an instability factor and needs to be coped with. This study follows the approach of reference [8] to prevent wheel skidding, and in addition, considers the uncertainties of tire/road condition and aerodynamic drag for developing the autonomous control.

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2.1.2 Wheel dynamics

\[ I_{w} \frac{d\omega}{dt} = T - r_{o} \left[ f_{a} \cos \delta - \sin \delta \right] \]  

(2)

where \( I_{w} \), \( r_{o} \) represent the inertia and effective radius of wheel \( j \); \( T \), \( \delta \) are the wheel torque and wheel steering angle used for control scheme.

2.1.3 Interaction between vehicle and reference path

Kinematics of the interaction can be expressed as [10]

\[ \dot{y} = -v \sin \phi \]  

(3a)

\[ \dot{\phi} = \frac{v}{(l/r_{o}) + y} \cos \phi + \beta + \gamma \]  

(3b)

where the path curvature \( \rho_{ref} \), assumed to be constant, is given.

2.1.4 Ranges of \( \sigma_{aero} \) and \( \rho_{ref} \)

In accordance with the changes of environment, we assume the coefficient \( \sigma_{aero} \) in Eq. (1b) varies in a known interval \( \Omega_{\sigma} \subset \mathbb{R} \) with

\[ |\sigma_{aero} - \sigma_{aero}| \leq \bar{\mu} \sigma_{aero}, \ \forall \sigma_{aero} \in \Omega_{\sigma} \]  

(4)

where \( \sigma_{aero} \) is the midpoint of the interval, and \( \bar{\mu} \geq 0 \) is the normalized radius. We also assume the path curvature \( \rho_{ref} \) falls in a set \( \Omega_{\rho} \subset \mathbb{R} \) which contains all the curvatures of a standard road. In general, the curvatures of a standard road lie in a range of \([-1/400, 1/400]\) [3].

2.2 Calculation of friction force

To model the friction forces in Eq. (1), we need the following normal load transfer, combined wheel slip, and friction coefficient (see [2], [3], [7] and references therein).

2.2.1 Normal load transfer

The normal loads on the four wheels can be expressed as

\[ f_{ni} = \frac{1}{2} \left[ \frac{1}{l_{i} + l_{f}} \left( l_{mg} - (h_{aero} - h)f_{aero} \right) \right] \]  

(5a)

\[ f_{fi} = \frac{1}{2} \left[ \frac{1}{l_{i} + l_{f}} \left( l_{mg} - (h_{aero} - h)f_{aero} \right) \right] \sin \phi \]  

(5b)

\[ f_{ri} = \frac{1}{2} \left[ \frac{1}{l_{i} + l_{r}} \left( l_{mg} + (h_{aero} - h)f_{aero} \right) \right] \sin \phi \]  

(5c)

\[ f_{rf} = \frac{1}{2} \left[ \frac{1}{l_{i} + l_{r}} \left( l_{mg} + (h_{aero} - h)f_{aero} \right) \right] \cos \phi \]  

(5d)

In Eq. (5), \( f_{aero} = \sigma_{aero} v^{2} \) is the aerodynamic drag force; \( h_{aero} \) is the height of the drag force acting on the vehicle body; \( h \) is the height of the CG; \( g \) is the gravitational constant; \( k_{l} \) and \( k_{s} \) are respectively the front and rear roll stiffness. For the normal loads (5), it is remarkable that the effect of the moment \( (h_{aero} - h)f_{aero} \) arising from the drag force, is quite smaller than that of the moment \( l_{mg} \) resulting from the vehicle weight.

2.2.2 Combined Wheel slip

The combined wheel slip \( S_{j} \) of each wheel \( j \) is defined as

\[ S_{j} = \max \left\{ r_{o} \omega_{o} \cos \alpha_{j}, -V_{j} \right\} \]  

(6)

where \( V_{j} \), \( \alpha_{j} \) are respectively the wheel center velocity and slip angle given as

\[ V_{j} = \left[ \begin{array}{c} v_{j1} \\ v_{j2} \\ v_{j3} \end{array} \right] = \left[ \begin{array}{c} \cos \beta - l_{g} \gamma \\ \sin \beta + l_{g} \gamma \\ \sin \beta + \gamma \end{array} \right] \]  

(7)

where \( \alpha_{j} = \delta_{j} - \beta_{j} \), \( \beta_{j} = \tan^{-1} (v_{j1}/v_{j2}) \), \( j = 1, ..., 4 \).

2.2.3 Friction coefficient and force

The friction force of wheel \( j \) is given as

\[ f_{ij} = f_{o} \left[ \mu_{o} \mu_{j} \right] \]  

(8)

where \( f_{o} \) is normal load (5), and \( \mu_{o}, \mu_{j} \) is friction coefficient obtained by using Eq. (6) with the following coordinate transform

\[ \mu_{j} = \left[ \begin{array}{c} \cos \beta_{j} - \sin \beta_{j} \\ \cos \beta_{j} \cos \beta_{j} \end{array} \right] \frac{1}{k_{j}} \frac{\mu_{h} \left( \| S \| X \right)}{S} S_{j} \]  

(9)

In Eq. (8), \( k_{j} \in [0.9, 0.95] \) is an attenuation factor used when tire tread profile is present. \( \mu_{h} \) is a scalar saturation function depending on the magnitude of combined wheel slip \( \| S \| \) and road condition \( X \) [2]. It is known that when \( \| S \| \) exceeds a threshold associated with road condition, the corresponding \( \mu_{h} \) saturates and so does the related friction force (7).

Conversely, the saturation of friction force can be avoided, provided

\[ \| S \| \]  

is limited below the related threshold. For each road condition \( X \), it is commonly assumed

\[ \begin{array}{l}
(9a) \ 
\mu_{h} (0, X) = 0 \\
(9b) \ 
k_{j} = \frac{\| S \| X}{F_{s}} = 0 = \lim \frac{\mu_{h} (\| S \| X)}{\| S \|} 
\end{array} \]

The initial slope \( k_{j} \) in Eq. (9) depends mainly on road conditions. A better road condition gives a larger slope \( k_{j} \) and in turn provides a larger friction coefficient.

3 Linear design model and assumptions

3.1 Linearization with order reduction

We neglect the effect of the moment \( (h_{aero} - h)f_{aero} \) in Eq. (5) and linearize the vehicle system around the operating point:

\[ \mathbf{\rho}_{ref} = 0, \left[ \begin{array}{c} v_{i} \\ \beta_{i} \\ y_{i} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right], \left[ \begin{array}{c} v_{0} \\ \phi_{0} \\ \delta_{0} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], T_{j} = 0, j = 1, ..., 4 \]  

(10)

Using vehicle data [1] to characterize the linearization shows that the linearized wheel subsystem converges much faster and exponentially to its quasi-steady state below [8]

\[ \frac{\partial \omega_{j}}{\partial t} = \frac{1}{r_{o}^{2} \lambda_{j}} \left( \partial v + \partial \gamma \right) + \frac{v_{j}}{r_{o} \lambda_{j}} \lambda_{j} \]  

(11)

and based on singular perturbation theory for order reduction, we replace the linearized wheel subsystem with its quasi-steady state (11), and finally arrive at the following design model:

(a) The linearized vehicle system:

\[ \begin{array}{c}
\dot{\hat{v}} \\
\dot{\beta} \\
\dot{y}_{c} \\
\dot{\phi}_{c}
\end{array} = \begin{array}{cccc}
-2 \sigma_{aero} v_{0} / m & 0 & 0 & 0 \\
0 & \sigma_{aero} v_{0} / m & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \begin{array}{c}
\hat{v} \\
\beta \\
y_{c} \\
\phi_{c}
\end{array} + \begin{array}{c}
\partial \gamma \\
\partial \gamma \\
\partial y_{c} \\
\partial \phi_{c}
\end{array} \]  

(12)
(b) The simplified tire/road friction model \((f_{u1}, f_{u2}, M_{u})\): Using replacement (11), the friction \((f_{u1}, f_{u2}, M_{u})\) in Eq. (12) can be expressed using the following cascade form:

- From the control input \((T_{j}, \delta_{j})\) to the following quasi-steady-state wheel slip \(\dot{S}_{j}\):

\[
\begin{pmatrix}
\frac{T_{1}}{\zeta_{j} f_{u1}} \\
\frac{T_{2}}{\zeta_{j} f_{u2}} \\
\frac{T_{3}}{\zeta_{j} f_{u3}} \\
\frac{T_{4}}{\zeta_{j} f_{u4}}
\end{pmatrix}
\begin{pmatrix}
\dot{S}_{1} \\
\dot{S}_{2} \\
\dot{S}_{3} \\
\dot{S}_{4}
\end{pmatrix}
= \begin{pmatrix}
\frac{-\beta_{j} f_{u1}}{v_{0} + \delta_{j}} \\
\frac{-\beta_{j} f_{u2}}{v_{0} + \delta_{j}} \\
\frac{-\beta_{j} f_{u3}}{v_{0} + \delta_{j}} \\
\frac{-\beta_{j} f_{u4}}{v_{0} + \delta_{j}}
\end{pmatrix}
\begin{pmatrix}
\dot{S}_{1} + \delta_{j} \\
\dot{S}_{2} + \delta_{j} \\
\dot{S}_{3} + \delta_{j} \\
\dot{S}_{4} + \delta_{j}
\end{pmatrix}
\]

(13)

- From the quasi-steady-state wheel slip \(\dot{S}_{j}\) to the following friction output \((f_{u1}, f_{u2}, M_{u})\):

\[
\begin{pmatrix}
f_{u1} \\
f_{u2} \\
M_{u1}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
-\delta_{j} & 1 & -\delta_{j}
\end{pmatrix}
\begin{pmatrix}
f_{u3} \\
f_{u4} \\
M_{u3}
\end{pmatrix}

\begin{pmatrix}
\dot{f}_{u1} \\
\dot{f}_{u2} \\
\dot{M}_{u1}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
-\delta_{j} & 1 & -\delta_{j}
\end{pmatrix}
\begin{pmatrix}
\dot{f}_{u3} \\
\dot{f}_{u4} \\
\dot{M}_{u3}
\end{pmatrix}
\]

(14)

In Eq. (14), \(k_{j}\) and \(k_{1}\) are defined in Eqs. (8) and (9); \(f_{u2}\) are the static normal loads on the four wheels, defined as

\[
\begin{pmatrix}
f_{u1} \\
f_{u2} \\
f_{u3} \\
f_{u4}
\end{pmatrix}
= \frac{mg}{2}
\begin{pmatrix}
l_{1} & l_{1} & l_{1} & l_{1} \\
l_{2} & l_{2} & l_{2} & l_{2} \\
l_{3} & l_{3} & l_{3} & l_{3} \\
l_{4} & l_{4} & l_{4} & l_{4}
\end{pmatrix}
\]

(15)

In other words, when the vehicle system operates around condition (10), the combination of wheel subsystem (2) and tire/road friction (5)-(8) can be regarded as the actuator, of which the BIBO stable and comparatively faster dynamics is neglected and replaced by the related quasi-steady state instead. Also notice that if substituting Eqs. (13)-(15) into vehicle system (12), the resulting system takes a subsystem \((T_{j}, \delta_{j})\) identical to the linear two-track mode commonly used in literature.

3.2 System assumptions

Some assumptions are given as follows:

Assumption 1. All the wheels have the same effective radius \(r_{e}\) and attenuation factor \(k_{j}\). Hence in Eqs. (13) and (14), we have \(r_{e} = r_{e}, \ k_{j} = k_{j}, \ j = 1, ..., 4\).

Assumption 2. The vehicle runs on a uniform road condition. More specially, the initial slopes (9) have the same value as \(k\). Hence in Eqs. (13) and (14), we have \(k_{j} = k_{j}, \ j = 1, ..., 4\).

Assumption 3. The wheel subsystem starts close to its quasi-steady-state. We also assume that the exact value of the above tire/road parameter \((r_{e}, k_{j}, k)\) is not known, and only the estimated value \((\hat{r}_{e}, \hat{k}_{j}, \hat{k})\) is used to construct the proposed control.

Suppose the design model (12)-(15) can be stabilized by the control input \((T_{j}, \delta_{j})\) with state feedback; and meanwhile, the magnitude of \(\dot{S}_{j}\) is ensured below a pre-specified constraint \(S_{j}\). Then, the original high order nonlinear vehicle system can also be locally stabilized around the operating point (10) using the same control scheme, since Jacobian linearization plus singular perturbation theory ensures the local stability can be concluded by using the stability of the reduced (slow) system [9]. Moreover, the magnitude of wheel slip \(S_{j}\) can be limited approximately below the constraint \(S_{j}\), it is because singular perturbation theory guarantees that trajectories of a fast and BIBO stable subsystem can remain within an \(O(\varepsilon)\) neighborhood of the quasi-steady state, if the trajectories start in the \(O(\varepsilon)\) neighborhood [9]. Hence in the following sections, we will present a \((T_{j}, \delta_{j})\) control scheme to achieve the concept.

4 Controller design

4.1 Control objective

For guiding a vehicle to approach and track a reference path of curvature \(\rho_{ref} \in \Omega_{\rho}\) in a constant speed \(v_{0}\), we choose the following state

\[
x_{u} = \begin{pmatrix}
\hat{v}_{u} \\
\beta_{u} \\
\gamma_{u} \\
\Phi_{u}
\end{pmatrix} = \begin{pmatrix} 0 \ 0 \ v_{0} \rho_{ref} \ 0 \ 0 \end{pmatrix}
\]

(16)
as the set point. Under uncertain tire/road condition \((r_{e}, k_{j}, k)\) and uncertain coefficient of aerodynamic drag \(\sigma_{\alpha\alpha, \varepsilon} \in \Omega_{\sigma}\), the control objective is two-fold: one is to achieve state regulation \(S_{j}\) to a ball neighborhood \([9]\). Hence in the following sections, we will present a \((T_{j}, \delta_{j})\) control scheme to achieve the concept.

4.2 Control structure of wheel torque and wheel steering

Let \((\hat{r}_{e}, \hat{k}_{j}, \hat{k}, \sigma_{\alpha\alpha})\) be the estimated values of \((r_{e}, k_{j}, k, \sigma_{\alpha\alpha})\). Define a unit-ball saturation function \(S_{ball} : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}\) as

\[
S_{ball}(z) = \begin{pmatrix}
z_{1} \\
z_{2}
\end{pmatrix}, \quad \begin{pmatrix}|z_{1}| \ |z_{2}|
\end{pmatrix} \leq 1, \ z \in \mathbb{R}^{2}
\]

(17)

As shown in Fig.2, the control structure of wheel torque and wheel steering is given as [8]:

\[
\begin{pmatrix}
T_{1} \\
T_{2} \\
T_{3} \\
T_{4}
\end{pmatrix}
= \begin{pmatrix}
\frac{\rho_{ref}}{v_{0}} & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
1
\end{pmatrix}

S_{ball}(\hat{S}_{j}^{-1} u) = \begin{pmatrix}
l_{1} \\
l_{2} \\
l_{3} \\
l_{4}
\end{pmatrix}
\]

(18a)

\[
l_{1} = l_{2} = l_{3} = l_{4} = l_{0}, \ u_{1} = u_{2}, \ u_{3} = u_{4} = u_{g}\n\]

(18b)

where \(f_{u2}\) is static normal load (15), and \((u_{e}, u_{g})\) is the bounded controller to be defined in the next section.
Substituting control structure (18) into design model (12)-(15), with Assumptions 1 and 2, gives
\[
\dot{x} = (A + \Delta_u) x + B(I_4 + \Delta_d) \left[ \frac{S_{\text{ball}}(x_{u}^{-1} u_{r})}{\sigma_{\text{ball}}(x_{u}^{-1} u_{r})} \right] + d
\]  
(19)
where \( I_4 \) is a \( 4 \times 4 \) identity matrix and \( x = \left[ \begin{array}{c} x_{u} \beta \gamma \phi \end{array} \right] \). 
\[
d = -\sigma_{\text{aero}} \nu_{aero} \left[ \begin{array}{c} m \ 0 \ 0 \ 0 \ -v_{s} \rho_{\text{ref}} \end{array} \right],
\]
\[
A = \begin{bmatrix}
-2 \sigma_{\text{aero}} \nu_{aero} m^{-1} & 0 & 0 & 0 \\
0 & \sigma_{\text{aero}} \nu_{aero} m^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -v_{aero}
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
\frac{(f_{u} + f_{a})}{m} & 0 & \frac{(f_{u} + f_{a})}{m} & 0 \\
0 & \frac{(f_{u} + f_{a})}{m} & \frac{(f_{u} + f_{a})}{m} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(f_{u} + f_{a})}{m}
\end{bmatrix}
\]
\[
\Delta_{\text{a}} = \frac{\left( \sigma_{\text{aero}} - \sigma_{\text{aero}} \right) v_{aero}}{m},
\]
\[
\Delta_{\text{u}} = \text{diag} \left( \frac{\left( \dot{r}_{a} - \dot{r}_{a} \right)}{\nu_{aero}} , \left( k_{s} - k_{s} \right) / \nu_{aero} , \left( \dot{r}_{a} - \dot{r}_{a} \right) / \nu_{aero} , \left( k_{s} - k_{s} \right) / \nu_{aero} \right)
\]
In system (19), the uncertainties \( \Delta_{\text{a}} \) and \( \Delta_{\text{u}} \) respectively reflect the mismatching between the estimated and the exact values of the aerodynamic coefficient and those of the tire/road data. The disturbance \( d \) arises from the influences of the aerodynamic drag, tracking curvature, and tracking speed. Some properties of control structure (18) and system (19) are summarized as follows:

(I) Property of control structure (18)

(P1) The control (18) is a decoupling scheme, and manipulating the term \( S_{\text{ball}}(x_{u}^{-1} u_{r}) \) inside it can be interpreted as manipulating the quasi-steady-state wheel slip \( S_{\nu_{aero}} \) inside friction model (13)-(15). This approximation \( S_{\nu_{aero}} \approx S_{\text{ball}}(x_{u}^{-1} u_{r}) \) can be observed by substituting Eq. (18) into Eq. (13) with \( \dot{r}_{a} \approx \dot{r}_{a} \) in particular, then the constraint objective \( \| S_{\nu_{aero}} \| \leq s_{\nu_{aero}} \) is achieved. Moreover, The arrangement of structure (18) can be used to cope with the \( \nu \)-split road condition [8] (i.e., a road condition which has different surface properties on the left and right sides).

(II) Property of system (19)

(P2) The pair \( (A, B) \) is controllable.

(P3) Concerning the uncertainty \( \Delta_{u} \) : For set point (16),
\[
\Delta_{u} x_{j} = 0_{3,1}
\]  
(20)
Moreover from inequality (4), the following matrix
\[
\bar{D} = \frac{\mu \sigma_{\text{aero}} \nu_{aero}}{m}
\]
\[
\begin{bmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
satisfies
\[
\Delta_{\nu_{aero}} \| D \| \leq \| D \| \dot{1} < 1
\]  
(21)

(P4) Concerning the uncertainty \( \Delta_{b} \) : The uncertainty \( \Delta_{b} \) satisfies
\[
\Delta_{b} = \Delta_{b} \geq 0_{4,4}
\]
provided that the estimated parameter \( (\hat{r}_{a}, \hat{k}_{s}, \hat{k}_{s}) \) is chosen with \( \hat{r}_{a} \approx r_{a}, \hat{k}_{s} \leq k_{s} \).

(P5) Concerning the disturbance \( d \) : The matrix equation
\[ Ax + B + d = 0_{5,1} \]  
(22)
has a solution
\[
\Gamma = \begin{bmatrix}
\sigma_{\text{aero}} v_{aero} & m v_{aero} \rho_{\text{ref}} & \sigma_{\text{aero}} v_{aero} & m v_{aero} \rho_{\text{ref}} & \sigma_{\text{aero}} v_{aero} & m v_{aero} \rho_{\text{ref}}
\end{bmatrix}
\]  
(23)
Consider a vector function \( \Xi(\vec{S}_{\nu}) = \sum_{j=1}^{n} [f_{\nu_{aero}} f_{\nu_{aero}}] \) defined by summing the terms in Eq. (14) with \( (k_{s}, k_{s}) \) being replaced by \( (\hat{k}_{s}, \hat{k}_{s}) \). By mapping all elements in the four sets \( \Xi(\vec{S}_{\nu}) \) through this surjective function \( \Xi(\vec{S}_{\nu}) \). Fig. 3 depicts the resulting projection, which can be verified as an elliptic shape taking semi-axes \( a = \sum_{i=1}^{n} f_{\nu_{aero}} k_{s} \) and \( b = \sum_{i=1}^{n} f_{\nu_{aero}} k_{s} \). In this figure, the aerodynamic drag \( \sigma_{\text{aero}} v_{aero} \), centrifugal force \( m v_{aero} \rho_{\text{ref}} \), and their resultant force \( \eta \) are also plotted.

Figure 3. Friction circle subject to wheel slip constraint

As subjected to the wheel slip constraint \( \dot{r}_{a} \), this friction ellipse illustrates the full range of all the possible estimated longitudinal and lateral tire/road friction forces (i.e., \( \sum_{i=1}^{n} f_{\nu_{aero}} \) and \( \sum_{i=1}^{n} f_{\nu_{aero}} \)) and their resultant \( \zeta \). Obviously, for achieving the path tracking under the wheel slip constraint, the limited friction force must be able to conquer the external force \( \eta \) that vehicles encounter during the tracking. That is to say, the external force \( \eta \) must lie inside this friction ellipse. And more specifically, there exists \( 0 \leq \delta < 1 \) such that
\[
\left( \frac{\sigma_{\text{aero}} v_{aero}}{m} \right) \leq \delta^2 < 1,
\]
\[
\forall (\sigma_{\text{aero}}, \rho_{\text{ref}}) \in \Omega_{\nu} \times \Omega_{\rho},
\]
Assumption 4 below gives an equivalent statement of Eq. (24).

Assumption 4. There exists \( 0 \leq \delta < 1 \) such that the matrix
\[ \Gamma \]  
(25)
with \( E_{1} \equiv [I_{2} \ 0_{2,2}], \ E_{2} \equiv [0_{2,2} \ I_{2}] \)  
(26)
4.3 Robust bounded controller design

For set point (16), define regulation error as
\[ e = x - x_d \]  
(27)

By using system (19), Eqs. (20) and (22), the dynamical error equation can be obtained as

\[ \dot{e} = (A + \Delta_A) e + B(I_4 + \Delta_B) \left[ \begin{array}{c} \delta_{\text{ball}}(\delta_{\text{ball}}^* e_{\text{ball}}) \\ \delta_{\text{ball}}(\delta_{\text{ball}}^* e_{\text{ball}}) \end{array} \right] - B\Gamma \]  
(28)

Accordingly, the bounded controller \((u_x, u_y)\) is proposed as

\[ \begin{align*}
\dot{u}_x &= -\varepsilon (1 + \gamma_H) E_B P e \\
\dot{u}_y &= -\varepsilon (1 + \gamma_H) E_B P e 
\end{align*} \]  
(29)

where \( P > P^*_e > 0 \) and \( \gamma_H > 0 \) are design parameters. The matrix \( P \) is a solution of the Riccati matrix inequality

\[ (A + \Delta_A) P^*_e + P^*_e (A + \Delta_A) - P^*_e B B^T P_e + \varepsilon I_5 \leq 0_{5 \times 5}, \forall \sigma_{\text{acc}} \in \Omega_e \]  
(30)

with \( \varepsilon > 0 \) being sufficiently small. Or sufficiently by using matrix inequality (21), the matrix \( P \) can be chosen as a solution of the following algebraic Riccati equation (ARE) [6]

\[ A^T P_e + P_e A - P_e B B^T P_e + \varepsilon I_5 = 0_{5 \times 5} \]  
(31)

where \( h_i > 0 \) is an auxiliary parameter for tuning \( \gamma_H \). LMI (31) is always satisfied, because its right-hand side is fixed, while its left-hand side can be arbitrarily small as enlarging \( \gamma_H \).

Choosing \((\varepsilon, P, \gamma_H)\) in the above manner is known as a low-and-high gain technique developed for bounded input control [12]-[13]. By this technique, utilization of the constrained control (i.e., the constrained wheel slip in this study) can be enhanced to robustly attenuate the tracking error (27).

### 4.4 Stability analysis

Consider a suitable solution \((\varepsilon, h_i, P, \gamma_H, h_i)\) chosen from ARE (30) and LMI (31). Let \( V(e) = e^T P e \) be the Lyapunov function candidate. Then, the Lyapunov derivative of system (28) under control (29) can be expressed as

\[ \dot{V}(e) = e^T (A + \Delta_A) (A + \Delta_A) - P_e B B^T P_e + \varepsilon I_5 \leq 0_{5 \times 5} \]  
(32)

Theorem 1 below summarizes the main result in this study.

**Theorem 1.** Let Assumptions (1)-(4) hold. Then, with the solution \((\varepsilon, h_i, P, \gamma_H, h_i)\), Lyapunov derivative (32) can be estimated as

(1) \[ \dot{V}(e) < 0, \quad \forall e \in \mathbb{R}^5 : c_i < e^T P e < c_i \]  

\[ c_i \triangleq h_i / \sqrt{\gamma_H}, \quad c_2 \triangleq (1 - \delta)^2 / \lambda_{\text{max}}(B^T B) \].

Moreover,

(2) \[ \gamma_H \rightarrow \infty, \quad c_i \rightarrow 0 \].

**Proof.** Omitted for brevity.

References

2. Kiencke, U. and Nielsen, L., Automotive Control Systems,