Asymptotic Robust Adaptive Tracking of Parametric Strict-Feedback Systems with Additive Disturbance

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Abstract: This paper deals with the tracking control of multi-input/multi-output nonlinear parametric strict-feedback systems in the presence of additive disturbances and parametric uncertainties. For such systems, robust adaptive controllers usually cannot ensure asymptotic tracking or even regulation. In this work, under the assumption the disturbances are bounded with bounded time derivatives, we present a C^2 robust adaptive control construction that guarantees the tracking error is asymptotically driven to zero.

1 Introduction

Robust adaptive control laws in the presence of additive disturbances can generally ensure closed-loop signal boundedness and convergence of the tracking error (state) to a residual bounded set with size of the order of the disturbance magnitude, but not asymptotic tracking (or regulation). See [3, 4, 5, 6, 7, 8, 10] for examples of such results. In this paper, we consider multi-input/multi-output (MIMO) nonlinear parametric strict-feedback systems subjected to bounded additive disturbances that are twice continuously differentiable and have bounded time derivatives. For these systems, we present a continuous robust adaptive control construction that guarantees asymptotic tracking. The proposed construction is based on the nonlinear robust control technique of [9], which was originally used to compensate for unstructured uncertainties. Here, we use it as a robustifying mechanism for adaptive controllers. That is, adaptation is used to compensate for structured (parametric) uncertainties while the robust mechanism compensates for disturbances, hence recovering the disturbance-free, asymptotic tracking property of the adaptive controller. The standard adaptive backstepping design is judiciously modified to allow the use of the robust control technique of [9]. Also instrumental to our new construction is the use of the sufficiently smooth projection-based adaptation law recently introduced in [2]. This allows the adaptive stabilizing functions of the backstepping design to be differentiable as many times as necessary. The stability analysis shows the proposed robust adaptive controller yields semi-global asymptotic tracking.

2 Problem Statement

We consider a class of parametric strict-feedback systems of the form

\[
\begin{align*}
\dot{x}_1 &= \varphi_1^T(x_1, x_2) \theta + x_2 + u \\
& \quad \vdots \\
\dot{x}_i &= \varphi_i^T(x_1, \ldots, x_i) \theta + x_{i+1} + u \quad \text{(1b)}
\end{align*}
\]

where \(x_i(t) \in \mathbb{R}^m\), \(i = 1, \ldots, n\) are the system states, \(\varphi_i \in \mathbb{R}^{p \times m}\), \(i = 1, \ldots, n\) are known nonlinearities, \(\theta \in \mathbb{R}^p\) is an uncertain constant parameter vector, \(d(t) \in \mathbb{R}^m\) is an uncertain additive disturbance, \(u(t) \in \mathbb{R}^m\) is the control input, and \(y = x_1\) is the system output. We make the following assumptions regarding the system:

A1. \(\varphi_i \in C^{n+1-i}, i = 1, \ldots, n\).

A2. \(d \in C^2\) and \(\|d(t)\|_{\mathbb{L}_\infty} \leq \bar{d}_0\), \(\|\dot{d}(t)\|_{\mathbb{L}_\infty} \leq \bar{d}_1\), and \(\|\ddot{d}(t)\|_{\mathbb{L}_\infty} \leq \bar{d}_2\) where \(\bar{d}_0, \bar{d}_1, \bar{d}_2\) are known positive constants.

A3. The parameter vector \(\theta\) belongs to a compact convex set \(\Omega := \{\theta : \|\theta\| \leq \theta_0\}\) where \(\theta_0\) is a known positive constant.

Let the output tracking error be defined as

\[
e := y - y_r
\]

where the \(C^{n+1}\) reference trajectory \(y_r(t) \in \mathbb{R}^m\) is such that

\[
y_r^{(i)}(t) \in \mathbb{L}_\infty, \quad i = 0, \ldots, n + 1,
\]

and \(\cdot^{(i)}(t)\) denotes the \(i\)th derivative with respect to time. Our goal is to construct a state feedback control \(u(x_1, x_2, \ldots, x_n)\) that ensures \(e(t) \to 0\) as \(t \to \infty\) and the boundedness of all closed-loop signals. The following notation will be used throughout the paper: \(\hat{\theta}_i(t) \in \mathbb{R}^p, i = 1, \ldots, n - 1\) are parameter estimates;

\[
\hat{\theta}_i(t) := \theta - \hat{\theta}_i(t), \quad i = 1, \ldots, n - 1
\]

and \(\cdot^{(i)}(t)\) denotes the corresponding parameter estimation errors; \(\text{Proj}(\mu_i, \hat{\theta}_i) \in \mathbb{R}^p, i = 1, \ldots, n - 1\) denote \(C^{n-i-1}\) projection operators used to ensure \(\hat{\theta}_i(t) \in \mathbb{L}_\infty\) independent of the stability analysis (see Appendix A for details); \(\Gamma_i \in \mathbb{R}^{p \times p}, i = 1, \ldots, n - 1\) are constant, diagonal, positive-definite matrices; and \(c_i, i = 1, \ldots, n + 1\) are positive constants. To reduce the notational complexity and facilitate the readability of the paper, the control construction that follows is presented for the case when \(m = 1\). Note, however, that the main result is readily applicable to the MIMO case. The following inequalities will also be used throughout the paper:

\[
\ln(\cosh(\|\xi\|)) \leq \sum_{i=1}^{q} \ln(\cosh(\xi_i)) \leq \|\xi\|^2, \quad \forall \xi \in \mathbb{R}
\]

\[
\|\xi\| \leq (\|\xi\| + 1) \|\tanh(\xi)\|, \quad \forall \xi \in \mathbb{R}.
\]
3 Construction of Control Law

Step 1

We begin by differentiating (2) and substituting from (1a) to obtain
\[
\dot{\epsilon} = \varphi_1^T(y)\theta + x_2 - \dot{y}_r. \tag{7}
\]
After adding and subtracting the term \(\varphi_1^T(y_r)\theta\) to (7), the error system becomes
\[
\dot{\epsilon} = \varphi_1^T(y_r)\theta + x_2 - \dot{y}_r + \dot{\omega}_1, \tag{8}
\]
where
\[
\dot{\omega}_1 = (\varphi_1^T(y) - \varphi_1^T(y_r))\theta \tag{9}
\]

Remark 1 Due to assumption A1 and (6), we can use the Mean Value theorem to show
\[
\left\| \dot{\omega} \right\| \leq \rho_{11} (|\epsilon|) \| \tanh (\epsilon) \| \tag{10}
\]
where \(\rho_{11} (\cdot) \in \mathbb{R}_{\geq 0}\) is some globally invertible, nondecreasing function.

Let
\[
\eta_2 := \frac{1}{c_1} (x_2 - \alpha_1) \tag{11}
\]
where \(\alpha_1\) is a stabilizing function yet to be designed. To facilitate the notation, let
\[
\eta_1 := \epsilon, \tag{12}
\]
and define
\[
V_1 = \ln (\cosh (\eta_1)) + \frac{1}{2} \tilde{\theta}_1^T \Gamma_1^{-1} \tilde{\theta}_1. \tag{13}
\]
Differentiating (13) along (8) gives
\[
\dot{V}_1 = \tanh (\eta_1) (\varphi_1^T(y_r)\theta - \dot{y}_r + \alpha_1 + \dot{\omega}_1 + c_1 \eta_2) - \tilde{\theta}_1^T \Gamma_1^{-1} \tilde{\theta}_1. \tag{14}
\]
Based on (14), we design the stabilizing function and parameter update law as follows
\[
\alpha_1 = -c_1 \tanh (\eta_1) - \varphi_1^T(y_r)\tilde{\theta}_1 + \dot{y}_r, \tag{15}
\]
\[
\dot{\tilde{\theta}}_1 = \Gamma_1 \text{Proj}(\mu_1, \tilde{\theta}_1), \quad \mu_1 = \varphi_1(y_r)\tanh (\eta_1) \tag{16}
\]
Substituting (15) and (16) into (14) gives
\[
\dot{V}_1 \leq -c_1 \tanh^2 (\eta_1) + \tanh (\eta_1) (c_1 \eta_2 + \dot{\omega}_1) \tag{17}
\]
where property P2 of the projection operator was used (see Appendix A).

Remark 2 The final step of our design will require that \(\tilde{\theta}_i(t) \in \mathcal{L}_\infty, \ i = 1, \ldots, n-1\) independently of the stability analysis. This motivates the use of the term \(\ln (\cosh (\eta_i))\) in the Lyapunov function of the first \(n-1\) steps of the backstepping procedure. In particular, due to property P3 of the projection operator (see Appendix A), we know \(\tilde{\theta}_i(t) \in \mathcal{L}_\infty\) if \(\mu_i(t) \in \mathcal{L}_\infty\). The boundedness of \(\mu_i\) is facilitated by the fact that \(\partial \ln (\cosh (\eta_i)) / \partial \eta_i = \tanh (\eta_i)\).

Remark 3 Using (11) and (15), the state \(x_2\) can be decomposed into
\[
x_2 = \frac{c_1 \eta_2 - c_1 \tanh (\eta_1) - \varphi_1^T(y_r)\dot{\theta}_1 + \dot{y}_r}{\zeta_{\eta_1} + \zeta_{\tilde{\theta}_1}} \tag{18}
\]
where the term \(\zeta_{\eta_1}\) is a function of \(\eta_1\) and \(\eta_2\), and the term \(\zeta_{\tilde{\theta}_1}\) is bounded. The usefulness of this decomposition will become apparent in the next step.

Step i \(2 \leq i \leq n-1\)

Let
\[
\eta_{i+1} := \frac{1}{c_i} (x_{i+1} - \alpha_i) \tag{19}
\]
where \(\alpha_i\) is a stabilizing function, and differentiate \(\eta_i := \frac{1}{c_{i-1}} (x_i - \alpha_{i-1})\) to obtain
\[
\dot{\eta}_i = \frac{1}{c_{i-1}} \left( \varphi_{i-1}^T \theta + c_i \eta_{i+1} + \alpha_i - \alpha_{i-1} \right) \tag{20}
\]
where (1b) was used. The derivative of \(\alpha_{i-1}(x_1, \ldots, x_{i-1}, \theta_1, \ldots, \theta_{i-1}, y_r, \ldots, y_{(i-1)}^{(i-1)})\) can be written as
\[
\dot{\alpha}_{i-1} = \Psi_{i-1}(x_1, \ldots, x_i, \tilde{\theta}_1, \ldots, \tilde{\theta}_{i-1}, y_d, \ldots, y_d^{(i)}) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j^T \theta \tag{21}
\]
where
\[
\dot{\Psi}_{i-1} = \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_j} \dot{x}_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \theta_j} \dot{\theta}_j \right) + \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j-1)}} y_d^{(j)} \tag{22}
\]
is known. Using (21), we can rewrite (20) as
\[
\dot{\eta}_i = \frac{1}{c_{i-1}} \left[ -\Psi_{i-1} + \alpha_i + c_i \eta_{i+1} + \varphi_{i-1}^T \left( -\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j^T \theta \right) \right]. \tag{22}
\]
Adding and subtracting the term \(w_b \theta / c_{i-1}\) to (22) yields
\[
\dot{\eta}_i = \frac{1}{c_{i-1}} \left( w_b \theta - \Psi_{i-1} + \alpha_i + c_i \eta_{i+1} + \dot{\omega}_i \right) \tag{23}
\]
where \(w_b := w_i(y_r, \zeta_{b_{i-1}}, \tilde{\theta}_1, \ldots, \tilde{\theta}_{i-1})\) and
\[
\dot{\omega}_i = (w_i - w_b) \theta. \tag{24}
\]
Remark 4 Using assumption A1 and (6), we can show
\[ \|\tilde{w}_i\| \leq c_{i-1} \sum_{j=1}^{i} \rho_{ij} (\|\tilde{\eta}_i\|) \|\tanh(\eta_j)\| \] (25)
where \( \rho_{ij} (\cdot) \in \mathbb{R}_{>0}, \ j = 1, \ldots, i \) are some globally invertible, nondecreasing functions and \( \tilde{\eta}_i := (\eta_1, \ldots, \eta_i)^T \). Note that the calculation of (25) is facilitated by the fact that \( x_i - \zeta_{b(i-1)} = \zeta_{\eta(i-1)} \) (see Remark 3).

Define
\[ V_i = V_{i-1} + \ln (\cosh(\eta_i)) + \frac{1}{2} \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i \] (26)
whose derivative is
\[ \dot{V}_i \leq \sum_{j=1}^{i-1} \frac{\tanh(\eta_j)}{c_{j-1}} \left( c_j \eta_{j+1} + \tilde{w}_j \right) - \frac{c_j \tanh^2(\eta_j)}{c_{j-1}} \] (27)
where \( c_0 = 1 \). Based on (27), we design
\[ \alpha_i = -c_i \tanh(\eta_i) - w_i \hat{\theta}_i + \Psi_i \] (28)
\[ \dot{\hat{\theta}}_i = \Gamma_i \text{Proj}(\mu_i, \hat{\theta}_i), \ \mu_i = \frac{1}{c_i} w_i^T \tanh(\eta_i) \] (29)
Substituting (28) and (29) into (27) gives
\[ \dot{V}_i \leq \sum_{j=1}^{i-1} \frac{\tanh(\eta_j)}{c_{j-1}} \left( c_j \eta_{j+1} + \tilde{w}_j \right) - \frac{c_j \tanh^2(\eta_j)}{c_{j-1}} \] (30)
\[ \text{Remark 5 Using (19) and (28), the state } x_{i+1} \text{ can be written as} \]
\[ x_{i+1} = \zeta_{\eta_i} + \zeta_{b_i} \] (31)
where
\[ \zeta_{\eta_i} = c_i \eta_{i+1} - c_i \tanh(\eta_i) + \Psi_{b(i-1)} - \Psi_i, \] (32)
\[ \zeta_{b_i} = \Psi_{b(i-1)} - w_i \hat{\theta}_i, \] (33)
where \( \Psi_{b(i-1)} := \Psi_i (y_r, \zeta_{b_1}, \ldots, \zeta_{b(i-1)}, \hat{\theta}_1, \ldots, \hat{\theta}_{i-1}, \ y_r, \ldots, y_r^{(i-1)}) \), \( \zeta_{\eta_i} \) is a function of \( \eta_1, \ldots, \eta_{i+1} \), and \( \zeta_{b_i} \) is bounded.

\[ \text{Step } n \]
In the last step, we modify the standard backstepping procedure in order to use the new robust control mechanism of [9] to deal with the additive disturbance in (1c). Let the variable \( r(t) \) be defined as
\[ r := \frac{1}{c_n} \tilde{\eta}_n + \eta_n \] (34)
where
\[ \eta_n := \frac{1}{c_{n-1}} (x_n - \alpha_{n-1}) \] (35)
After differentiating (34), we obtain
\[ \dot{r} = \frac{1}{c_n} \dot{\eta}_n + c_n r - c_n \eta_n \] (36)
Differentiating \( \eta_n \) twice produces
\[ \dot{\eta}_n = \frac{1}{c_{n-1}} \left( \phi_n^* \theta + d + \dot{u} - \alpha_{n-1} \right) \] (37)
where (1c) was used.

We first concentrate on the calculation of the term \( \dot{\alpha}_{n-1} \). To end, we have
\[ \dot{\alpha}_{n-1} = \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j^* (x_1, \ldots, x_j) \theta + \Phi \] (38)
where
\[ \Phi = \sum_{j=1}^{n-1} \left( \frac{\partial \alpha_{n-1}}{\partial x_j} \dot{x}_{j+1} + \frac{\partial \alpha_{n-1}}{\partial \theta_j} \dot{\theta}_j \right) + \sum_{j=1}^{n} \frac{\partial \alpha_{n-1}}{\partial y_d^{(j-1)}} y_d \] (39)
is known. Differentiating (38) gives
\[ \dot{\alpha}_{n-1} = \Phi + f_1 (x_1, \ldots, x_n, \hat{\theta}_1, \ldots, \hat{\theta}_{n-1}, y_d, \ldots, y_d^{(n-1)}) \]
\[ + g_1 (x_1, \ldots, x_n, \hat{\theta}_1, \ldots, \hat{\theta}_{n-1}, y_d, \ldots, y_d^{(n)}) \] (40)
where
\[ f_1 (\cdot) = \sum_{j=1}^{n-1} \left[ \left( \frac{\partial \alpha_{n-1}}{\partial x_j} \right) \left( x_{k+1} + \phi_k^* \theta \right) \right] \]
\[ + \sum_{k=1}^{n} \frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial y_d} y_d^{(k)} + \sum_{k=1}^{n} \left( \frac{\partial^2 \alpha_{n-1}}{\partial \theta_j \gamma_{bk}} \phi_j \right) \] (41)
\[ g_1 (\cdot) = \sum_{j=1}^{n-1} \left( \sum_{k=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial \gamma_{bk}} \phi_j \right) \phi_j \theta, \] (42)
and property P3 of the projection operator was used.

We now turn our attention to the calculation of the term \( \phi_\theta^T \) in (37). To end, we have
\[ \phi_\theta \theta = g_2 (x_1, \ldots, x_n, \hat{\theta}_1, \ldots, \hat{\theta}_{n-1}, y_d, \ldots, y_d^{(n)}) \] (43)
where
\[ g_2 (\cdot) = \frac{\partial \phi_\theta}{\partial x_j} \sum_{j=1}^{n-1} \left( \frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j \theta + \frac{\partial \alpha_{n-1}}{\partial x_j} x_{j+1} \right) \]
\[ + \frac{\partial \alpha_{n-1}}{\partial \theta_j} \gamma_{bj} + \frac{\partial \alpha_{n-1}}{\partial y_d^{(j-1)}} y_d^{(j)} \]
\[ + \sum_{j=1}^{n-1} \frac{\partial \phi_\theta}{\partial x_j} (\phi_j \theta + x_{j+1}) \] (44)
and
\[ f_2(\cdot) = \frac{\partial \rho_n^T}{\partial \nu_n} (c_{n-1}^2c_n^3 - c_{n-1}c_n^2 \eta_n ) + \sum_{j=1}^{n-1} \frac{\partial \nu_{n-1}}{\partial \nu_j} \gamma_{n_j} \right) . \] (45)

Substituting (40), (43), and (37) into (36), we obtain
\[ \dot{r} = \frac{1}{c_{n-1}c_n} \left( f_2 - f_1 + \dot{u} - \Phi + g_2 - g_1 \right) + c_n r - c_n \eta_n , \] (46)

We now add and subtract the terms \( g_{b1} = g_1(yd, \zeta_{b1}, \ldots, \zeta_{b(n-1)}, \bar{\theta}_1, \ldots, \bar{\theta}_{n-1}, yd, \ldots, y_d^{(n)}) \) and
\( g_{b2} = g_{2}(yd, \zeta_{b1}, \ldots, \zeta_{b(n-1)}, \bar{\theta}_1, \ldots, \bar{\theta}_{n-1}, yd, \ldots, y_d^{(n)}) \) to the right-hand side of (46) to obtain
\[ \dot{r} = c_n r - c_n \eta_n + \frac{1}{c_{n-1}c_n} \left( f_2 - f_1 + \dot{u} - \Phi + g_2 - g_1 + \left( g_{b2} - g_{b1} \right) \right) \] (47)

where \( h(yd, \zeta_{b1}, \ldots, \zeta_{b(n-1)}, \bar{\theta}_1, \ldots, \bar{\theta}_{n-1}, yd, \ldots, y_d^{(n)}) \) has the special property that
\[ h(t), \dot{h}(t) \in L_{\infty} \] (48)
due to (3) and properties P1 and P3 of the projection operator. Finally, we rewrite (47) as
\[ \dot{r} = \frac{1}{c_{n-1}c_n} \left( -\Phi + h + \dot{u} + \dot{\Phi} \right) + f + c_n r - c_n \eta_n \] (49)
where \( f := (f_2 - f_1 + f_3) / c_{n-1}c_n \).

**Remark 6** Using (84) and (6), we can show \( f \) has the special property that
\[ \|f\| \leq \rho_f (\|z\|) \] (50)
where \( \rho_f (\cdot) \in \mathbb{R}_{>0} \) is some globally invertible, non-decreasing function, and
\[ x = (\eta_1, \eta_2, \ldots, \eta_n, r)^T \]
\[ z = (\tanh(\eta_1), \tanh(\eta_2), \ldots, \tanh(\eta_{n-1}), \eta_n, r)^T . \] (51)

Based on (49), we design \( \dot{u} \) as [9]
\[ \dot{u} = \Phi - \beta \text{sgn}(\eta_n) - c_{n-1}c_n (c_n + c_n + 1) r \] (52)
where \( \beta > 0 \). The actual \( C^0 \) control input can be written from (52) as follows
\[ u(t) = -\int_0^t \left[ c_{n-1}c_n (c_n + c_n + 1) \eta_n (\tau) + \beta \text{sgn}(\eta_n(\tau)) \right] d\tau + \Phi(t) - \Phi(0) - c_{n-1}c_n (c_n + c_n + 1) (\eta_n(t) - \eta_n(0)) \] (53)
where \( u(0) = 0 \). After substituting (52) into (49), we obtain the closed-loop system
\[ \dot{r} = -r - c_n \eta_n + f - c_{n-1}r + \frac{1}{c_{n-1}c_n} \left( h + \dot{d} - \beta \text{sgn}(\eta_n) \right) . \] (54)

## 4 Main Result

Before presenting the main result in Theorem 1, we state a technical lemma.

**Lemma 1** (See [9] for proof.) Let the function \( L(t) \in \mathbb{R} \) be defined as follows
\[ L := \frac{r}{c_{n-1}c_n} \left( h + \dot{d} - \beta \text{sgn}(\eta_n) \right). \] (55)

If the control gain \( \beta \) is selected to satisfy the following sufficient condition
\[ \beta > \|h(t)\|_{L_{\infty}} + \|\dot{d}(t)\|_{L_{\infty}} + \frac{1}{c_n} \left( \|h(t)\|_{L_{\infty}} + \|\dot{d}(t)\|_{L_{\infty}} \right) , \] (56)
then
\[ \int_0^t L(\tau) d\tau \leq \zeta_b \] (57)
where the positive constant \( \zeta_b \) is defined as
\[ \zeta_b = \frac{1}{c_{n-1}c_n^2} \left[ \beta |\eta_n(0)| + \eta_n(0) \left( h(0) + \dot{d}(0) \right) \right] . \] (58)

**Theorem 1** The control law (53) ensures that all system signals are bounded and \( e(t) \to 0 \) as \( t \to \infty \), provided \( \beta \) is adjusted according to (56), \( c_n > (c_{n-1}/2c_n - 2)^2 \), and the control gains \( c_i, i = 1, \ldots, n + 1 \) are selected sufficiently large relative to the system initial conditions.

**Proof:** Let the function \( P(t) \in \mathbb{R} \) be defined as follows
\[ P(t) := \zeta_b - \int_0^t L(\tau) d\tau \] (59)
where \( \zeta_b \) and \( L(t) \) were defined in Lemma 1. If \( \beta \) is selected to according to (56), it follows from Lemma 1 that \( P(t) \geq 0 \).

We now define the following function \( V \)
\[ V := V_{n-1} + \frac{1}{2} \eta_n^2 + \frac{1}{2} r^2 + P . \] (60)

Using (5), we can bound (60) as follows
\[ \lambda_1 \ln(\cosh(\|s\|)) \leq V \leq \lambda_2 \|s\|^2 \] (61)
where
\[ s = \left[ x^T \bar{\theta}_1^T \ldots \bar{\theta}_{n-1}^T \sqrt{P} \right]^T , \]
\[ \lambda_1 = \frac{1}{2} \min \{ 1, \lambda_{\text{min}} (\Gamma_i^{-1}) \} , \]
\[ \lambda_2 = \max \left\{ \frac{1}{2} \lambda_{\text{max}} (\Gamma_i^{-1}), 1 \right\} , \] (62)
and \( x \) was defined in (51).

Taking the time derivative of (60) and substituting from (34), (54), and (59), we obtain
\[ \dot{V} = \dot{V}_{n-1} - c_n \eta_n^2 - r^2 + rf - c_n r^2 \] (63)

If \( f(s,t) \) denotes the right-hand side of the closed-loop system on which the stability analysis is being performed, notice from (54) and (55) that \( f(s,t) \) has a discontinuity on the set of Lebesgue measure zero \( \{ (s,t) : \eta_n = 0 \} \). Since Lemma 77 requires that a solution exist for \( \dot{s} = f(s,t) \), see [9] for a discussion on the existence of Filippov’s generalized solution.
upon use of (55). Substituting now from (30) for \( j = n - 1 \) and (50) gives

\[
\dot{V} \leq \sum_{j=1}^{n-1} \left[ \frac{\tanh(\eta_j)}{c_{j-1}} (c_j \eta_{j+1} + \tilde{w}_j) - \frac{c_j \tanh^2(\eta_j)}{c_{j-1}} \right] - c_n \eta_n^2 - r^2 + \left[ \rho_f(\|x\|) \|z\| \|r\| - c_n r^2 \right] \tag{64}
\]

We can upper bound (64) by using (25) as follows

\[
\dot{V} \leq - \sum_{j=1}^{n-1} \left( k_j - \rho_j (\|\tilde{\eta}_j\|) \right) \tanh^2(\eta_j) - \left( \sigma - \frac{\rho_f^2(\|x\|)}{4c_n+1} \right) \|z\|^2. \tag{71}
\]

It follows from (71) that

\[
\dot{V} \leq - \gamma \|z\|^2 \tag{72}
\]

where the constant \( \gamma \) is such that \( 0 < \gamma < 1 \). where the constant \( \gamma \) satisfies \( 0 < \gamma < 1 \).

From (61) and (72), let

\[
W_1(s) := \lambda_1 \ln(\cosh(\|s\|)), \quad W_2(s) := \lambda_2 \|s\|^2, \quad W(s) := \gamma \|x\|^2. \tag{74}
\]

From (73), we define the sets \( \mathcal{D} \) and \( \mathcal{S} \) as follows

\[
\mathcal{D} := \{ s : \|s\| < s_{\min} \} \quad \text{and} \quad \mathcal{S} := \{ s \in \mathcal{D} : W_2(s) < \lambda_1 \ln(\cosh(s_{\min})) \} \tag{75}
\]

where \( s_{\min} = \min \left\{ \rho_j^{-1}(k_j), \rho_f^{-1}(2\sqrt{c_n+1}) \right\} \). We can now invoke Lemma 2 of [1] to state that such a \( s(t) \) is in \( \mathcal{L}_\infty \). From (3), we then know \( x_1(t) \in \mathcal{L}_\infty \). From (15), (16), and property P3 of the projection operator, we know \( \alpha_1(t), \theta_1(t) \in \mathcal{L}_\infty \). We can now use (11) to show \( x_2(t) \in \mathcal{L}_\infty \). From (1a), we know \( \dot{x}_1(t) \in \mathcal{L}_\infty \). Continuing with this procedure, we can show \( \alpha_i(t), x_{i-1}(t), \in \mathcal{L}_\infty, i = 2, \ldots, n-1 \). We can then state \( \eta_i(t), \alpha_{i-1}(t) \in \mathcal{L}_\infty, i = 1, \ldots, n-1 \) by using (22). Using (54) and assumption A2, we can show \( r(t) \in \mathcal{L}_\infty \). From (34) and (38), we know \( \tilde{\eta}_i(t), \alpha_{i-1}(t) \in \mathcal{L}_\infty \); hence, from the time derivative of (35), we have \( \dot{x}_n(t) \in \mathcal{L}_\infty \). Finally, we can use (1c) to show that \( u(t) \in \mathcal{L}_\infty \).

Now, it is clear from (74) that \( \dot{W}(s(t)) \in \mathcal{L}_\infty \), which is a sufficient condition for \( W(s) \) being uniformly continuous. It then follows from Lemma 2 of [1] that \( \gamma \|x(t)\|^2 \to 0 \) as \( t \to \infty \) \( \forall s(0) \in \mathcal{S} \), which implies from (51) that \( e(t) \to 0 \) as \( t \to \infty \) \( \forall s(0) \in \mathcal{S} \).

Note that the region of stability in (76) can be made arbitrarily large to include any initial conditions by increasing the control gains \( c_i, i = 1, \ldots, n+1 \) (i.e., a semi-global stability result). Specifically, we can use the second equation in (74) and (76) to calculate the
region of stability as follows
\[ k_i > \rho_i \left( \cosh^{-1} \left( \exp \left( \frac{\lambda_2}{\lambda_1} \|s(0)\|^2 \right) \right) \right), \quad i = 1, \ldots, n - 1, \quad \text{and} \]
\[ c_{n+1} > \frac{1}{4\sigma f_2^2} \left( \cosh^{-1} \left( \exp \left( \frac{\lambda_2}{\lambda_1} \|s(0)\|^2 \right) \right) \right) \]
where
\[ \|s(0)\| = \sqrt{\sum_{i=1}^{n} \eta_i^2(0) + r^2(0) + \sum_{i=1}^{n-1} \hat{\theta}_i^2(0) \hat{\theta}_i(0) + P(0)} \]  

Note that the inequalities in (70) and (77) can be satisfied for large enough gains \( c_i, i = 1, \ldots, n + 1 \) because: i) \( \rho_i(\cdot) \) is not a function of \( c_i \), ii) \( \eta_i(0), r(0), \) and \( P(0) \) are only a function of \( 1/c_i \), and iii) \( r(0) \) is not a function of \( c_{n+1} \) since \( u(0) = 0 \).

5 Conclusion
This work considered the tracking problem for MIMO nonlinear parametric strict-feedback systems in the presence of additive disturbances and parametric uncertainties. The continuous robust adaptive controller, whose construction is founded on the fusion of an adaptation law and a dynamic robust control mechanism, exploits the two times continuous differentiability of the disturbances. The robust adaptive control law guarantees the semi-global asymptotic convergence of the tracking error to zero and boundedness of all signals.

References

A Projection Operator
We use the following projection operator, which was recently proposed in [2]:
\[ \text{Proj}(\mu_i, \hat{\theta}_i) = \mu_i - \frac{\pi_1 \pi_2 \nabla p(\hat{\theta}_i)}{4(\varepsilon^2 + 2\varepsilon \theta_0)^{n-1} \theta_0^2}, \quad i = 1, \ldots, n - 1 \]  
where
\[ p(\hat{\theta}_i) = \hat{\theta}_i^T \hat{\theta}_i - \theta_0^2, \]
\[ \pi_1 = \begin{cases} \left( \varepsilon^{n-i} \hat{\theta}_i \right) & \text{if } p(\hat{\theta}_i) > 0 \\ 0 & \text{otherwise} \end{cases} \]
\[ \pi_2 = \frac{1}{2} \nabla p(\hat{\theta}_i) \mu_i + \left( \frac{1}{2} \nabla p(\hat{\theta}_i) \mu_i \right)^2 + \delta^2. \]
\( \mu_i \) was defined in (29), \( \nabla \) is the gradient operator, \( \varepsilon, \delta \) are arbitrary positive constants, and \( \theta_0 \) was defined in assumption A3. It can be proven that the above projection operator has the following properties [2]:

**P1.** \[ \|\hat{\theta}_i(t)\| \leq \theta_0 + \varepsilon \forall t \geq 0; \]

**P2.** \[ \hat{\theta}_i^T \text{Proj}(\mu_i, \hat{\theta}_i) \delta \geq \hat{\theta}_i^T \mu_i; \]

**P3.** \[ \text{Proj}(\mu_i, \hat{\theta}_i) = \gamma_{\mu i} + \gamma_{bi} \quad \text{and} \quad \|\text{Proj}(\mu_i, \hat{\theta}_i)\| \leq \|\mu_i\| \left[ 1 + \left( \frac{\theta_0 + \varepsilon}{\theta_0} \right)^2 \right] + \frac{\theta_0 + \varepsilon}{\theta_0^2} \delta, \]  
where
\[ \gamma_{\mu i} = \mu_i - \frac{\pi_1 (\pi_2 - \delta) \nabla p(\hat{\theta}_i)}{4(\varepsilon^2 + 2\varepsilon \theta_0)^{n-1} \theta_0^2}, \]
\[ \gamma_{bi} = -\frac{\pi_1 \delta \nabla p(\hat{\theta}_i)}{4(\varepsilon^2 + 2\varepsilon \theta_0)^{n-1} \theta_0^2}, \]
and
\[ \|\gamma_{\mu i}\| \leq \|\mu_i\| \left[ 1 + \left( \frac{\theta_0 + \varepsilon}{\theta_0} \right)^2 \right], \]
\[ \|\gamma_{bi}\| \leq \frac{\theta_0 + \varepsilon}{\theta_0^2} \delta. \]  

**P4.** \[ \text{Proj}(\mu_i, \hat{\theta}_i) \in \mathbb{C}^{n-i-1}. \]