Rejection of sinusoids from nonlinearly perturbed uncertain regular linear systems

Vivek Natarajan and Joseph Bentsman, Member, IEEE

Abstract — An application-motivated system class - nonlinearly perturbed regular linear systems (NPRLS) – is considered, and the response of the latter to periodic inputs is characterized. A recently proposed robust control scheme for tracking bandlimited periodic signals by uncertain exponentially stable regular linear systems (RLS) is then applied to an uncertain system belonging to the NPRLS class, whose linearization is an exponentially stable RLS, to reject internally generated sinusoids from the system output. Assuming the uncertain NPRLS to be unknown, but its gain at the frequency of interest known and bounded away from zero, and using the aforementioned characterization, the stability and disturbance rejection of the resulting topology are shown to be guaranteed for sufficiently small nonlinearity.

Index Terms—Internal model principle, regular linear systems, nonlinear perturbation, periodic response.

I. INTRODUCTION

Internal model principle [1] has been used to track and reject periodic signals in finite dimensional linear [2]-[6] and nonlinear systems [7]-[9] when the plant model is known. When the plant model is unknown, this principle has been applied using only the transfer function (TF) gains at known. When the plant model is unknown, this principle has been applied using only the transfer function (TF) gains at

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II. NOTATION AND BACKGROUND ON RLS AND NPRLS

The paper uses the following notation:

- Space of bounded linear operators from $X$ to $Y$. Let $\mathcal{L}(X,Y)$.
- $L^2([t,T),X) / L^\infty([t,T),X) / L^\infty_0([0,\infty),X)$. Space of square integrable / essentially bounded functions from $[t,T)$ to $X$ with the usual norm / Space of locally square integrable functions from $[0,\infty)$ to $X$.
- $D(A) / \rho(A)$. Domain / resolvent set of an operator $A$.
- $\|\cdot\|_X$ - Norm in space $X$.
- $\mathbb{R}/\mathbb{C}$ - Space of real/complex numbers.
- $\|\cdot\|_{L^2([0,T),X)}$ - Same as $\|\cdot\|_{L^2([0,T),X)}$ when $X$ is clear.
- $\|\cdot\|_{X,\infty}$ - Same as $\|\cdot\|_{L^\infty([0,\infty),X)}$ and $C_B([0,\infty),X)$ - Banach space of continuous bounded functions from $[0,\infty)$ to $X$ with $\|\cdot\|_{X,\infty}$ as norm.

Regular Linear Systems (RLS). The following definitions and background on RLS can be found in [15]-[17] and references therein and are reproduced below to enhance the clarity of presentation. Let $U$, $X$ and $Y$ be Hilbert spaces and $\Omega = L^2([0,\infty),U)$ and $\Gamma = L^2([0,\infty),Y)$ be input and output spaces, respectively. An abstract linear system on $\Omega$, $X$ and $\Gamma$ is a quadruple $\Sigma = (\mathcal{T},\Phi,\mathcal{L},\mathcal{F})$ where $\mathcal{T} = (T_t)_{t \geq 0}$
is a $C_0$-semigroup on $X$, $\Phi = (\Phi_t)_{t \geq 0}$, $L = (L_t)_{t \geq 0}$, $F = (F_t)_{t \geq 0}$ are families of bounded linear operators from $\Omega$ to $X$ (input to state), $X$ to $\Gamma$ (initial state to output) and $\Omega$ to $Y$ (input to output), respectively. The operators $\Phi$, $L$ and $F$ can be extended naturally to the Frechet space of inputs $\Omega = L^2_{\text{loc}}([0,\infty), U)$ and outputs $F = L^2_{\text{loc}}([0,\infty), Y)$, with the extensions denoted by $\Phi_\omega$, $L_\omega$ and $F_\omega$, respectively.

Let $A$ be the infinitesimal generator of $T$. The Hilbert spaces $X_1$ and $X_{-1}$ are defined as follows: $X_1$ is $D(A)$ with the norm $\|x\|_{X_1} = \|B(A)x\|_X$, where $B \in \rho(A)$ is fixed and $X_{-1}$ is the completion of $X$ with respect to the norm $\|x\|_{X_{-1}} = \|(B(A)^{-1}x\|_X$. The semigroup $T$ can be extended to a semigroup on $X_{-1}$ isomorphic to $T$ and will be denoted by the same symbol. For any abstract linear system, there exists an unique $B \in \mathcal{L}(U, X_{-1})$ called the control operator such that $\Phi_{t}u = \int_{0}^{t}T_{t-t_{0}}Bu(t)\,dt$, for all inputs $u \in \tilde{\Omega}$ and all $t \geq 0$. Then, for any initial state $x_0$ the state $x$ of $\Sigma$ at time $t \geq 0$ is expressed as

$$x(t) = T_{t}x_{0} + \Phi_{t}u = T_{t}x_{0} + \int_{0}^{t}T_{t-t_{0}}Bu(t)\,dt. \tag{1}$$

The state $x : [0, \infty) \to X$, as defined, is a continuous function. In general, $\hat{B} \in \mathcal{L}(U, X_{-1})$ is called a $T$-admissible control operator if the map

$$\int_{0}^{t}T_{t-t_{0}}\hat{B}u(t)\,dt \in \mathcal{L}(\Omega, X) \quad \text{for each} \quad t \geq 0. \tag{2}$$

Associated with every abstract linear system is a unique $C \in \mathcal{L}(X_1, Y)$ called the observation operator such that $\forall x \in X_1, (L_{-}x)(t) = CT_{t}x$. $L_{-}$ is completely determined by this equation due to the density of $X_1$ in $X$. The $A$-extension of $C$ is defined by

$$C_{\Lambda}x' = \lim_{\lambda \to 0} C\lambda(AI - A)^{-1}x'$$

with $\lambda$ real and for all $x'$ in the domain

$$D(C_{\Lambda}) = \{x' \in X : \text{the limit above exists}\}. \tag{3}$$

The input-to-output operator of any abstract linear system can be described by a TF which is an operator valued analytic function defined and bounded on some right half complex plane. Let $G$ denote the TF of $\Sigma$. $G$ is called regular if the following limit exists $\forall u \in U$,

$$Dv = \lim_{\lambda \to \infty} G(\lambda)v, \quad \lambda \text{ real}. \tag{4}$$

Then $D \in \mathcal{L}(U, Y)$ is called the feedthrough operator of $G$.

In this case, the state $x(t)$ given in (1) and the output of $\Sigma$,

$$y = L_{\omega}x_{0} + F_{\omega}u, \quad \text{where} \quad x_{0} \text{ is the initial state and} \quad u \in \tilde{\Omega} \text{ is the input, satisfy pointwise almost everywhere (a.e.) in time, the equation}$$

$$y(t) = C_{\Lambda}x(t) + Du(t). \tag{5}$$

If $G$ is regular, $\Sigma$ is called RLS. $(A, B, C, D)$ as introduced are the generating operators (GOs) of $\Sigma$. A RLS $\Sigma$ is called exponentially stable if the associated semigroup $T$ satisfies

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{-\alpha t}, \quad \forall t \geq 0, \quad \text{for some} \quad M \geq 1 \quad \text{and} \quad \alpha > 0.$$

In the rest of the paper, an exponentially stable RLS $\Sigma_{p} = (T, \Phi, L, F)$, with GOs $(A_{p}, B_{p}, C_{p}, D_{p})$ and TF $G_{p}(s)$ is considered.

**Nonlinearly Perturbed Regular Linear Systems (NPRLS).** The NPRLS, $\Sigma_{p,N}$, corresponding to the above RLS $\Sigma_{p}$, is defined as being represented by the differential equation

$$\begin{align*}
\dot{x}_{p} &= A_{p}x_{p} + B_{p}u + \varepsilon g(x_{p}), \\
y_{p} &= C_{p,A}x_{p} + D_{p}u,
\end{align*} \tag{6}$$

where $C_{p,A}$ is the $A$-extension of $C_{p}$, $\varepsilon > 0$ is a small constant, $g$ is a nonlinear term, and (3) satisfies the following:

**Assumption 1:** $T$, the $C_0$-semigroup on $X$ generated by $A_{p}$, is exponentially stable and satisfies

$$\|T(t)\|_{\mathcal{L}(X)} < Me^{-\alpha t} \quad \text{with} \quad a > 0 \quad \text{and} \quad M \geq 1.$$

**Assumption 2:** $g(x) : X \to X$ satisfies the Lipschitz condition that, for any bounded domain $D \subset X$, $\exists L(D)$:

$$\|g(x) - g(y)\|_{X} < L(D)\|x - y\|_{X}, \quad \forall x, y \in D \quad \text{and} \quad g(0) = 0.$$

**Assumption 3:** $(A_{p}, I_{X}, C_{p}, 0)$ are the GOs of an RLS $(T, \Phi^{\mathcal{F}}, L^{\mathcal{F}}, F^{\mathcal{F}})$ with state $X$ and input and output taking values in $X$ and $Y$ respectively. $I_{X}$ is the identity operator on $X$.

Under Assumptions 1-2, Lemma 3 shows that for a given $u \in L^{\infty}([0,\infty), U)$ and $x_{0} \in X$, $\exists \mathcal{F} : \forall \mathcal{F} < \mathcal{F}$, the state equation in (3) has a unique continuous solution, bounded in $X$, and satisfying a variation of parameters formula (like (1)). The existence proof is along the lines of the proof in chapter 6 of [18] and is sketched here since, though $B_{p} \in \mathcal{L}(U, X_{-1})$ it may not belong to $\mathcal{L}(U, X)$.

**Assumption 3** enables treating the nonlinearity as an input and renders the output equation in (3) meaningful.

III. **PROBLEM STATEMENT**

Consider an NPRLS satisfying Assumptions 1-3 that tracks the reference sinusoid of frequency $\omega$ satisfactorily, but for
the presence of an internally generated small magnitude higher harmonic at \( r\omega \), an \( r \)-integer multiple of \( \omega \).

**Objective.** Design a control scheme that eliminates the harmonic at \( r\omega \) from the NPRLS output without affecting system stability and tracking performance at frequency \( \omega \).

This problem statement is motivated by an industrial application represented in Fig. 1, where an actuator modeled as a linear finite dimensional system perturbed by a smooth nonlinearity (Section V.A [14]) is coupled to an infinite dimensional beam. In this application, the input is a sinusoid of frequency \( \omega \) and the output contains small amplitude higher harmonics, usually negligible, due to the nonlinearity. But, though small, the sinusoid at \( \omega \) in the output is particularly undesirable and must be eliminated since \( \omega \) is a resonance frequency of the beam. This was accomplished in [14] assuming a finite dimensional plant. Given the infinite dimensional dynamics of the beam, in the present work the plant model is more appropriately assumed to be a NPRLS \( \Sigma_{P,N} \) with \( U = Y = \mathbb{R} \).

**Fig. 1. Schematic representation of motivating application**

**IV. RESPONSE OF NPRLS TO PERIODIC excitation**

This section contains a series of lemmas that characterize the response of NPRLS to periodic excitation. Consider \( \Sigma_P \) and its GOs as defined in Section II. Since \( T \) is exponentially stable (Assumption 1), \( \Phi : \Omega \to X \) is continuous for all \( t \) and \( \|\Phi_t(u)\|_X \leq N\|u\|_{L^2[0,1]} \), \( \forall u \in \tilde{\Omega} \) [19]. For bounded input \( u \), a bound on \( \Phi_t(u) \) depending on \( \|u\|_{U,\infty} \) is desired and can be given by the following lemma.

**Lemma 1:** If \( u \in L^\infty([0,\infty),U) \), then \( \|\Phi_t(u)\|_X \leq N\|u\|_{L^2[0,1]} \) for some fixed \( N > 0 \) independent of \( t \).

**Proof:** Consider the exponentially stable \( C_0 \)-semigroup \( \tilde{T}_t = \text{e}^{at} \) generated by \( A_p + bI \) with \( 0 < b < a \). Fix any \( t \geq 0 \) and \( v \in \Omega \). Then, \( \Phi_t v = \int_0^t \tilde{T}_{t-s} B_p v(t) \, dt = \int_0^t \tilde{T}_s B_p \left( e^{b(t-s)} v(t) \right) \, dt \)

\[
= \int_0^t \tilde{T}_{t-s} B_p v(t) \, dt = \Phi_t v,
\]

where \( v(t) = e^{b(t-t)} v(t) \in L^2([0,t],U) \). \( \Phi_t \in \mathcal{L}(\Omega,X) \), since \( \|\Phi_t v\|_X = \|\Phi_t v\|_0 \leq N\|v\|_{L^2[0,1]} \leq N e^{bt} \|v\|_{L^2[0,1]} \) for any \( v \in \Omega \). Since this holds for all \( t \), \( B_p \) is a \( \tilde{T} \)-admissible control operator and \( \tilde{T}, \Phi \) is an abstract linear control system [19] where \( \tilde{\Phi} = (\tilde{T}, \Phi) \) is an abstract linear control system. Moreover, since \( \tilde{T} \) is exponentially stable, \( \tilde{\Phi} \) is uniformly bounded with \( \|\tilde{\Phi}_t\|_{\mathcal{L}(\Omega,X)} \leq \tilde{N} \). Therefore,

\[
\|\Phi_t u\|_X = \|\tilde{\Phi}_t e^{b(t-t)} u(t)\|_X \leq \tilde{N}\|\tilde{\Phi}_t e^{b(t-t)} u\|_{L^2[0,1]} \leq \tilde{N}\|u\|_{U,\infty}
\]

and the lemma is proven. \( \Box \)

**Lemma 2:** Let \( u : [0,\infty) \to U \) and \( v : [0,\infty) \to X \) be continuous \( T \)-periodic functions. Then under Assumption 1, \( w_u(t) = \int_0^t \tilde{T}_{t-s} B_p u(t) \, dt \) and \( w_v(t) = \int_0^t \tilde{T}_{t-s} v(t) \, dt \) converge in \( \|\cdot\|_X \), asymptotically, to continuous \( T \)-periodic functions. The rate of convergence depends on \( \|u\|_{U,\infty} \) and \( \|v\|_{U,\infty} \) but not on the particular functions.

**Proof:** Since \( B_p \in \mathcal{L}(U,X_{-1}) \), the integration for \( w_u \) is in \( X_{-1} \). That \( w_u(t) \in X \), \( \forall t \), and the continuity of \( w_u \) in \( X \) follows from the properties of RLS. Using Lemma 1 and a change of variables, it follows that for \( n > m \), \( p = n - m \) and \( \forall s \in [0,T] \),

\[
\left\| \int_0^T \int_0^{T_{nT+s-t}} B_p u(t) \, dt \right\|_{X} = \left\| \int_0^T \int_0^{T_{nT+s-t}} B_p u(t) \, dt \right\|_{X} = \left\| \int_0^T \int_0^{T_{nT+s-t}} B_p u(t) \, dt \right\|_{X} = \left\| \int_0^T \int_0^{T_{nT+s-t}} B_p u(t) \, dt \right\|_{X}
\]

Hence the sequence of continuous functions \( f_n \) defined as \( f_n(s) = w_u(nT+s) \), \( \forall s \in [0,T] \), is Cauchy in the supremum norm. Hence \( f_n(s) \) converges to a function \( f(s) \) uniformly \( \forall s \in [0,T] \), which implies that \( w_u(t) \) converges to a continuous \( T \)-periodic function asymptotically. A similar argument establishes the result for \( w_v \). The claim about the rate of convergence follows from the proof. \( \Box \)

Lemma 3 concludes that the state of the periodically excited NPRLS in (3) is periodic, asymptotically, with the same period.
Lemma 3: Consider the RLS $\Sigma_p$, the corresponding NPRLS $\Sigma_{p,N}$ given in (3) and the associated nonintegrable integral map $q = \mathbb{I}(p)$ defined $\forall t \in [0,\infty)$ as:

$$q(t) = p(t) - \int_0^t \mathcal{T}_x x_0 - \int_0^t \mathcal{T}_{-\tau} B_p u(t) d\tau - \varepsilon \int_0^t \mathcal{T}_{-\tau} g(p(\tau)) d\tau.$$ 

Let $\|x_0\|_X < C_0$ and $u : [0,\infty) \rightarrow U$ be a continuous $T$-periodic function. Let Assumptions 1-2 hold. Then, since $\Sigma_p$ is exponentially stable, $\|\Phi(t)u\|_X \leq N\|u\|_U$ for some $N > 0$ and $\forall t$. Hence the map $I$ is well defined from $C_B([0,\infty),X)$ to itself. Assume that $\varepsilon$ is small so that for some $0 < \varepsilon < 1$ and $D_0 = \{x \in X : \|x\|_X \leq (C_0 + (MC + \bar{N}u)\varepsilon)\}|/(1-\varepsilon) = d_0\}$, $(\varepsilon M d_0)/\alpha < \varepsilon$. Here $L_0$ is the Lipschitz constant of $g(x)$ for the domain $D_0$. Then, $\exists x_p, \hat{x}_p \in C_B([0,\infty),X)$ with $I(x_p) = 0$ and $\hat{x}_p$ a $T$-periodic function such that for any given $\lambda > 0$, $\exists \gamma > 0$ such that $\|\hat{x}_p[t,\infty) - \hat{x}_p[t,\infty)\|_X < \lambda$.

Proof: The main ideas of the proof are sketched below. Step 1 is to establish the existence of a unique solution to $I(p) = 0$, which would then be the variation of parameters solution to (3). This is done by first showing that if the initial condition satisfies $\|x_0\|_X < d_0$, a unique solution $x_p(t)$ exists for $t \in [0,\sigma)$. Next, arguing that $\|x_p(t)\|_X < d_0$, $\forall t \in [0,\sigma]$, enables extending the solution to $[\sigma,2\sigma]$. Repeating this argument shows that $x_p(t)$ exists $\forall t \in [0,\infty)$. Step 2 proves asymptotic convergence of $x_p(t)$ to a $T$-periodic function, $\hat{x}_p(t)$. Let $\int_0^t \mathcal{T}_{-\tau} B_p u(t) d\tau$ and $\int_0^t \mathcal{T}_{-\tau} g(v_0(\tau)) d\tau$ converge asymptotically to $T$-periodic functions $v_0(t)$ and $v_1(t)$, respectively. For $k > 1$, let $v_k$ be the $T$-periodic function to which $\int_0^t \mathcal{T}_{-\tau} g(v_0(\tau) + v_{k-1}(\tau)) d\tau$ converges asymptotically.

The sequence of $T$-periodic functions $\{v_0 + v_1\}$ is shown to be Cauchy in $C_B([0,\infty),X)$ and its limit $\hat{x}_p$ is the function to which $x_p(t)$ converges asymptotically. □

Remark 1: With $x_p(t)$ and $\hat{x}_p(t)$ as in Lemma 3, it can be shown that for a fixed $0 < \delta < 1$, a $T$ can be found such that $\forall t > \delta T$, $\|\hat{x}_p(t) - x_p(t)\|_X < \delta^n$ for every integer $n > 0$. Hence $\hat{x}_p(t) - x_p(t) \in L^2((0,\infty),X)$.

If $C_p \in \mathcal{L}(X,Y)$ in (3), it follows from Lemma 3 that $y(t)$ is continuous and tends to a $T$-periodic function asymptotically. But when $C_p \in \mathcal{L}(X_1,Y_1)$, as is the case for a general RLS, Lemma 4 clarifies the behavior of $y(t)$.

Lemma 4: Let the assumptions in Lemma 3 hold. Then the output $y_p(t)$ of (3) can be written as $y_p(t) = y_{1,p}(t) + y_{nl,p}(t) + y_{0,p}(t)$ where $y_{1,p}$ and $y_{nl,p}$ are the $T$-periodic components of the output in the absence and presence of the nonlinearity, respectively and $y_{0,p} \in L^2((0,\infty),Y)$.

Proof: From Remark 1 and Assumptions 1-3, since $x_p(t)$ is known and bounded, the NPRLS (3) can be regarded as a RLS with 2 inputs, $u$ and $\varepsilon g(x_p)$. Let $n$ satisfy $\|T(nT)\|_{\mathcal{L}(X)} < 1$. Consider the system

$$\hat{x}_p = A_p x_p + B_p u + \varepsilon g(x_p), \quad y_p = C_p x_p + D_p u, \quad \hat{x}_p(0) = (I - T(nT))^{-1}(\Phi n u + \Phi n^\varepsilon g(x_p)).$$

Equations (3) and (4) differ in the initial conditions and the second input. But the difference in the second input is in $L^2((0,\infty),X)$ since $\|\hat{x}_p - x_p\|_X \leq L_0 \|\hat{x}_p - x_p\|_X$, (use Remark 1), $\|\hat{x}_p\|_{L^2,X} \leq d_0$, $\|y_p\|_{L^2,X} \leq d_0$. From the properties of exponentially stable RLS the difference in the outputs of (3) and (4), $y_{0,p} \in L^2((0,\infty),Y)$ clearly, $\|y_{nl,p}\|_{L^2,Y} < \delta$ for every integer $n > 0$. Hence $\hat{x}_p(t) - x_p(t) \in L^2((0,\infty),X)$.

V. CONTROLLER DESIGN

In Theorem 1, the controller designed in [13] for tracking of finite number of sinusoids by RLS is applied to the NPRLS in (3) to address the problem statement of Section III. In the present section, the input and outputs take values in $\mathbb{R}$, i.e. $U = Y = \mathbb{R}$. For all TFs $G(s)$ and $\forall \tau \in \mathbb{R}$, it is assumed that $G(-j\tau)$ is the complex conjugate of $G(j\tau)$ in Lemmas 3 and 4, if $u$ is a sinusoid of period $T$, then using the theory of TFs for RLS [17], $y_{1,p}$, the $T$-periodic component of the output in the absence of the nonlinearity, can be shown to be a sinusoid
Theorem 1: Consider the RLS $\Sigma_p$ and the NPRLS $\Sigma_{p,N}$ 

\[
\begin{bmatrix}
\begin{array}{c}
A_p \\
B_p \\
C_p \\
D_p \\
\end{array}
\end{bmatrix}
\begin{bmatrix}
X_p \\
Y_p \\
U_p \\
V_p
\end{bmatrix}
\begin{bmatrix}
\Lambda_p \\
\Lambda_{p,N} \\
\Lambda_{p,F} \\
\Lambda_{p,T}
\end{bmatrix}
\] 

such that if $\epsilon < \epsilon^*$ the augmented system in Fig. 3, now 

\[
\begin{align*}
\Sigma_{p,N} & : \begin{cases}
Ax + Bu & = g(x) + x \in L^2([0,\infty), \mathbb{R}) \\
y & = Cx + Du
\end{cases} \\
\Sigma_p & : \begin{cases}
Ax & = g(x) + x \in L^2([0,\infty), \mathbb{R}) \\
y & = Cx
\end{cases}
\end{align*}
\]

\[
\begin{bmatrix}
\begin{array}{c}
A_p \\
B_p \\
C_p \\
D_p \\
\end{array}
\end{bmatrix}
\begin{bmatrix}
X_p \\
Y_p \\
U_p \\
V_p
\end{bmatrix}
\begin{bmatrix}
\Lambda_p \\
\Lambda_{p,N} \\
\Lambda_{p,F} \\
\Lambda_{p,T}
\end{bmatrix}
\] 

is stable and has a unique output that can be written as $y_p^c = y_{f,p}^c + y_{u,p}^c + y_{0,p}^c$ where $y_{f,p}^c$, a sinusoid of frequency $\omega_0$, and $y_{u,p}^c + y_{0,p}^c$, a T-periodic function, are periodic components of the output of $\Sigma_{p,N}$ in the absence and presence of the nonlinearity, respectively and $y_{0,p}^c \in L^2([0,\infty), \mathbb{R})$. The $L^2$-norm over a period for $y_{u,p}^c$

Fig. 2. Unaugmented system

Fig. 3. Augmented system

Proof: Let $G_p(j\omega)$ be the TF gain of $\Sigma_p$ at $\omega$. From Lemma 4 $\left|G_p(j\omega) - G_{p,N}(j\omega)\right|$ is small for small $\epsilon$, and hence $\left|1-G_p(j\omega)G_{p,N}(j\omega)\right| < 1$. From Theorem 1 in [13] ($R$ is the same as 1-Q in [13]; The additional assumption in this paper that $R$ be strictly proper does not alter the results in [13]) it follows that $\exists \epsilon^*: \forall \epsilon \in (0, \epsilon^*)$, the augmented system in Fig. 3 with $\Sigma_{p,N}$ replaced by $\Sigma_p$, denoted as $\Sigma_p(\epsilon)$, is an exponentially stable RLS; the superscript $c$ indicates ‘closed loop’. The state space and GOs for $\Sigma_p(\epsilon)$ are $x^c = X^c \times \mathbb{R}^2 \times \mathbb{R}^3$ and $\left(A_p^c(\epsilon), B_p^c(\epsilon), C_p^c, D_p^c\right)$ respectively, with input $u$ and output $y_p$ where:

$$A_p^c = \begin{bmatrix}
A_p & 0 & B_pC_F \\
B_p^c & A_p & B_pD_pC_F \\
0 & -\epsilon B_p^c & A_p(\epsilon) + \epsilon B_p^cC_F
\end{bmatrix},
B_p^c = \begin{bmatrix}
B_p \\
B_pD_p \\
\epsilon B_p^c
\end{bmatrix},$$

Choose $R$ so that its strictly proper TF, $G_R$, satisfies $\left|1-G_{p,N}(j\omega_0)G_R(j\omega_0)\right| = 0$. In the augmented system (Fig. 3), $F$ is the linear stable SISO system with TF

$$G_F(s) = \frac{\epsilon^2}{s^2 + 2\epsilon \omega_0 s + \omega_0^2},$$

with $0 < \epsilon < 1$, a parameter to be chosen. Let the GOs of $F$ be $(A_F(\epsilon), C_F, F, 0)$ where $A_F(\epsilon)$ is a linear function of $\epsilon$, and $B_F$ and $C_F$ are independent of $\epsilon$. Then, for small $\epsilon$, $\exists \epsilon^*: \forall \epsilon \in (0, \epsilon^*)$ the augmented system in Fig. 3 with $\Sigma_{p,N}$ replaced by $\Sigma_p$ is an exponentially stable RLS.
is small (proportional to $\varepsilon$). Comparing the TFs, from $u$ to $y_p$, of $\Sigma_P(\zeta)$ and $\Sigma_P(\zeta)$ and noting that the gain of $F$ at all frequencies away from $\omega_0$ (like $\omega$) is small (proportional to $\zeta$), it can be shown that $y_{\xi,p}^c - y_{i,p}$ is small, implying that the augmented system tracks the input sinusoid of frequency $\omega$ akin to the unaugmented system. The loop containing $F$ in Fig. 3 can be replaced by $1/(1-F)$, which has poles at $\pm j\omega_0$, with output $m = u - C_R x_R + C_F x_F$ and input $u - y_R$. Hence the presence of the harmonic at $\omega_0$ in $y_{nl,p}^c$ and so in $y_R$ would lead to $m$ becoming unbounded. Since this contradicts the fact that the states of the stable NPRLS $\Sigma_{P,N}$ are bounded, $y_{nl,p}^c$ must have no harmonic at $\omega_0$.

In Lemma 3, the characterization of stability and periodic response depended on the nonlinearity ($\varepsilon$ and $L$) and the growth bounds on the semigroup $(M$ and $a)$. Let $M(\zeta) = 2\sup_{t \geq 0} \|T^c_t(\zeta)\|_{L^\infty(X)}$ and $a(\zeta)$ the largest positive number such that $\|T^c_t(\zeta)\|_{L^\infty(X)} \leq M(\zeta)e^{-a(\zeta)t}$. When $\zeta \neq 0$, one can use the perturbation theory of $C_0$-semigroups [18] to show that the growth bounds for $T^c_t(\zeta)$ change continuously in the interval $(0,\lambda^*]$. Consequently, there exists a continuous function of $\zeta$, $\varepsilon_m(\zeta) > 0$, such that for each $\zeta \in (0,\lambda^*)$ if $\varepsilon < \varepsilon_m(\zeta)$, then the augmented system is stable. But at $\zeta = 0$, the perturbation theory cannot be used. Instead a transfer function based approach can be used to show that in a neighborhood of $\zeta = 0$, $\varepsilon_m(\zeta) \geq \varepsilon > 0$. Hence $\varepsilon > 0$ such that if $\varepsilon < \varepsilon^*$, then for each $\zeta \in (0,\lambda^*)$ the augmented system in Fig. 3 (with $\Sigma_{P,N}$) is stable. $\Box$

VI. CONCLUSION

Nonlinear perturbations are considered in the framework of regular linear systems (RLS) and the stability and response of the resulting NPRLS to periodic excitation are characterized. On this basis, analysis of the stability and disturbance rejection performance of a control scheme developed for RLS when applied to a NPRLS is carried out. It is shown that for sufficiently small perturbations, the feedback loop is stable and eliminates the internally generated harmonic, while preserving the plant response at the input frequency.

REFERENCES


