System transformation of unstable systems
induced by a shift-invariant subspace

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Abstract—Given an inner function, the orthogonal complement of the corresponding shift invariant subspace induces a system transformation for linear time-invariant systems, which is a generalization of the lifting technique for the sample-data control and Hambo-transform in the sense the inner function is arbitrary. This paper extends the transformation for systems with unstable eigenvalues, and derives a unified formula for transformation operators for both stable and anti-stable systems. A potential application is in the area of closed-loop system identification, where an unstable system is identified under the stabilizing feedback connection. The application to closed-loop system identification will be presented elsewhere.

I. INTRODUCTION

Transformation is a versatile tool in the systems and control theory. The Fourier transform and the Laplace transform are essential in developing the theory. For example, it is a common knowledge that the Laplace transform of signals for a system described by a linear differential equation with constant coefficients leads to the notion of transfer functions.

A shift invariant subspace of the space of square integrable functions \( L_2(0, \infty) \) is an important notion. For example, the lifting technique in sampled-data control [1], [12] uses the orthogonal complement to define a “lifted system,” which becomes a fundamental tool to study the \( H^2 \) and \( H^\infty \) control problems. Another example is the Hambo transform induced by the generalized orthonormal basis functions [4], [5]. Actually, the two notions coincide when we work in an abstract way to define the system transform [8] using the orthogonal complement of a shift-invariant subspace corresponding to an inner function. Indeed, if the inner function is pure delay, it yields the lifting technique for sampled-data control, and if the inner function is rational, it yields the Hambo transform. Thus we can regard the transformation as a generalization of the lifting technique.

This extended version of the lifting technique has a number of applications. One such instance is the \( H^\infty \) control problem for a class of infinite dimensional systems. For such a class, so-called Hamiltonian formula characterizing the minimal achievable \( H^\infty \) norm was derived in [13], [7], [9]. The approximation by a lower order model for of high order systems are studied using the transformation and filtered signals [3], [9]. Another important area is the system identification. It was shown in [10] that the transformation can be extended to stochastic systems where the signals are random processes and that standard subspace algorithms such as MOESP [11] can be employed.

The purpose of the paper is to extend the transformation method for systems having unstable eigenvalues. Most of the works in the literature assumes that the system is stable [5], [10], with an exception of [9] where the transformation of the adjoint system of a stable system was considered. However, the formulae for stable and anti-stable systems are seemingly different.

This paper derives a unified formula for transformation operators for both stable and anti-stable systems. Moreover, we consider a feedback connection and prove that the transformation of the feedback system is the feedback of the transformed systems. An application for such results is the closed-loop system identification problem, and it will be discussed in elsewhere [2].

II. PRELIMINARIES

A. Signal spaces in time and frequency domains

Let \( L^2(j\mathbb{R}) \) be the space of square integrable functions of frequency \( j\omega \in j\mathbb{R} \) with the inner product

\[
\langle u, v \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(j\omega)u(j\omega) \, d\omega.
\]

The space \( H^2 \) is the space of analytic functions in the right half plane with the norm

\[
\|u\| = \sup_{\nu > 0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |u(\nu + j\omega)|^2 \, d\omega \right)^{1/2} < \infty. \tag{1}
\]

If \( u \in H^2 \), then non-tangential limits exists at almost every points on the imaginary axis, and the boundary value function is in \( L^2(j\mathbb{R}) \). With this relation, we identify \( H^2 \) as a subspace of \( L^2(j\mathbb{R}) \). It can be shown that the orthogonal complement of \( H^2 \) in \( L^2(j\mathbb{R}) \) is the space of analytic functions in the left half plane with the norm similarly defined as (1) except that \( \nu < 0 \), which is denoted as \( H^2_+ \).

The Fourier transform is the isomorphism between the signal spaces of time and frequency domains. The space \( L^2(-\infty, \infty) \) of square integrable functions of time \(-\infty < t < \infty \) is isomorphic to \( L^2(j\mathbb{R}) \) via the Fourier transform. Similarly, the spaces \( L^2(0, \infty) \) and \( L^2(-\infty, 0) \) of square integrable functions of time \( 0 < t < \infty \) and \(-\infty < t < 0 \) are isomorphic to \( H^2 \) and \( H^2_+ \), respectively.

B. Multiplicative operator

A bounded function on the imaginary axis induces a multiplication operator on \( L^2(j\mathbb{R}) \) as a transfer function. The space of such bounded functions is \( L^\infty \) with the norm

\[
\|h\|_\infty = \text{ess sup}_{\omega} |h(j\omega)|.
\]
For the sake of simplicity, we denote the multiplicative operator induced by a function \( h \in L^\infty \) as \( h \) as well. It is easy to verify that the induced norm of the multiplicative operator \( h : L^2(j\mathbb{R}) \to L^2(j\mathbb{R}) \) is equal to \( \|h\|_\infty \).

The space of bounded analytic functions in the right half plane is \( H^\infty \). If \( h \in H^\infty \), then the multiplicative operator \( h : L^2(j\mathbb{R}) \to L^2(j\mathbb{R}) \) leaves \( H^2 \) invariant.

A function \( \phi \in H^\infty \) is called inner if \( |\phi(j\omega)| = 1 \) for almost all \( \omega \). Let \( \phi^\sim(s) = \phi(-\tilde{s}) \) be the para-conjugate of \( \phi \). Then the statement \( \phi(s)\phi^\sim(s) = 1 \) is equivalent to \( \phi \) is inner. Furthermore, \( \phi \) as a multiplicative operator on \( L^2(j\mathbb{R}) \) is unitary.

C. Shift-invariant subspace and its orthogonal complement

If \( \phi \in H^\infty \) is inner, then the space \( \phi H^2 \) is a closed subspace of \( H^2 \), called shift-invariant subspace. The orthogonal complement of \( \phi H^2 \) with respect to \( H^2 \) is denoted as \( S = H^2 \ominus \phi H^2 \), which plays an instrumental role in the subsequent discussion.

If \( \phi \) is a rational inner function, say
\[
\phi(s) = \frac{(p_1 - s) \cdots (p_r - s)}{(p_1 + s) \cdots (p_r + s)},
\]
with distinct zeros, then \( S \) is spanned by
\[
\left\{ \frac{1}{p_1 + s}, \ldots, \frac{1}{p_r + s} \right\}.
\]
If \( \phi(s) = e^{-hs}, h > 0 \), then the subspace \( S \) is nothing but the image of the Fourier transform of the squarely integrable functions supported to the interval \((0, h)\).

It is obvious that the spaces \( L^2(j\mathbb{R}), H^2 \), and \( H^2_\perp \) have the following decompositions:
\[
L^2(j\mathbb{R}) = \oplus_{k=-\infty}^{\infty} \phi^k S, \quad H^2 = \oplus_{k=0}^{\infty} \phi^k S, \quad H^2_\perp = \oplus_{k=-\infty}^{-1} \phi^k S,
\]
where \( \phi^k = (\phi^\sim)^{-k} \) if \( k < 0 \).

From (2), any \( u \in L^2(j\mathbb{R}) \) has the expression
\[
u = \sum_{k=-\infty}^{\infty} \phi^k u_k, \quad u_k \in S.
\]
Furthermore, \( \|u\|^2 = \sum_{k=-\infty}^{\infty} \|u_k\|^2 \). In this sense, we can identify \( L^2(j\mathbb{R}) \) and \( \ell^2(S) \).

If the signal \( u \) is vector-valued, we can apply the transformation component-wise. Thus (2) and (3) are valid if \( u_k \) is interpreted as an \( S \)-valued vector function.

III. TRANSFORMATION OF SYSTEMS

A. Transformed system

Consider a linear system
\[
\frac{d}{dt} x = Ax + Bu
\]
\[
y = Cx + Du,
\]
where \( A \in \mathbb{R}^{n \times n} \), and \( B, C, \) and \( D \) are matrices of compatible sizes. The system (4), (5) is denoted as \((A, B, C, D)\) for simplicity. We assume that \( A \) does not have eigenvalues on the imaginary axis. Then the transfer function
\[
h(s) = D + C(sI - A)^{-1} B
\]
does not have poles on the imaginary axis, either. Hence \( h \in L^\infty \), and it defines a multiplicative operator on \( L^2(j\mathbb{R}) \).

Because \( u \) and \( y \) are in \( L^2(j\mathbb{R}) \), the isomorphism between \( L^2(j\mathbb{R}) \) and \( \ell^2(S) \) induces a bounded map \( h_D \) by the commutative diagram:
\[
\begin{array}{ccc}
L^2(j\mathbb{R}) & \xrightarrow{h} & L^2(j\mathbb{R}) \\
\downarrow & & \downarrow \\
\ell^2(S) & \xrightarrow{h_D} & \ell^2(S)
\end{array}
\]
The following theorem (Theorem 1) shows that the map \( h_D \) is shift-invariant and has a state space realization
\[
\begin{alignat}{2}
\xi_{t+1} &= A\xi_t + Bu_t, & \quad \xi_t &\in \mathbb{R}^n, \\
y_t &= C\xi_t + Du_t. & \quad y_t &\in \mathbb{R}.
\end{alignat}
\]

Theorem 1: Let \( h : L^2(j\mathbb{R}) \to L^2(j\mathbb{R}) \) be defined by the state space representation (4)-(5). Suppose that \( \phi \) and \( \phi^\sim \) are analytic at the spectrum of \( A \). Then the map \( h_D : \ell^2(S) \to \ell^2(S) \) defined by (7) has the realization (8)-(9), where the operators \( A : \mathbb{R}^n \to \mathbb{R}^n, B : S \to \mathbb{R}^n, C : \mathbb{R}^n \to \mathbb{S}, \) and \( D : S \to S \) are defined by
\[
\begin{alignat}{2}
A\xi &= \phi^\sim(A)\xi, & \quad A\xi &\in \mathbb{R}^n, \\
Bu &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \phi^\sim(A)(j\omega I + A)^{-1} B \\
&\quad - \phi(j\omega)(j\omega I + A)^{-1} B \right) u(j\omega) d\omega, & \quad Bu &\in \mathbb{R}^n \\
(C\xi)(s) &= \left( C(sI - A)^{-1} \right) \xi, & \quad (C\xi)(s) &\in \mathbb{R}^n \\
(Du)(s) &= h(s)u(s) - \phi(s)C(sI - A)^{-1} Bu, & \quad (Du)(s) &\in \mathbb{R}^n
\end{alignat}
\]

Proof. If the matrix \( A \) is stable, the theorem was proved in [9] except for the expression of the operator \( B \). From [9, Lemma 3],
\[
Bu = \int_{-\infty}^{\infty} \exp(-At)B(F^{-1}\phi^\sim u)(t)dt
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( -j\omega I - A \right)^{-1} B\phi^\sim(j\omega) u(j\omega) d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} -\phi(j\omega)(j\omega I + A)^{-1} Bu(j\omega) d\omega
\]
where \( F^{-1} \) is the inverse Fourier transform. Note that \( \phi^\sim(A)(sI + A)^{-1} B \in H^2_\perp \), and hence it is orthogonal to \( u \in H^2 \). This implies that the operator \( B \) is given by (11).

If the matrix \( A \) is anti-stable, the system \((-A, -B, C, D)\) is a stable system whose transfer function is \( h(-s) = D - C(sI + A)^{-1} B \). Note that this corresponds to reversing the time axis. Let \( \tilde{u} \) be the Fourier transform of the reversed
signal of the inverse Fourier transform of \( u \in L^2(j\mathbb{R}) \), i.e., 
\[ \hat{u}(j\omega) = u(-j\omega). \] 
Express \( u \) as in (3). Then 
\[ \hat{u} = \sum_{k=-\infty}^{\infty} \phi^k \hat{u}_k, \quad \hat{u}_k = \tau u_{-k+1} \quad (14) \]
where \( \tau : S \rightarrow S \) is defined by 
\[ (\tau u)(s) = \phi(s)u(-s), \quad u \in S. \] 
(15)
Notice that \( \tau^2 = I \). Suppose that \( u \) and \( y \) are the input and the output of the system \((A, B, C, D)\), respectively. Then \( \hat{u} \) and \( \hat{y} \) are the input and the output of \((-A, -B, C, D)\), respectively, where \( \hat{y} \) is defined similarly. Let \( A_s, B_s, C_s, \) and \( D_s \) be the operators defined for the stable system \((-A, -B, C, D)\), or
\[
\begin{align*}
A_s &= \phi^\sim (-A) = \phi(A) \\
B_s u &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( -\phi^\sim (-A) (j\omega I - A)^{-1} B + \phi(j\omega) (j\omega I - A)^{-1} B \right) u(j\omega) d\omega, \\
(C_s \xi)(s) &= \left( C (sI + A)^{-1} - \phi(s)C (sI + A)^{-1} \phi^\sim (-A) \right) \xi, \\
(D_s u)(s) &= h(-s)u(s) - \phi(s)C (sI + A)^{-1} B_s u.
\end{align*}
\]
(16)
Putting \( \xi_t = \zeta_{-t+2} \) and substituting (14), we have
\[
\begin{align*}
\xi_t &= A_s \xi_{t+1} + B_s \hat{u}_{t-1} \\
\tau y_t &= C_s \xi_{t+1} + D_s \hat{u}_{t+1}.
\end{align*}
\]
(17)
Since \( A \) is analytic at \( \phi^\sim \), \( A_s \) is invertible. Hence the input \( u \) and the output \( y \) satisfy
\[
\begin{align*}
\xi_{t+1} &= A_s^{-1} \xi_t - A_s^{-1} B_s \tau u_t \\
y_t &= \tau C_s \xi_{t+1} + \tau D_s \tau u_t \\
&= \tau C_s A_s^{-1} \xi_t + (\tau D_s \tau - \tau C_s A_s^{-1} B_s \tau) u_t.
\end{align*}
\]
From this, we have
\[
\begin{align*}
A \xi &= A_s^{-1} \xi = \phi^\sim (-A) \xi \\
B u &= -A_s^{-1} B_s \tau u
\end{align*}
\]
\[
\begin{align*}
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} -\phi^\sim (-A) (j\omega I - A)^{-1} B + \phi(j\omega) (j\omega I - A)^{-1} B \phi(-j\omega) u(j\omega) d\omega, \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi^\sim (-A) (j\omega I + A)^{-1} B - \phi(j\omega) (j\omega I + A)^{-1} B u(j\omega) d\omega,
\end{align*}
\]
(18)
which turns out the same formulae as for a stable matrix. If the matrix \( A \) has both stable and anti-stable eigenvalues, then there is a non-singular matrix \( T \) such that 
\[
A = T \begin{bmatrix} A_s & 0 \\ 0 & A_a \end{bmatrix} T^{-1},
\]
where \( A_s \) is stable and \( A_a \) is anti-stable. From the block diagonal structure, we conclude that the operators \( A, B, C, \) and \( D \) are written exactly the same. Q.E.D.

Remark 1: Notice that the formulae for the operator \( B \) are seemingly different for stable and anti-stable systems in [9]. Theorem 1 uses the frequency domain, which proves useful for unifying the formulas for both stable and anti-stable systems.

B. Inverse

Consider the system \((A, B, C, D)\) with the state space realization (4),(5) having the transfer function \( h(s) \) as in (6). Let \( (A, B, C, D) \) be the operators of the transformed system.

Let \( K \) be a matrix of appropriate size. Then the operators of the transformed systems for the transfer functions \( Kh(s), h(s)K, \) and \( K + h(s) \) are easily derived as follows:

It is obvious that the transfer function \( Kh(s) \) has a realization \((A, B, KC, KD)\), and the corresponding operators \((A, B, KC, KBD)\). The transfer function \( h(s)K \) has a realization \((A, BK, C, DK)\), and the corresponding operators \((A, BK, C, BDK)\). The transfer function \( K + h(s) \) has a realization \((A, B, C, K + D)\), and the corresponding operators \((A, B, C, K + D)\).

Less obvious is the inverse of a system. The following lemma shows that the inverse of the transformed system is the transformation of the inverse system.

Lemma 1: Consider the system \((A, B, C, D)\) with the state space realization (4),(5) having the transfer function \( h(s) \) as in (6). Assume that \( D \) is invertible, \( h(s)^{-1} \) has the realization \((A_-, B_-, C_-, D_-) = (A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})\). Assume that \( A \) and \( A_- \) do not have eigenvalues on the imaginary axis. Let \( \phi \) be an inner function, and \( S = H^2 \cap \phi H^2 \). Assume that \( \phi \) and \( \phi^\sim \) are analytic at the spectra of \( A \) and \( A_- \).
Let \((A, B, C, D)\) and \((A_-, B_-, C_-, D_-)\) be the operators (10)-(13) for the systems for the systems \((A, B, C, D)\) and \((A_-, B_-, C_-, D_-)\), respectively. Then \(D\) is invertible. Furthermore \(A_- = A - BD_-^{-1}C, \ B_- = BD_-^{-1},\ C_- = -D_-^{-1}C, \text{ and } D_- = D^{-1}.

**Proof** First, a straightforward calculation shows that
\[
(D - Du)(s) = h^{-1}(s)h(s)u(s) - \phi(s)h^{-1}(s)C(sI - A)^{-1}Bu
+ \phi(s)D^{-1}C(sI - A_-)^{-1}B_-Du
= u(s) - \phi(s)D^{-1}C(sI - A_-)(Bu - B_-Du).
\]

We shall prove \(B = B_- D\). Notice that
\[
B_- Du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \phi^-(A_-)(j\omega I - A)^{-1}BD^{-1} - \phi(A_-)(j\omega I - A)^{-1}BD^{-1} \right\} du
\]
\[
\times \left\{ \left( D + C(j\omega I - A)^{-1}B \right) u(j\omega)
- \phi(j\omega)C(j\omega I - A)^{-1}Bu \right\} d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \phi^-(A_-)(j\omega I - A)^{-1}Bu(j\omega)
- \phi^-(A_-)(j\omega I - A)^{-1}Bu(j\omega) \right\} d\omega
+ \frac{1}{2\pi} \int_{\Gamma} \left\{ (sI - A_-) - \phi^-(A_-)\phi(s)(sI - A_-)
- (sI - A) + \phi^-(A_-)\phi(s)(sI - A) \right\} dsBu.
\]

where \(\Gamma\) is a closed contour in the right half plane that encircles clock-wise the anti-stable eigenvalues of \(A\) and \(A_-\) (see Fig. 1). Let \(E_-\) and \(E\) be the projection matrices on the anti-stable eigenspaces of \(A_-\) and \(A\), respectively. Then, it follows that
\[
\frac{1}{2\pi} \int_{\Gamma} (sI - A_-)^{-1} ds = -E_-,
\]
\[
\frac{1}{2\pi} \int_{\Gamma} \phi(s)(sI - A_-)^{-1} ds = -\phi(A_-)E_-,
\]
\[
\frac{1}{2\pi} \int_{\Gamma} (sI - A)^{-1} ds = -E,
\]
\[
\frac{1}{2\pi} \int_{\Gamma} \phi(s)(sI - A)^{-1} ds = -\phi(A)E.
\]

Hence form (11) and (18),
\[
\begin{align*}
(B - B_- D)u &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \phi^-(A_-)(j\omega I - A)^{-1}Bu(j\omega)
- \phi^-(A)(j\omega I - A)^{-1}Bu(j\omega) \right\} d\omega
+ \left( I - \phi^-(A_-)\phi(A) \right) EBu.
\end{align*}
\]

If \(A\) is stable, then \(-j\omega I - A)^{-1}B \in H_2^2\) and \(E = 0\). Thus \((B - B_- D)u = 0\). If \(A\) is anti-stable, then \(E = I\) and
\[
Bu = -\phi^-(A) \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega I - A)^{-1}Bu(j\omega)d\omega.
\]

If \(A\) has both stable and anti-stable eigenvalues, then the block diagonalization proves that \((B - B_- D)u = 0\). This proves \(D_- D = I\). Hence \(D_- = D_-^{-1}\) and \(B_- = BD_-^{-1}\).

We can similarly prove \(B_- C = A_- A_-\) and \(C_- = -D_- C\).

**Remark 2:** It should be noted that when the inner function is rational and the matrices \(A\) and \(A_-\) are stable the result was shown in [5]. Lemma 1 does not assume that the inner function is rational, and the system matrices may have unstable eigenvalues.

**C. Feedback connection**

Consider the feedback connection shown of a plant \(P\) and a controller \(C\) as shown in Fig. 2, where \(y_p\) and \(y_c\) are outputs, \(u_p\) and \(u_c\) are inputs, and \(r_p\) and \(r_c\) are exogenous inputs of the plant and the controller, respectively. Suppose that \(P\) and \(C\) are described by state-space realizations
\[
\begin{align*}
\frac{dx_p}{dt} &= A_p x_p + B_p u_p + B_p r_p \\
y_p &= C_p x_p + D_p u_p + D_p r_p \\
\frac{dx_c}{dt} &= A_c x_c + B_c u_c + B_c r_c \\
y_c &= C_c x_c + D_c u_c + D_c r_c.
\end{align*}
\]

Stack the variables
\[
\begin{align*}
x_{cl} &= \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad y_{cl} = \begin{bmatrix} y_p \\ y_c \end{bmatrix}, \quad u_{cl} = \begin{bmatrix} u_p \\ u_c \end{bmatrix}, \quad r_{cl} = \begin{bmatrix} r_p \\ r_c \end{bmatrix},
\end{align*}
\]
and let
\[
\begin{align*}
A &= \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{pi} & 0 \\ 0 & B_{ci} \end{bmatrix}, \quad i = 1, 2, 3
\end{align*}
\]
and
\[
\begin{align*}
C &= \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}, \quad D_i = \begin{bmatrix} D_{pi} & 0 \\ 0 & D_{ci} \end{bmatrix}, \quad i = 1, 2, 3.
\end{align*}
\]
Notice that the feedback connection imposes the relation
\[ u_{cl} = J y_{cl}, \quad J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (19) \]
We assume that the feedback connection is well-posed and hence \( I - JD \) is invertible. Hence the feedback connection in Fig. 2 when the input is \( u_{cl} \) and the output is \( y_{cl} \) has a state-space representation
\[
\frac{dx_{cl}}{dt} = A_{cl}x_{cl} + B_{cl}r_{cl},
\]
\[ y_{cl} = C_{cl}x_{cl} + D_{cl}r_{cl}, \quad (20) \]
where
\[
A_{cl} = A + JB_1(I - JD_1)^{-1} C, \quad (22)
\]
\[
B_{cl} = B_1(I - JD_1)^{-1} JD_2 + B_2, \quad (23)
\]
\[
C_{cl} = (I - D_1J)^{-1} C, \quad D_{cl} = (I - D_1J)^{-1} D_2. \quad (24)
\]
Let \( P \) and \( C \) have transformed system representations (8)-(9) using the operators \((A_p, B_{p1}, B_{p2}, C_p, D_p)\) and \((A_c, B_{cl}, B_{c2}, C_c, D_c)\), respectively. We would like to ask whether the transformed system of the feedback connection can be constructed from the transformed systems.

Let \((A, B_i, C, D_i)\) and \((A_{cl}, B_{cl}, C_{cl}, D_{cl})\) be the operators of the transformed systems of \((A, B_i, C, D_i)\) and \((A_{cl}, B_{cl}, C_{cl}, D_{cl})\), respectively. It is obvious that \( A, B, C, D \) satisfy
\[
A = \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{pi} & 0 \\ 0 & B_{ci} \end{bmatrix}, \quad i = 1, 2,
\]
\[
C = \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}, \quad D_i = \begin{bmatrix} D_{pi} & 0 \\ 0 & D_{ci} \end{bmatrix}, \quad i = 1, 2.
\]

**Theorem 2:** Consider the feedback connection in Fig. 2. Assume that \( A_p, A_c, \) and \( A_{cl} \) do not have eigenvalues on the imaginary axis. Let \( \phi \) be an inner function, and \( S = H^2 \otimes \phi H^2 \). Assume that \( \phi \) and \( \phi^* \) are analytic at the spectra of \( A_p, A_c, \) and \( A_{cl} \). Then the operators for the feedback system obey the following equations:
\[
A_{cl} = A + JB_1(I - JD_1)^{-1} C, \quad (25)
\]
\[
B_{cl} = B_1(I - JD_1)^{-1} JD_2 + B_2, \quad (26)
\]
\[
C_{cl} = (I - D_1J)^{-1} C, \quad D_{cl} = (I - D_1J)^{-1} D_2. \quad (27)
\]

**Proof** The operators of the transformed system of the closed loop system is calculated by using Lemma 1. Details are omitted. Q.E.D.

If \( C \) is a stabilizing controller and the exogenous inputs are in \( H^2 \), then so are the inputs and the outputs. In this case, the transformed signals satisfy the following forward state equation even if the plant has unstable eigenvalues. More precisely, if \( r_{cl} \in H^2 \), write
\[ r_{cl} = \sum_{k=0}^{\infty} \phi^k r_k, \quad r_k = \begin{bmatrix} r_{p,k} \\ r_{c,k} \end{bmatrix} \in S. \]

Then we have
\[
\xi_{p,t+1} = A_p \xi_{p,t} + B_p u_{p,t} + B_p \phi \delta_{p,t}, \quad (29)
\]
\[
y_{p,t} = C_p \xi_{p,t} + D_p u_{p,t} + D_p \phi \delta_{p,t}, \quad (30)
\]
\[
\xi_{c,t+1} = A_c \xi_{c,t} + B_c u_{c,t} + B_c \phi \delta_{c,t}, \quad (31)
\]
\[
y_{c,t} = C_c \xi_{c,t} + D_c u_{c,t} + D_c \phi \delta_{c,t}, \quad (32)
\]
with the feedback connection
\[
\begin{bmatrix} u_{p,t} \\ u_{c,t} \end{bmatrix} = \begin{bmatrix} y_{c,t} \\ y_{p,t} \end{bmatrix}, \quad (33)
\]

**Remark 3:** When \( \phi \) is rational and \( A_p, A_c, \) and \( A_{cl} \) are stable matrices, the results of this section was already proven in [5]. In this paper, we need not have to assume that \( \phi \) is rational. Furthermore, we show that the assumptions on \( A_p, A_c, \) and \( A_{cl} \) are not necessary to obtain the result.

**Remark 4:** In [9], a stable system and an anti-stable system is connected in a special way to compute Schmidt pairs of a Hankel operator for a class of infinite dimensional systems. Theorem 2 considers the standard feedback connection.

**D. Stochastic system**

Consider the feedback system in Fig. 2 consisting of a plant and a controller having stochastic inputs. Describe the system by the following state equations:
\[
dx_p = A_p x_p dt + B_p \xi + B_p \phi \delta, \quad (34)
\]
\[
d\eta = C_p x_p dt + D_p \xi + D_p \phi \delta, \quad (35)
\]
\[
dx_c = A_c x_c dt + B_c \eta, \quad (36)
\]
\[
d\zeta = C_c x_c dt + D_c \eta, \quad (37)
\]
Define the signals
\[
x_{cl} = \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad \eta_{cl} = \begin{bmatrix} \eta_p \\ \eta_c \end{bmatrix}, \quad \zeta_{cl} = \begin{bmatrix} \zeta_p \\ \zeta_c \end{bmatrix}.
\]

Then the closed-loop system is described by
\[
dx_{cl} = A_{cl} x_{cl} dt + B_{cl} \phi \delta dt, \quad (38)
\]
\[
d\eta_{cl} = C_{cl} x_{cl} dt + D_{cl} \phi \delta dt. \quad (39)
\]
where \( A_{cl}, B_{cl}, C_{cl}, D_{cl} \) are exactly as in (22), (23), and (24).

Notice that if the controller stabilizes the feedback loop, then \( A_{cl} \) is stable. Thus the closed loop signals obey the results in [10]. In what follows, we will shall show that the stochastic signals in the transformed domain satisfy the discrete-time state-space equation even if the plant has unstable eigenvalues.

Let
\[
w_{\delta,h}(t) = \begin{cases} \frac{w(t) - w(t - \delta)}{\delta} & 0 < t \leq h, \\ 0 & t > h. \end{cases}
\]

Then \( w_{\delta,h} \) is in \( L^2(0, \infty) \) with probability 1. Let \( y_{\delta,h} \) be the response of the system (20), (21) when the input \( w_{\delta,h} \) is applied. If we consider the limit \( \delta \to 0 \), then the response of the system (38), (39) when the processes are terminated.
at time $t = h$ is recovered using the transformed system (29)-(33).

When $\phi$ is rational, then the space $S$ is finite dimensional. Let

$$w_{\phi,h} = \sum_{k=0}^{\infty} \phi^k w_{\phi,h,k}, \quad y_{\phi,h} = \sum_{k=0}^{\infty} \phi^k y_{\phi,h,k}$$

We can show that as $\delta \to 0$ and $h \to \infty$, $w_{\phi,h,k}$ and $y_{\phi,h,k}$ are convergent sequences. Though the limits $\lim_{\delta \to 0, h \to \infty} w_{\phi,j,k}$ and $\lim_{\delta \to 0, h \to \infty} y_{\phi,j,k}$ are not squarely summable, they are the input and the output of the transformed system. Hence the transformed signals of (38), (39) satisfies the transformed system (29)-(33).

IV. CONCLUSIONS

This paper extended the results in [5], [10] to systems with unstable eigenvalues. It was shown that there is a unified formulae of transformation for stable and anti-stable systems, and that the transformed system can be described by a forward discrete-time system when the feedback system is stabilized even if the plant and/or the controller are unstable. The result can be applied to closed-loop system identification.

REFERENCES