2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Optimal design of a class of hybrid systems with uncertain parameters

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Abstract— **Many modern engineering systems can be mathematically modeled as hybrid systems. For many such systems, there may be uncertain parameters and also parameters that can be adjusted so that the system achieves some optimal performance. It is important to develop efficient numerical tools and software to optimize for these adjustable parameters. We focus on a specific class of hybrid systems where mode transitions are dependent only on the amount of time spent in a mode (or equivalently a clock value). The amount of time spent in each mode is assumed to be a random variable with a known distribution. We aim to design or choose values for the free parameters in each mode of the hybrid system so that the expected value of some meaningful cost-function is minimized. This can be framed as a stochastic optimization problem. We use the** ***sample average approximation*** **method to solve the resulting stochastic optimization problem. We illustrate the method for the optimal design of a thermal management system of a prototypical aircraft.**

I. Introduction

Hybrid systems are a useful abstraction for systems that have a combination of discrete and continuous dynamics. They typically consist of digital programs that interact with each other and with an analog environment. Examples include manufacturing controllers, aircraft management systems, unmanned aerial vehicles and robots. For plants that can be modeled as hybrid systems, we may also need a controller that is hybrid in nature. Design of such hybrid controllers or computation of optimal controls for hybrid systems in general is a difficult task. For some results, see [1] and [2]. In [1], the authors present a method to compute approximations to optimal feedback control laws using discretizations of Bellman type inequalities for lower bounds on the optimal value function. In [2], the authors discuss necessary conditions for optimality for a class of hybrid systems and show how this leads to non-smooth optimization problems.

However, efficient algorithms for solving these non-smooth optimization problems are still not available. In [3], the author discusses algorithms which efficiently solve control synthesis problems for constrained linear systems and constrained linear hybrid systems. There are also tools like HYSDEL ([4]) that provide a computational framework for modeling hybrid systems with linear affine dynamics in discrete time. Models described in HYSDEL can be used by a set of tools that solve reachability analysis and control problems. There are few or no tools currently available for optimal control of nonlinear hybrid systems or stochastic hybrid systems.

In this paper, we focus on numerical methods for a class of nonlinear hybrid systems where the mode transitions are enabled by a clock value. i.e., mode transitions are dependent only on the amount of time spent in a mode and not on the actual state of the system. The amount of time spent in a mode of operation (the transition time) is also assumed to be a random variable with a known distribution. Thus the transition times can be thought of as uncertain parameters of the hybrid system. Note that these transitions times are not part of the design and are a characteristic of the system to be controlled. The hybrid system also has some free adjustable parameters corresponding to each mode of operation that can be chosen by the designer. The objective is to pick values for these adjustable parameters of the system that minimizes the expected value of some meaningful cost-function that captures the behavior of the system over a finite time-horizon. Thus the cost-function may include finite-time-integrals of some function of the states of the system and the controls.

We show an optimal control synthesis technique to minimize the expected value of a cost-function. This can be framed as a stochastic optimization problem. We use the *sample average approximation* method described in [5] to solve the stochastic optimization problem. The basic idea behind the *sample average approximation* method is simple. It is a Monte Carlo sampling-based approach to stochastic optimization problems. A random sample of the uncertain parameters is generated and the expected value function is approximated by the corresponding sample average function. The resulting sample average optimization problem is solved using standard optimization techniques. We use the optimization software IPOPT ([6]) to numerically solve the resulting sample average optimization problem. One could also use software like CPLEX ([7]) to solve the resulting optimization problem.

As a case study, we illustrate the method for the design of a thermal management system of a prototypical aircraft. The parameters to be optimized for are the fuel flow rates of the system during various phases of the mission and the uncertain parameters are the time durations of various phases of the mission. In principle it would also be possible to take into account the effects due to the implementation platform such as delays in computation and communication, and reliability of communication.

This work was supported by DARPA under contract #FA9550-10-C-0116.
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II. Optimal design of hybrid systems with time-triggered mode transitions

In this section, we discuss an approach to compute optimal controls for discrete-time hybrid systems with uncertain parameters. In particular, we look at discrete-time hybrid systems with time-triggered mode transitions. i.e., the switching of the system from one mode to another depends only on the amount of time spent in that mode (or equivalently a clock value). The amount of time spent in each mode is a random variable with some known distribution. Note that for such systems, the sequence of modes that the system operates in is known beforehand. Thus the system we consider has dynamics of the form

$$
\begin{aligned}
x(k+1) &= T(1,x(k),p_0) \text{ for } j_0 \le k < j_1 \\
x(k+1) &= T(2,x(k),p_1) \text{ for } j_1 \le k < j_2 \\
x(k+1) &= T(3,x(k),p_2) \text{ for } j_2 \le k < j_3 \\
&\ldots.. \\
x(k+1) &= T(m,x(k),p_m) \text{ for } j_{m-1} \le k < j_m.
\end{aligned} \tag{1}
$$

Here k represents the time-index. The amount of time spent in each mode $\Delta_q = j_q - j_{q-1}$, is a random variable with some known distribution $\mathscr{W}_q$. The system has m modes and the $m-th$ is considered to be the 'final' mode. The time duration of a sample trajectory is given as

$$L = \sum_{q=1}^{m} \Delta_q. \tag{2}$$

Systems described above are particularly useful to model processes where the sequence of modes of operation are fixed and known in advance, but the transition times are uncertain. A good example for such a system is the typical mission of an aircraft which has various modes of operation like *Taxing*, *Take-off*, *Flying* and *Landing*. A typical aircraft mission follows a fixed sequence of modes, but the time spent in modes like *Taxing* and *Flying* may be uncertain.

The variables p_q are parameters that need to be optimized for in the operation of each mode. In what follows, we describe how one can do such an optimization. The cost-function we use is the expected value of some functional of the sample trajectories. The optimization problem can be written as

$$\min_{p_q \in U_q} \left\{ C := \mathbb{E}_\Delta \left[\sum_{q=1}^{m} \sum_{k=j_{q-1}}^{j_q} F_q\left(x(k), p_q\right) \right] \right\}. \tag{3}$$

where $x(k)$ is subject to the dynamics described in (1). F_q is assumed to be a differentiable function of $x(k)$ and p_q. Δ is the vector of transition times. i.e. Δ_q is the $q-th$ component of the vector Δ. The expectation is taken over the uncertain transition times Δ_q. To solve the stochastic optimization problem described above, the *sample average approximation* (SAA) method (see [5]) is a natural choice. We briefly review the SAA method in the following subsection.

A. Review of the sample average approximation method

In the most general form, stochastic optimization problems take the form

$$\min_{u \in U} \left\{ g(u) := \mathbb{E}_P G(u,W) \right\}. \tag{4}$$

Here W is a random vector having probability distribution P. U is the set from which the variables u can be chosen from. For the optimal design problem we described before, the random vector W would correspond to the vector Δ whose elements are the random transition times for each mode. The variables u include both the parameters p_q that need to be optimized for and the states $x(k)$ that are subject to the constraints imposed by the system dynamics. $\mathbb{E}_P G(u,W) = \int G(u,W)P(dw)$ is the expected value of the objective function $G(u,W)$. The SAA method is suitable for optimization problems that have the following characteristics.

- The expected value function $g(u) := \mathbb{E}_P G(u,W)$ cannot be written in a closed form and its value cannot be easily calculated.
- The function $G(u,W)$ is easily computable for given u and W.
- The set of feasible solutions U is very large so that enumeration is not feasible.

The optimal design problem we described before has all these characteristics and therefore makes the *sample average approximation* method a natural approach to this problem. The basic idea of *sample average approximation* is simple indeed. It is a Monte Carlo sampling-based approach to stochastic optimization problems. A random sample of W is generated and the expected value function is approximated by the corresponding sample average function. The obtained sample average optimization problem is solved, and the procedure is repeated several times until a stopping criterion is satisfied.

Let $W^1, \ldots, W^N$ be an independently and identically distributed (i.i.d) random sample of N realizations of the random vector W. Consider the sample average function

$$\bar{g}_N(u) := \frac{1}{N} \sum_{i=1}^{N} G(u, W^i). \tag{5}$$

The sample average approximation (SAA) problem is

$$\min_{u \in U} \bar{g}_N(u). \tag{6}$$

It has been shown that the solution of the sample average approximation problem (6) converges to the solution of the original problem (4) with probability one. Also roughly speaking, the optimal value for the objective function obtained from the approximate problem converges exponentially fast to the true optimal value for the objective function as $N \to \infty$. For more theoretical details on the convergence of the SAA method, see [5].

B. Application of SAA to optimization of time-triggered hybrid systems

For hybrid systems of the type described in (1) at the beginning of this section, optimal design can be done using the SAA method described before. One can generate a finite number of samples for the random vector of transition times Δ and then find the optimal parameters p_q that minimize the corresponding sample average function. The variables u that need to be optimized for include both the state variables x_k^i corresponding to each sample Δ^i and the parameters p_q. In our notation, Δ^i for $i = 1,2,...,N$ are N i.i.d. samples for the vector of transitions times and x_k^i is the state of the system at time k for a trajectory corresponding to the sample Δ^i. The state variables x_k^i are subject to the constraints imposed by the system dynamics given as

$$\begin{aligned}
g_k^i &= x_{k+1}^i - T(1,x_k^i,p_0) = 0 \text{ for } j_0 \le k < j_1^i \\
g_k^i &= x_{k+1}^i - T(2,x_k^i,p_1) = 0 \text{ for } j_1^i \le k < j_2^i \\
g_k^i &= x_{k+1}^i - T(3,x_k^i,p_2) = 0 \text{ for } j_2^i \le k < j_3^i \\
&\cdots\cdots \\
g_k^i &= x_{k+1}^i - T(m,x_k^i,p_m) = 0 \text{ for } j_{m-1}^i \le k < j_m^i.
\end{aligned} \tag{7}$$

where $j_q^i - j_{q-1}^i = \Delta_q^i$. The cost-function is approximated by the sample average given as

$$\bar{C} = \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{q=1}^{m}\sum_{k=j_{q-1}^i}^{j_q^i} F_q\left(x_k^i, p_q\right)\right]. \tag{8}$$

To find the state variables x_k^i and the parameters p_q that minimizes the cost-function $\bar{C}$ subject to the constraints in (7), we use the optimization software IPOPT (see [6]). IPOPT is a software package for large-scale nonlinear optimization. It uses an interior point line search filter method to find local solutions to nonlinear optimization problems. For the IPOPT software, it is necessary to compute the gradient of the cost-function $\bar{C}$ with respect to the variables x_k^i and p_q. These derivatives are given as

$$\frac{\partial \bar{C}}{\partial x_k^i} = \frac{1}{N}\frac{\partial F_q}{\partial x_k^i} \tag{9}$$

$$\frac{\partial \bar{C}}{\partial p_q} = \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{k=j_{q-1}^i}^{j_q^i}\frac{\partial F_q}{\partial p_q}\right]. \tag{10}$$

It is also necessary to compute the Jacobian of the constraint equations. The elements of the Jacobian of the constraint equations are computed as

$$\begin{aligned}
\frac{\partial g_k^i}{\partial x_{k+1}^i} &= 1.0 \\
\frac{\partial g_k^i}{\partial x_k^i} &= \frac{-\partial T(q,.,.)}{\partial x_k^i} \\
\frac{\partial g_k^i}{\partial p_q} &= \frac{-\partial T(q,.,.)}{\partial p_q}.
\end{aligned} \tag{11}$$

The IPOPT software takes in as input the user-provided routines that compute the cost-function, the gradient of the cost-function and the Jacobian of the constraint equations and returns optimal values for the parameters p_q.

III. Application to optimal design of aircraft thermal management system

As a case study, we apply the SAA method described before to the optimal design of the thermal management system of a prototypical aircraft. In a typical aircraft, the fuel flow rates for each mode of operation is maintained at a steady state. A pump is used to push fuel from the fuel tank into the fuel circuit. The fuel is then used to reject the heat produced by the environmental control system (ECS) and the electric power system (EPS). Only part of the fuel flowing in the circuit is sent to the nozzles for consumption. Part of the fuel goes back to the fuel tank. For a schematic of the thermal management system, see Figure 1.

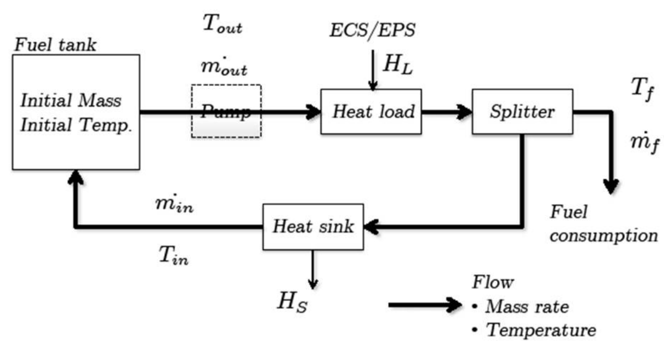


Fig. 1. Model of thermal management system.

The heat loads and fuel consumption rates during various modes of the aircraft mission are different. This leads to some interesting design problems. One interesting design problem is to find the optimal fuel flow rates for each mode so that the temperature of the fuel that goes into the nozzles stays close to an optimal temperature. For the purposes of this paper, we consider the design of the TMS as if there were only two modes - *Taxing* and *Flying*. Indeed, these modes are the two most crucial modes of a typical mission as the amount of time spent in other modes like *take-off* and *landing* is extremely short compared to the overall mission time and the contribution of design parameters in these modes to relevant cost-functions is negligible. Therefore we focus on the problem of choosing the optimal fuel flow rates during the *taxing* and *flying* modes and treat the problem as if there were only two modes.

We consider two dynamic variables - the mass (M) and temperature (T) of the fuel remaining in the tank. The temperature of the fuel after it absorbs heat from the EPS and ECS is denoted by T_f. The increase in temperature ($Tf - T$) is related to the fuel flow rate m_{out} through the heat-balance equation

$$m_{out}C_{sp}\left(T_f - T\right) = H_L, \tag{12}$$

where H_L is the heat load coming from the ECS/EPS. C_{sp} is the specific heat of the fuel. The fuel that is returned to the fuel-tank is cooled by the air-fuel heat exchanger. The temperature of the fuel after it is cooled is denoted by T_{in}. The drop in temperature ($T_f - T_{in}$) is assumed to be a fraction of the difference between Tf and the outside air temperature T_{air}. i.e.,

$$T_f - T_{in} = f(Tf - T_{air}), \tag{13}$$

where f is referred to as the heat sink efficiency. If m_f is the rate at which fuel is consumed, then the rate at which fuel is returned to the fuel-tank after re-circulation is given as $m_{in} = m_{out} - m_f$. The rate of change of the temperature of the fuel remaining in the tank is derived from the heat-balance equation

$$\begin{aligned} m_{in}C_{sp}T_{in} - m_{out}C_{sp}T &= \frac{d}{dt}\left(MC_{sp}T\right) \\ &= -m_f C_{sp}T + MC_{sp}\dot{T}. \end{aligned} \tag{14}$$

Discretizing the above equations, the discrete-time dynamics for the temperature (T) and mass (M) variables can be written as

$$\begin{aligned} &M(k+1) = M(k) - \delta . m_f(k) \\ &T(k+1) = T(k) + \\ &\qquad \frac{\delta}{M(k)}\left(m_{in}(k)T_{in}(k) - m_{out}(k)T(k) + m_f(k)T(k)\right). \end{aligned} \tag{15}$$

Here δ is the size of the discrete time-step and from the above equations we have

$$\begin{aligned} T_{in}(k) &= Tf(k) + f(T_{air} - Tf(k)) \\ \text{and where } Tf(k) &= T(k) + \frac{H_L}{m_{out}C_{sp}}. \end{aligned} \tag{16}$$

Note that m_f and m_{out} are considered to be constant within each mode. More precisely

$$m_{out}(k) = \begin{cases} m_{taxi} \text{ if } 0 \le k < \Delta_{taxi} \\ m_{fly} \text{ if } \Delta_{taxi} \le k < \Delta_{taxi} + \Delta_{fly}. \end{cases} \tag{17}$$

m_{taxi} and m_{fly} are the parameters that need to be chosen so that we get some desirable thermal behavior. Δ_{taxi} and Δ_{fly} are random variables uniformly distributed within the intervals $[300s, 900s]$ and $[3600s, 4500s]$ respectively. H_L is also considered to be constant within each mode. The cost-function that we are going to use for this problem is a combination of the quality of the fuel temperature going into the combustor (Tf) and the control effort in terms of the fuel flow rates. The cost-function is

$$C = \mathbb{E}\left[\frac{1}{2}\sum_k \left(Tf(k) - T_{set}\right)^2 + \frac{W}{2}\sum_k m_{out}^2(k)\right] \tag{18}$$

T_{set} is a set-point temperature at which we desire the fuel-combustor temperature (Tf) to be close to. W is a parameter that decides how much the control effort in terms of the fuel flow rates should be penalized.

As described for the SAA method, we generate a finite number of samples for the taxing and flying times. The samples are Δ^i_{taxi} and Δ^i_{fly} for $i = 1, 2, ..., N$. The sample average cost-function is given as

$$\bar{C} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\sum_k \left(Tf^i_k - T_{set}\right)^2 + \frac{W}{2}\frac{1}{N}\sum_{i=1}^{N}\Delta^i_{taxi}m^2_{taxi} + \Delta^i_{fly}m^2_{fly}. \tag{19}$$

The gradient of the sample average cost-function with respect to the optimization variables are given as

$$\begin{aligned} \frac{\partial \bar{C}}{\partial M^i_k} &= 0.0, \\ \frac{\partial \bar{C}}{\partial T^i_k} &= \frac{1}{N}\left(Tf^i_k - T_{set}\right), \\ \frac{\partial \bar{C}}{m_{taxi}} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{\Delta^i_{taxi}}(Tf^i_k - T_{set}).\frac{-H_L}{m^2_{taxi}C_{sp}} + \frac{W}{N}\sum_{i=1}^{N}\Delta^i_{taxi}m_{taxi}, \\ \frac{\partial \bar{C}}{m_{fly}} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{k=\Delta^i_{taxi}}^{\Delta^i_{taxi}+\Delta^i_{fly}}(Tf^i_k - T_{set}).\frac{-H_L}{m^2_{fly}C_{sp}} + \frac{W}{N}\sum_{i=1}^{N}\Delta^i_{fly}m_{fly}. \end{aligned} \tag{20}$$

The constraints on the variables M^i_k and T^i_k are derived from the dynamics as described in equation (7) of Section II-B. Also the elements of the Jacobian of the constraint equations are computed as described in equation (11) of Section II-B.

A. *Results*

We solved the SAA problem for different number of samples. For 50 samples, the number of constraint equations derived from the system dynamics is around 450,000. For a problem of this size, IPOPT takes about 4 minutes to solve the optimization problem on a laptop with a Intel Core2 Duo CPU P9400 running at 2.40 GHz and 2.95 GB RAM. The values for various fixed parameters in the TMS model are shown in Table I. Figure 2 shows some optimization results obtained using IPOPT for different values of W. The plots shows how the optimal fuel flow rates obtained change as the number of samples are increased. As you can see, the optimal values converge very quickly with respect to the number of samples.

For $W = 1.0$, the optimal *taxing* fuel flow-rate (m_{taxi}) is roughly 3 times bigger than the fuel consumption rate (m_f). The value of the cost-function is lowest for this higher flow-rate because the penalty on the control effort is small and for this higher fuel-rate, the resulting fuel temperature (Tf) after it absorbs heat from the ECS/EPS is made lower (and closer to T_{set}). However, if the fuel flow-rate is increased beyond this optimal value, the cost-function would increase because the resulting temperature Tf may go way below the set-point temperature T_{set}. For $W = 250.0$, the optimal value for m_{taxi} is roughly 2 times bigger than m_f. This is because the penalty on the control effort is higher and the fuel-flow rates have to be reduced resulting in higher values for Tf.

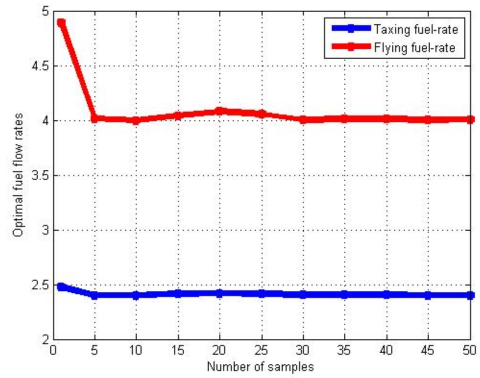
Figure 3 shows the variation of the objective function and optimal fuel flow-rates with respect to the weighting parameter W. As described before, the objective function has two components. The first component reflects the quality of the fuel temperature going into the combustor and is given as

$$C_1 = \mathbb{E}\left[\frac{1}{2}\sum_k \left(Tf(k) - T_{set}\right)^2\right]. \tag{21}$$

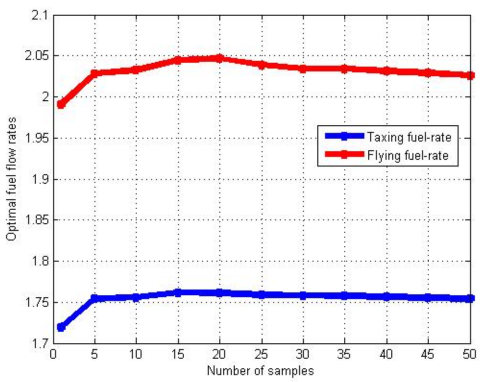
Variable	Value
H_L (Taxing)	$18.44(kW)$
H_L (Flying)	$20(kW)$
m_f (Taxing)	0.84 (kg/s)
m_f (Flying)	1.44 (kg/s)
C_{sp}	0.2 (kJ/kg K)
f	0.1
T_{set}	320K
$M(0)$	9000 kg
$T(0)$	280K

TABLE I

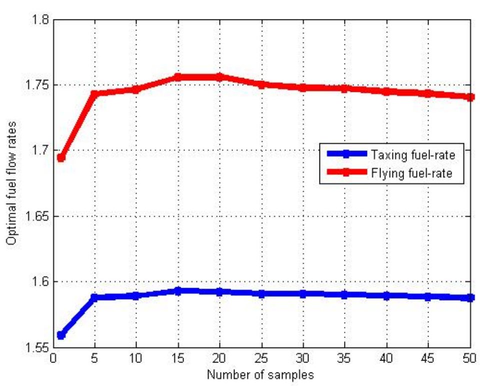
FIXED PARAMETER VALUES FOR TMS MODEL.



(a) W = 1.0



(b) W = 125.0



(c) W = 250.0

Fig. 2. Results obtained using the SAA method for the optimal design of TMS. These plots show results obtained for different values of W. The plots show how the optimal fuel-flow rates change as the number of samples are increased.

The second component reflects the control effort in terms of the fuel-flow rates and is given as

$$C_2 = \mathbb{E}\left[\frac{1}{2}\sum_k m_{out}^2(k)\right]. \tag{22}$$

Now for different values of W, we get different Pareto optimal solutions in the following sense. For a given value of W, let p_q^* be the optimal parameters that lead to optimal objective function values of C_1^* and C_2^*. Then for the system to perform such that the objective function $C_1 = C_1^*$, the minimum required value of C_2 is C_2^* and vice versa. Figure 3 shows a plot of (C_1^*, C_2^*) for different values of W. Figure 3 also shows how the optimal fuel flow rates change with respect to W. It can be clearly seen that the optimal flow rates decrease as W is increased.

IV. Summary and Future steps

We have described a simple but effective procedure to optimally design hybrid systems whose mode transitions are time-triggered. In particular, we assumed that the transitions times are random variables with known distributions and used stochastic optimization methods to design the hybrid system so that on average, its performance is optimal. We illustrated the method to optimize the fuel flow-rates in different modes of operations of the thermal management system of a prototypical aircraft. In the current setting, we assumed that the parameters to be optimized for are fixed for each mode. It is possible to formulate optimal control problems where the control has some feedback dependence on the state. The feedback dependence on the state can be expressed in terms of parameters and then one could in principle optimize for these parameters.

Acknowledgment

This work was supported by DARPA under contract #FA9550-10-C-0116. The views expressed are those of the authors and do not reflect the official policy or position of the Department of Defense or the U.S. Government. Distribution Statement “A” (Approved for Public Release, Distribution Unlimited).

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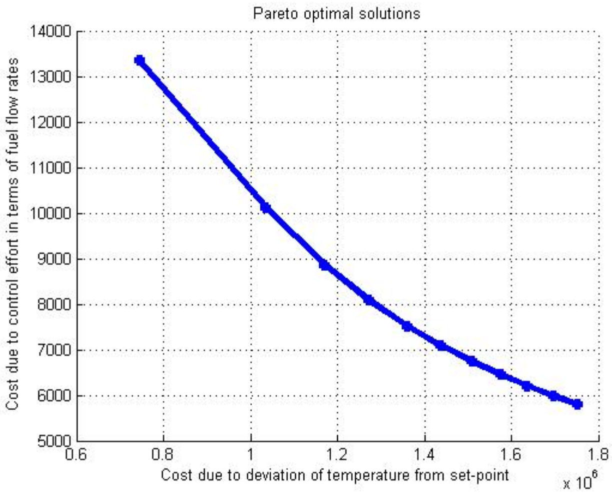
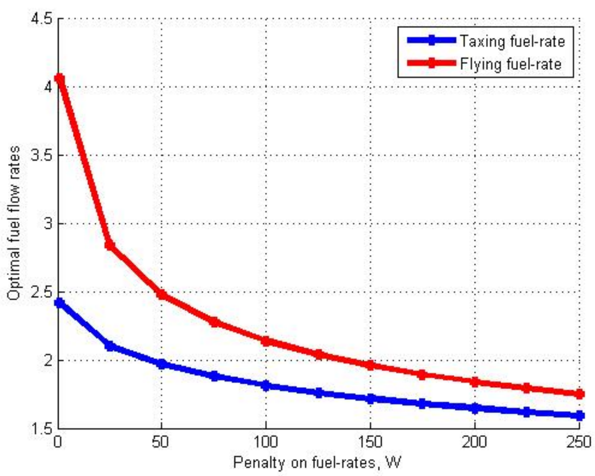


(a) Pareto optimal curve. The x-axis is for C_1^* as defined in (21) and the y-axis is for C_2^* as defined in (22).



(b) Variation of optimal fuel flow-rates with W.

Fig. 3. Results obtained using the SAA method for optimal design of TMS

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