Stability Analysis and $H_{\infty}$ Controller Design of a Class of Switched Discrete-Time Fuzzy Systems

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Abstract—In this paper, the problems of stability analysis and $H_{\infty}$ controller design of a class of switched nonlinear systems are investigated. In a classical way, the modeling of the systems is approached by switched fuzzy systems, and both fast switching and slow switching are considered there. In particular, for slow switching scheme, a new mode-dependent average dwell time switching is proposed for the underlying switched fuzzy systems. Based on a fuzzy-basis-dependent and mode-dependent Lyapunov function, the $H_{\infty}$ state-feedback controller is derived. A numerical example is given to show the validity and potential of the theoretical results.

I. INTRODUCTION

In the past two decades, switched systems have been extensively studied since switching feature exists in quite many practical systems subject to internal parameters variations or external environmental changes [1], [2]. Switching signals, either designable or not, are crucial in analysis and design of switched systems. They are often considered as either fast (arbitrary) or slow (dwell time or average dwell time) switching. As a typical slow-switching rule, the average dwell time (ADT) switching means that the number of switches in a finite interval is bounded and the average time between consecutive switching is not less than a constant. So far, many results on ADT switching have been obtained, see [3]–[6] for instance. However, the ADT switching commonly considered in the literature is independent of the system modes and probably leads to a certain conservativeness accordingly [7].

On the other hand, the most of studies are focused on switched linear systems with vast systematical results obtained, the investigation on switched nonlinear systems are relatively somewhat limited. It has been well recognized that, T-S fuzzy model provides an effective mathematical tool to describe nonlinear systems in a general framework [8]–[10]. Fuzzy logic control approach usually divide the nonlinear system into several local dynamics, which are represented by linear state-space models, and the overall model can be described by fuzzy fusion of these local fuzzy models. Numerous stability analysis and $H_{\infty}$ control results have been obtained in literature of fuzzy control and fuzzy systems, e.g., [11]–[14], in which basic stability and $H_{\infty}$ conditions are derived based on Lyapunov theory and formulated in terms of linear matrix inequalities.

Alternatively, to approximate the switched nonlinear systems, some works have been initiated by considering the switched fuzzy systems. Based on common Lyapunov function approach, the stability conditions for switched fuzzy systems have been obtained [15]. Furthermore, the relaxed stabilization conditions for discrete-time switched fuzzy systems by means of the switched quadratic Lyapunov function for all local fuzzy systems have been available in [16], [17]. In addition, some results on guaranteed cost control of switched fuzzy systems have been obtained, readers are referred to [18], [19] and the references therein for more details. It is worth mentioning that, however, for neither fast nor slow switching, the problems of $l_2$-gain analysis and $H_{\infty}$ controller design for discrete-time switched fuzzy systems have almost not been investigated so far.

In this paper, we are concerned with using a fuzzy-basis-dependent and mode-dependent Lyapunov function to study the problems of stability analysis and $H_{\infty}$ controller design for discrete-time switched fuzzy systems. The stability analysis and $H_{\infty}$ controller design for arbitrary switching are firstly derived and the corresponding results are extended to the case of mode-dependent average dwell time (MDADT). This paper is organized as follows. We review the definitions and lemmas on stability and $l_2$-gain of switched systems in Section II. The case of switched systems under arbitrary switching is first studied in Section III, the corresponding results are extended to the case of MDADT switching in Section IV. A numerical example is given in Section V, and the paper is concluded in Section VI.

Notation: The notation used in this paper is fairly standard. The superscript “T” stands for matrix transposition, $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space, respectively. In addition, in symmetric block matrices or long matrix expressions, we use * as an ellipsis for the terms that are induced by symmetry, and $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. $l_2[0, \infty)$ refers to the space of square summable infinite vector sequences. $C^1$ denotes the space of continuously differentiable functions. The notation $P > 0$ ($\geq 0$) means that $P$ is a real symmetric positive (semi-positive) definite matrix, and $I$ and $0$ represent, respectively, the identity matrix and zero matrix.
II. PRELIMINARIES

The T-S fuzzy systems, suggested by Takagi and Sugeno, can represent a general class of nonlinear systems modeled by switched fuzzy systems, if a switched nonlinear system is approximated as switched fuzzy systems. If a switched nonlinear system is modeled by switched fuzzy system, it can be represented by the forms (include system model and controller) as follows Subsystem i:

Local Plant Rule p:
If \( \xi_i(k) \) is \( M_{ip1} \), and ..., \( \xi_i(k) \) is \( M_{ipr} \), then

\[
x_{k+1} = A_{ip}x_k + B_{ip}u_k + F_{ip}w_k
\]

\[
z_k = C_{ip}x_k + D_{ip}u_k + G_{ip}w_k
\]

\[
u_k = K_{ip}x_k
\]

where \( \xi_i(k) \) are the premise variables, \( M_{ip} \) are the fuzzy sets, \( x_k \in \mathbb{R}^{n_x} \) is the state vector, \( u_k \in \mathbb{R}^{n_u} \) is the input vector, \( w_k \in \mathbb{R}^{n_w} \) is the disturbance input and belongs to \( l_2[0, \infty) \), \( z_k \in \mathbb{R}^{n_z} \) is the output vector. The subsystem number \( i \) is determined by switching signal \( \sigma \), which is a piecewise constant of time and takes its value in \( I = \{1, ..., s\} \), \( s > 1 \) is the number of subsystems. At an arbitrary time \( k \), \( \sigma \) may be dependent on \( k \) or \( x_k \), or both, or other logic rules. For a switching sequence \( k_0 < k_1 < k_2 < ... \), \( \sigma \) is continuous from right everywhere. When \( k \in [k_i, k_{i+1}) \), we say the \( \sigma_k \)-th subsystem is active. \( (A_{ip}, B_{ip}, F_{ip}, C_{ip}, D_{ip}, G_{ip}) \) is the \( p \)-th local model of the \( i \)-th subsystem. Through the "fuzzy fusion", the final closed-loop switched fuzzy system is given by

\[
x_{k+1} = \sum_{p=1}^{\beta(i)} h_{ip}x_{k+1} = A_{ip}x_k + B_{ip}u_k + F_{ip}w_k
\]

\[
z_k = \sum_{p=1}^{\beta(i)} h_{ip}x_k + C_{ip}x_k + D_{ip}u_k + G_{ip}w_k
\]

\[
u_k = \sum_{p=1}^{\beta(i)} h_{ip}x_k
\]

where \( A_{ipq} = A_{ip} + B_{ip}K_{ip}, C_{ipq} = C_{ip} + D_{ip}K_{ip}, \) and \( \eta_{ip} = \prod_{l=1}^{n_p} M_{ipl}(\xi_i(k)) \), \( h_{ip} = \frac{\eta_{ip}}{\sum_{p=1}^{\beta(i)} \eta_{ip}} \)

in which \( M_{ipl}(\xi_i(k)) \) is the grade of membership function of \( \xi_i(k) \) in \( M_{ipl} \). It is assumed that \( \eta_{ip} \geq 0 \) for all \( k \) and \( p = 1, 2, ..., \beta(i) \). Therefore, the normalized membership function \( h_{ip} \) satisfies

\[
h_{ip} \geq 0, \quad \sum_{p=1}^{\beta(i)} h_{ip} = 1, \quad \text{for all } k
\]

Our objective in this paper is to design a state-feedback \( H_\infty \) controller for the switched fuzzy system (4)-(5) under arbitrary or mode-dependent average dwell time switching. The following definitions are first recalled.

Definition 1: [7] For a switching signal \( \sigma(t) \) and any \( T \geq t \geq 0 \), let \( N_{\sigma}(t, T) \) be the switching numbers that the \( i \)-th subsystem is activated over the interval \( [t, T] \), and \( T_i \) denote the running time of \( i \)-th subsystem over the interval \( [t, T] \), \( i \in I \). We say that \( \sigma(t) \) has a mode-dependent average dwell time \( \tau_{ai} \), if there exist positive number \( N_{0i} \) (we call \( N_{0i} \) mode-dependent chatter bounds here), and \( \tau_{ai} \) such that

\[
N_{\sigma}(t, T) \leq N_{0i} + \frac{T_i}{\tau_{ai}}, \quad \forall T \geq t \geq 0
\]

Remark 1: Definition 1 has been proposed in [7], it means that if there exist positive numbers \( \tau_{ai}, i \in I \) such that a switching signal has the MDADT property, we only require the average time among the intervals associated with the \( i \)-th subsystem is larger than \( \tau_{ai} \).

Definition 2: [1] The switched system (4)-(5) with \( \omega_k = 0 \) is globally uniformly asymptotically stable (GUAS), if there exists a class \( K\) function \( \alpha \) such that for all initial condition \( x_0 \), the solutions of (4)-(5) satisfy the inequality \( \|x_k\| \leq \alpha(\|x_0\|, k), \forall k \geq 0 \).

Definition 3: For \( \gamma > 0 \), system (4)-(5) is said to be GUAS with \( l_2 \)-gain no greater than \( \gamma \), if under zero initial condition, system (4)-(5) is GUAS and the inequality \( \sum_{k=0}^{\infty} z_k^T z_k \leq \sum_{k=0}^{\infty} \gamma^2 \omega_k^T \omega_k \) holds for all nonzero \( \omega_k \in l_2[0, \infty) \).

Definition 4: [6] For \( \gamma > 0 \), system (4)-(5) is said to be GUAS with weighted \( l_2 \)-gain no greater than \( \gamma \), if under zero initial condition, system (4)-(5) is GUAS and the inequality \( \sum_{k=0}^{\infty} (1-k)^k z_k^T z_k \leq \sum_{k=0}^{\infty} \gamma^2 \omega_k^T \omega_k \), \( 0 < \lambda < 1 \) holds for all nonzero \( \omega_k \in l_2[0, \infty) \).

In addition, the following lemmas are required for later development.

Lemma 1: [5] The following arbitrarily switched system

\[
x_{k+1} = A_k x_k + F_k w_k
\]

\[
z_k = C_k x_k + G_k w_k
\]

is GUAS with \( l_2 \)-gain no greater than \( \gamma \), if there exist \( C \) functions \( V_{\sigma(k)} : \mathbb{R}^n \rightarrow \mathbb{R}, \sigma(k) \in I, \) class \( K_{\infty} \) functions \( \kappa_{1i} \) and \( \kappa_{2i}, i \in I, \) and a constant \( \gamma > 0 \), satisfying

\[
\kappa_{1i}(\|x_{k}\|) \leq V_{\sigma}(x_{k}) \leq \kappa_{2i}(\|x_{k}\|)
\]

\[
\Delta V_{\sigma}(x_k) < -\lambda_i V_{\sigma}(x_k)
\]

where

\[
\kappa_{1i}(\|x_{k}\|) \leq V_{1}(x_k) \leq \kappa_{2i}(\|x_{k}\|)
\]

\[
\Delta V_{i}(x_k) \leq -\lambda_i V_{i}(x_k)
\]

and \( \forall \sigma(k) = i, \sigma(k) = j \rightarrow I \times I, \) \( i \neq j \),

\[
V_{i}(x_{k_\sigma}) \leq \mu_i V_{j}(x_{k_\sigma})
\]

then the system (9)-(10) is GUAS with \( \omega_k = 0 \) for any switching signal with MDADT

\[
\tau_{ai} > \tau_{ai} = -\ln \mu_i / \ln(1 - \lambda_i)
\]

Lemma 3: Consider the switched system (9)-(10) with \( x_0 = 0 \), and let \( 0 < \lambda_i < 1, \mu_i > 1, i \in I \). If the system satisfying (13), (14), (15), and \( \forall i \in I \), denoting \( \Gamma(k) = z_k^T z_k - \gamma^2 \omega_k^T \omega_k \)

\[
\Delta V_{i}(x_k) + \lambda_i V_{i}(x_k) + \Gamma(k) \leq 0
\]
then the switched system (9)-(10) is GUAS and has weighted l2-gain \( \sum_{k=0}^{\infty} (1 - \lambda)^k z_k^T z_k < \gamma_s \sum_{k=0}^{\infty} \omega_k \) no greater than \( \gamma_s = \sqrt{\frac{\lambda}{1-\lambda}} \) for switching signal with MDADT (16), where \( \lambda = \max(\lambda_i), \Delta = \min(\lambda_i) \).

Remark 2: The proof of Lemma 3 is cumbersome. We omit them here due to space limitation, and we will include them in the journal version of the paper.

Lemma 4: Given two matrices \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times n} \), and symmetric positive definite matrix \( P \in \mathbb{R}^{m \times m} \), then

\[
A^T P B + B^T P A \leq A^T P A + B^T P B
\]

III. STABILITY ANALYSIS AND H∞ CONTROL DESIGN FOR FAST-SWITCHED SYSTEMS

A. Stability and l2-gain Analysis

The aim here is to find less-conservative conditions ensuring the globally uniformly asymptotic stability and l2-gain of system (4)-(5). As a natural choice, choosing a common quadratic Lyapunov function of the form \( V(x_k) = x_k^T P x_k, \) we can obtain these conditions. However, such an approach often yields overly conservative results. An improved method is to construct fuzzy-basis-dependent and mode-dependent quadratic Lyapunov function (FMDQLF) for switched fuzzy systems. The form of FMDQLF is given by \( V(x_k) = x_k^T P_{a_k}(\xi(k))x_k. \) Now, based on the FMDQLF, we shall present our first result on l2-gain conditions of switched fuzzy systems (4)-(5) in the following theorem.

Theorem 1: The closed-loop discrete-time switched fuzzy system (4)-(5) is GUAS with l2-gain no greater than \( \gamma \), if there exist a set of matrices \( P_{ip} > 0 \), symmetric matrices \( X_{ipq}, W_{ipq} \), and matrices \( U_{ipq} \), such that for every \( (i, j) \in I \times I \),

\[
\Delta_{ipq}^T \begin{bmatrix} \beta(j) & 0 \\ 0 & 0 \end{bmatrix} \Delta_{ipq} - \begin{bmatrix} P_{ip} & 0 \\ 0 & \gamma^2 I \end{bmatrix} + \Theta_{ipq} < 0
\]

\[
\hat{\Delta}_l = \begin{bmatrix} \Theta_{i_11} & \Theta_{i_12} & \cdots & \Theta_{i_1\beta(i)} \\ \Theta_{i_21} & \cdots & \Theta_{i_2\beta(i)} \\ \vdots & \ddots & \vdots \\ \Theta_{i\beta(i)1} & \cdots & \Theta_{i\beta(i)\beta(i)} \end{bmatrix} > 0
\]

where \( \Delta_{ipq} = (\Gamma_{ip} + \Gamma_{ipq}) / 2 \),

\[
\Gamma_{ipq} = \begin{bmatrix} A_{ipq} & F_{ipq} \\ C_{ipq} & G_{ipq} \end{bmatrix}, \quad \Theta_{ipq} = \begin{bmatrix} X_{ipq} & \ast \\ U_{ipq} & W_{ipq} \end{bmatrix}
\]

Proof. Assume \( P_l(\xi(k)) \) to be of the form \( P_l(\xi(k)) = \sum_{i=1}^{\xi(i)} h_{il} P_{il} \), where \( P_{il} > 0 \), thus, one obtains \( P_l(\xi(k)) > 0 \), and the FMDQLF satisfies

\[
\kappa_{i_1}(\|x_k\|) \leq V(x_k) \leq \kappa_{i_2}(\|x_k\|)
\]

for some class \( K_{\infty} \) functions \( \kappa_{i_1} \) and \( \kappa_{i_2} \).

Consider the closed-loop system (4)-(5) with \( \omega_k = 0, (i, j) \in I \times I \), taking the difference between \( V(x_{k+1}) \) and \( V(x_k) \), we have

\[
\Delta V = x_k^T \sum_{l=1}^{j}\sum_{p=1}^{\beta(j)} \sum_{q=1}^{\beta(i)} \sum_{m=1}^{\beta(i)} \sum_{n=1}^{\beta(i)} h_{ip} h_{iq} h_{im} h_{in} \times \left( A_{ipq}^T P_{jl} A_{lmn} - P_{ip} \right) x_k
\]

\[
= x_k^T \sum_{l=1}^{j}\sum_{p=1}^{\beta(j)} \sum_{q=1}^{\beta(i)} \sum_{m=1}^{\beta(i)} \sum_{n=1}^{\beta(i)} \frac{1}{4} h_{ip} h_{iq} h_{im} h_{in} \times \left( A_{ipq}^T P_{jl} (A_{lmn} + A_{imn}) - 4P_{ip} \right) x_k
\]

\[
= x_k^T \sum_{l=1}^{j}\sum_{p=1}^{\beta(j)} \sum_{q=1}^{\beta(i)} \sum_{m=1}^{\beta(i)} \sum_{n=1}^{\beta(i)} \frac{1}{8} h_{ip} h_{iq} h_{im} h_{in} \times \left( A_{ipq}^T P_{jl} (A_{lmn} + A_{imn}) - 8P_{ip} \right) x_k
\]

\[
+ \left( A_{lmn} + A_{imn} \right)^T P_{jl} (A_{ipq} + A_{iqp}) x_k
\]

then, according to Lemma 4, we obtain

\[
\Delta V \leq x_k^T \sum_{l=1}^{j}\sum_{p=1}^{\beta(j)} \sum_{q=1}^{\beta(i)} \sum_{m=1}^{\beta(i)} \sum_{n=1}^{\beta(i)} \frac{1}{4} h_{ip} h_{iq} h_{im} h_{in} \times \left( A_{ipq}^T P_{jl} (A_{lmn} + A_{imn}) - 8P_{ip} \right) x_k
\]

\[
= x_k^T \sum_{l=1}^{j}\sum_{p=1}^{\beta(j)} \sum_{q=1}^{\beta(i)} \sum_{m=1}^{\beta(i)} \sum_{n=1}^{\beta(i)} \frac{1}{4} h_{ip} h_{iq} h_{im} h_{in} \times \left( A_{ipq}^T P_{jl} (A_{lmn} + A_{imn}) - 8P_{ip} \right) x_k
\]

\[
+ \left( A_{lmn} + A_{imn} \right)^T P_{jl} (A_{ipq} + A_{iqp}) x_k
\]

from (19), it is not difficult to get

\[
(A_{ipq} + A_{iqp})^T P_{jl} (A_{ipq} + A_{iqp}) - 4P_{ip} + 4X_{ipq} < 0
\]

with the above inequality, we have

\[
\Delta V < -x_k^T \sum_{l=1}^{j}\sum_{p=1}^{\beta(j)} \sum_{q=1}^{\beta(i)} \sum_{m=1}^{\beta(i)} \sum_{n=1}^{\beta(i)} \frac{1}{4} h_{ip} h_{iq} h_{im} h_{in} \times \left( A_{ipq}^T P_{jl} (A_{lmn} + A_{imn}) - 8P_{ip} \right) x_k
\]

\[
= x_k^T \sum_{l=1}^{j}\sum_{p=1}^{\beta(j)} \sum_{q=1}^{\beta(i)} \sum_{m=1}^{\beta(i)} \sum_{n=1}^{\beta(i)} \frac{1}{8} h_{ip} h_{iq} h_{im} h_{in} \times \left( A_{ipq}^T P_{jl} (A_{lmn} + A_{imn}) - 8P_{ip} \right) x_k
\]

\[
+ \left( A_{lmn} + A_{imn} \right)^T P_{jl} (A_{ipq} + A_{iqp}) x_k
\]

\[
\hat{X}_i = \begin{bmatrix} X_{i11} & X_{i12} & \cdots & X_{i1\beta(i)} \\ \ast & X_{i22} & \cdots & X_{i2\beta(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \ast & \cdots & X_{i\beta(i)\beta(i)} \end{bmatrix}
\]
by inequality (20), one has
\[ \hat{X}_i > 0 \]
that is, \( \Delta V < 0 \), the stability of system (4)-(5) with \( \omega_k = 0 \) is guaranteed. On the other hand, by some mathematical operations, \( l_2 \)-gain of system (4)-(5) can be proved. Consider the closed-loop system (4)-(5) with zero initial condition, we have
\[
\Delta V = \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \beta(i,j) \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{m=1}^{n} \sum_{n=1}^{n} h_{ip} h_{iq} h_{im} h_{in} \times \left( \begin{array}{ccc}
A_{imn} & F_{im} \\
A_{imn} & G_{im}
\end{array} \right) - \left( \begin{array}{c}
P_{ip} \\
P_{ip}
\end{array} \right) \zeta_k
\]
It follows from Lemma 1 that
\[
\Delta V + z_k^T z_k - \gamma^2 \omega_k^T \omega_k \leq \zeta_k \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{m=1}^{n} \sum_{n=1}^{n} h_{ip} h_{iq} h_{im} h_{in} \times \left( \begin{array}{ccc}
A_{imn} & F_{im} \\
A_{imn} & G_{im}
\end{array} \right) - \left( \begin{array}{c}
P_{ip} \\
P_{ip}
\end{array} \right) \gamma^2 \omega_k^T \omega_k
\]
using the same technique as stability analysis, we can obtain (12). Then, according to Lemma 1, system (4)-(5) has the \( l_2 \)-gain no greater than \( \gamma \). This completes the proof.

B. \( H_\infty \) Controller Design

Now, based on stability and \( l_2 \)-gain results in Theorem 1, the conditions of a switched fuzzy-dependent controller ensuring the GUAS and \( H_\infty \) performance of system (4)-(5) can be obtained, which is given in the following theorem.

**Theorem 2**: If there exist a set of matrices \( Q_{ip} > 0 \), symmetric matrices \( X_{ipq} \), \( W_{ipq} \), and matrices \( \Omega_i \), \( U_{ipq} \), \( Y_{ipq} \), such that \( \forall (i,j) \in I \times I, 1 \leq p \leq q \leq \beta(i), 1 \leq i \leq \beta(j) \),
\[
\begin{bmatrix}
O_{ipq} & \ast & \ast & \ast \\
U_{ipq} & W_{ipq} - \gamma^2 I & \ast & \ast \\
M_{ipq} & G_{ipq}^T G_{ipq} & \ast & \ast \\
N_{ipq} & 0 & -Q_{ipq} & \ast
\end{bmatrix} < 0
\] (21)
holds. Then the closed-loop system (4)-(5) is GUAS with \( H_\infty \) performance index \( \gamma \), and the \( H_\infty \) controller is given by
\[
K_{ip} = Y_{ip} \Omega_i^{-1}
\] (23)

**Proof.** Consider the closed-loop system (4)-(5), applying the change of variable \( Y_{ip} = K_{ip} \Omega_i \), to (21), one has
\[
\begin{bmatrix}
O_{ipq} & \ast & \ast & \ast \\
U_{ipq} & W_{ipq} - \gamma^2 I & \ast & \ast \\
M_{ipq} & G_{ipq}^T G_{ipq} & \ast & \ast \\
N_{ipq} & 0 & -Q_{ipq} & \ast
\end{bmatrix} < 0
\]
with the inequality \( (Q_{ip} - \Omega_i)^T Q_{ip} (Q_{ip} - \Omega_i) \geq 0 \), which is equivalent to \( \Omega_i^T Q_{ip} \Omega_i \geq \Omega_i + \Omega_i^T - Q_{ip} \), yields
\[
\begin{bmatrix}
X_{ipq} - \Omega_i^T Q_{ip} \Omega_i & \ast & \ast & \ast \\
U_{ipq} & W_{ipq} - \gamma^2 I & \ast & \ast \\
M_{ipq} & G_{ipq}^T G_{ipq} & \ast & \ast \\
N_{ipq} & 0 & -Q_{ipq} & \ast
\end{bmatrix} < 0
\]

and
\[
\begin{bmatrix}
X_{ipq} - \Omega_i^T Q_{ip} \Omega_i & \ast & \ast & \ast \\
U_{ipq} & W_{ipq} - \gamma^2 I & \ast & \ast \\
M_{ipq} & G_{ipq}^T G_{ipq} & \ast & \ast \\
N_{ipq} & 0 & -Q_{ipq} & \ast
\end{bmatrix} < 0
\]

setting \( Q_{ip} = P_{ip} \), \( \Omega_i^T X_{ipq} \Omega_i^{-1} = \hat{X}_{ipq} \), \( U_{ipq} \), \( \Omega_i^T X_{ipq} \Omega_i^{-1} = \hat{U}_{ipq} \),

We have
\[
\begin{bmatrix}
X_{ipq} - P_{ip} & \ast & \ast & \ast \\
U_{ipq} & W_{ipq} - \gamma^2 I & \ast & \ast \\
M_{ipq} & G_{ipq}^T G_{ipq} & \ast & \ast \\
N_{ipq} & 0 & -P_{ip} & \ast
\end{bmatrix} < 0
\]
by Schur complement, the above inequality is equivalent to (19). In addition, pre-multiplying \( diag(\Omega_i^T, I, I, I, I, \ldots, \Omega_i^T) \) and post-multiplying \( diag(\Omega_i^{-1}, I, I, \ldots, \Omega_i^{-1}) \) to (22), one has (20). Therefore, by Theorem 1, switched fuzzy system (4)-(5) is GUAS and has a \( H_\infty \) performance index \( \gamma \).

**Remark 2**: Note that, if we omit subscript \( i \) and \( j \) in Theorem 2, we will get the corresponding \( H_\infty \) controller design conditions for non-switched fuzzy systems, which have been proposed in [13].

IV. EXTENSION TO SLOW-SWITCHED SYSTEMS

Based on the similar techniques used in the previous developments, we consider a class of MDADT switching defined as (8), and have the following propositions:

**Proposition 1**: Consider the closed-loop system (4)-(5), and let \( 0 < \lambda_i < 1 \), \( \mu_i \geq 1 \) be the given constant. If there exist matrices \( P_{ip} > 0 \), symmetric matrices \( X_{ipq} \),

\[ W_{ipq}, \text{ and matrices } U_{ipq}, \text{ such that } \forall (i, j) \in I \times I, i \neq j, \]
\[ 1 \leq p \leq q \leq \beta(i), 1 \leq l \leq \beta(i). \]
\[ \Delta T_{ipq} \left[ \begin{array}{ccc} P_{il} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \gamma^2 I \end{array} \right] \]
\[ + \left[ \begin{array}{c} X_{ipq} \\ U_{ipq} \\ W_{ipq} \end{array} \right] \leq 0 \]
\[ (24) \]
\[ \Delta T_{ipq} \left[ \begin{array}{ccc} P_{il} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \gamma^2 I \end{array} \right] \]
\[ + \left[ \begin{array}{c} X_{ipq} \\ U_{ipq} \\ W_{ipq} \end{array} \right] \leq 0 \]
\[ (25) \]
\[ P_{jl} - \mu_j P_{lp} \leq 0 \]
\[ (26) \]

where
\[ \gamma_s = \sqrt{2 \gamma_s} \Delta_{ipq} = \Gamma_{ipq}^2 + \Gamma_{ipq}, \quad \Gamma_{ipq} = \left[ \begin{array}{ccc} A_{ipq} & F_{ipq} \\ C_{ipq} & G_{ipq} \end{array} \right] \]

then system (4)-(5) under MDADT switching (16) is GUAS with weighted \( L_2 \)-gain no greater than \( \gamma_s \).

**Proposition 2:** Consider the closed-loop system (4)-(5), and let \( 0 < \lambda_i < 1, \mu_i \geq 1 \) be the given constant. If there exist a set of matrices \( \Omega_{ip}, \) symmetric matrices \( X_{ipq}, \) \( W_{ipq}, \) matrices \( \Omega_i, U_{ipq}, Y_{ipq} \) such that \( \forall (i, j) \in I \times I, i \neq j, \)
\[ 1 \leq p \leq q \leq \beta(i), 1 \leq l \leq \beta(i). \]
\[ \left[ \begin{array}{ccc} O_{ipq} & * & * \\ U_{ipq} & * & * \\ M_{ipq} & \frac{E_{ip} + F_{ip}}{2} - Q_{jl} & * \\ N_{ipq} & \frac{G_{ip}}{2} & 0 - I \end{array} \right] \]
\[ \leq 0 \]
\[ (27) \]

holds. Then system (4)-(5) is GUAS with weighted \( L_2 \)-gain no greater than \( \gamma_s = \sqrt{2 \gamma_s} \) under MDADT switching (16) and the controller is given by (23).

**Remark 4:** Incorporating the slow switching considered in Lemma 2 and Lemma 3, the proof for the aforementioned propositions can be obtained using the same techniques in previous section, and we will include them in the journal version of the paper.

V. NUMERICAL EXAMPLE

In this section, to illustrate the effectiveness of the results developed in previous sections, we consider the following discrete-time switched fuzzy system (4)-(5) consists of two subsystems, and each subsystem has two fuzzy rules.

\[ A_{11} = \begin{bmatrix} 0.9 & 0.28 \\ -0.02 & 0.72 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.8 & 0.1 \\ 0.09 & 0.84 \end{bmatrix}, \]
\[ A_{21} = \begin{bmatrix} 0.85 & 0.1 \\ 0.1 & 0.85 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.86 & 0.11 \\ 0.41 & 0.95 \end{bmatrix}, \]
\[ B_{11} = \begin{bmatrix} 0.6 \\ -0.6 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix}. \]

\[ F_{11} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad F_{12} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}, \quad F_{21} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad F_{22} = \begin{bmatrix} 0.6 \\ 2 \end{bmatrix}, \]
\[ C_{11} = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} 0.8 & 0.4 \\ 0.85 & 0.7 \end{bmatrix}, \]
\[ D_{11} = 0.1, \quad D_{12} = 0.72, \quad D_{21} = 0.3, \quad D_{22} = 0.92, \]
\[ G_{11} = 0.2, \quad G_{12} = 0.7, \quad G_{21} = 1.3, \quad G_{22} = 0.8. \]

The fuzzy membership functions are taken as
\[ h_{11} = \sin^2(x_1 + 0.5), \quad h_{12} = 1 - h_{11}, \]
\[ h_{21} = 1 - \frac{1}{1 + \exp(-3x_2/(0.5 - \pi/2))} \times 1 + \exp(-3x_2/(0.5 + \pi/2)), \quad h_{22} = 1 - h_{21} \]

Our purpose here is to design a set of fuzzy-basis-dependent and mode-dependent state-feedback controllers such that the resulting closed-loop system is stable with an optimized \( H_\infty \) disturbance attenuation performance.

By using the LMI toolbox in Matlab to solve the conditions in Theorem 2, we can get \( \gamma = 3.1371 \) and the \( H_\infty \) controllers gains as
\[ K_{11}=-[0.1.794 -0.0300], \quad K_{12}=-[0.7554 -0.5348], \]
\[ K_{21}=-[1.3135 -1.3078], \quad K_{22}=-[1.0707 -0.9476]. \]

Applying the controllers in (30) and generating an arbitrary switching sequence, one can get the steady-response of the resulting closed-loop system as shown in Fig.1 for \( \omega(k) = 1 \times \exp(-0.5k). \)

Then, we consider the MDADT switching. Given \( \lambda_1 = 0.1, \lambda_2 = 0.2, \mu_1 = 1.4, \mu_2 = 2 \) and solving the conditions in Proposition 2. We can obtain \( \gamma_s = 7.3746 \) and the controllers gains as
\[ K_{11}=-[0.0867 0.0176], \quad K_{12}=-[0.1182 -0.0147], \]
\[ K_{21}=-[1.0359 -1.1985], \quad K_{22}=-[1.0192 -0.9932]. \]

Using the controllers and generating switching sequences with MDADT property \( \tau_{1a} = 3.1935, \quad \tau_{2a} = 3.1063. \) The steady-response of the system is given in Fig.2 with \( \omega(k) = 1 \times \exp(-0.5k). \) Fig.3 gives the comparison on \( H_\infty \) performance indices that the resulting closed-loop systems can achieve when applying (30) and (31), respectively. It can be seen from Fig.3 that the designed controllers in (30) and (31) under the admissible switching signals are effective.

VI. CONCLUSIONS AND FUTURE WORKS

In this paper, the problems of stability analysis and \( H_\infty \) controller design of a class of switched nonlinear systems, which are switched discrete-time fuzzy systems, have been studied. A fuzzy-basis-dependent and mode-dependent Lyapunov function is proposed for the system, and a new \( H_\infty \)
controller is derived via linear matrix inequalities formulations. We also remarked that the obtained conditions cover the case of non-switched fuzzy systems, which has been widely studied. Future works will be concerned with the investigations on the applications of switched fuzzy systems to real systems, such as vehicle suspension systems, robotic fish systems, etc.

REFERENCES