Global Stabilization of Non–Globally Linearizable Triangular Systems: Application to Transient Stability of Power Systems

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Abstract—A general methodology to globally stabilize an equilibrium of a class of non-globally linearizable triangular systems is presented. The technique is applicable to all triangular systems described by analytic vector fields. The method is used to give an explicit solution to the challenging problem of transient stability of multimachine power systems with leaky transmission lines, for which only existence results are currently available.

I. INTRODUCTION

Nonlinear dynamical systems in triangular forms have been widely studied in the control literature. In particular, the so-called backstepping technique [1], [2], [3] has proven to be successful to globally stabilize a given equilibrium of globally feedback linearizable triangular systems. The main objective of this paper is to propose a methodology for global stabilization of a general class of non–globally feedback linearizable triangular systems.

Several works are dedicated to stabilization of triangular systems that are only locally feedback linearizable, see e.g. [4], [5], [6], [7], [8], [9], [10], [11]. All these works either provide only existence results for a stabilizing control law, or require additional assumptions in order to compute an explicit control law. In this paper, we consider a particular class of triangular systems, but show that all triangular systems described by analytic vector fields can be written in this form, thus giving to the method a general validity.

As an application of the general theory developed in the paper we consider the transient stability problem for multimachine power systems, consisting of $N$ generators, nonlinear loads and leaky transmission lines. Transient stability is concerned with the ability of the system to reach an acceptable steady-state following a fault, e.g., a short circuit or a generator outage, that is later cleared by the protective system operation. The fault modifies the circuit topology—driving the system away from the stable operating point—and the question is whether the trajectory remains in the basin of attraction of this (or another) equilibrium after the fault is cleared. The key analysis issue is then the evaluation of the domain of attraction of the system’s operating equilibrium, while the control objective is the enlargement of the latter, see [12], [13], and the references therein, for more details and a literature review.

Similarly to [13], the full 3$N$–dimensional model of the $N$–generator system with lossy transmission lines, loads and excitation controllers, is considered. In [13] the existence of a nonlinear static state feedback law that ensures asymptotic stability of the operating point with a well-defined estimate of the domain of attraction provided by a bona fide Lyapunov function is established. To the best of our knowledge, even in the lossless case, no explicit globally stabilizable state-feedback controller has yet been reported. Providing an affirmative answer to this problem, even for the lossy case, is a central contribution of this paper.

The remaining of the paper is organized as follows. To facilitate the understanding of the general methodology the material is presented in increasing degrees of complexity. First, a simple single-input, two-dimensional example is considered in Section II. Then, Sections III and IV are devoted to the extensions to general two- and three-dimensional systems, respectively. The final result is presented in Section V, where the $n$–dimensional, multi–input case is treated. Section VI is devoted to the application example, while Section VII contains some concluding remarks.

II. A MOTIVATING EXAMPLE

With the aim of providing the reader with an insight into the more general construction presented below, we begin by studying a simple triangular system. The system is described by the equations

$$
\dot{x}_1 = 1 - x_2^2, \quad \dot{x}_2 = u.
$$

This system is not globally feedback linearizable, since the relative degree is not defined for $x_2 = 0$. However, the set of equilibrium points for this system is $\{ (x_1, x_2) \in \mathbb{R}^2 : |x_2| = 1 \}$ and local feedback linearization is possible around each equilibrium point. Suppose that the equilibrium to be stabilized is $(x_1, x_2) = (0, 1)$; the classical backstepping approach provides a solution through the dynamics $\dot{z}_1 = z_2$ and $\dot{z}_2 = v$ for the variables $z_1 = x_1$ and $z_2 = 1 - x_2^2$ and the control $v = -2x_2u$. Selecting the stabilizing control for the $z$–dynamics as $v = -z_1 - z_2$, yields the control law

$$
u = -\frac{1}{2x_2}(-x_1 - 1 + x_2^2).
$$

This solution has two limitations; the first one, which is due to the fact that the system is only locally feedback
linearizable, is that the control law is not defined at \( x_2 = 0 \). The second limitation is that the solution does not distinguish between the points \((0,1)\) and \((0,-1)\), since we have both

\[
\dot{x}_1 \big|_{x_1 = 0, x_2 = 1} = 0, \quad \dot{x}_2 \big|_{x_1 = 0, x_2 = 1} = 0
\]

and

\[
\dot{x}_1 \big|_{x_1 = 0, x_2 = -1} = 0, \quad \dot{x}_2 \big|_{x_1 = 0, x_2 = -1} = 0.
\]

We now show a possible way to overcome both limitations. Note that other methods based on modified backstepping approach could be used to stabilize the equilibrium of system (1), for instance a saturated backstepping as in [14]; however, extending these techniques to \( n \)-dimensional systems (such as systems (16) and (23) below) may result in cumbersome structures. On the contrary, the extension of the method presented herein to MIMO system does not introduce further complexity (see, for comparison, the control laws (13), (22) and (26) below). To streamline our statements the following definition is introduced.

**Definition 2.1:** Let \( X \subset \mathbb{R}^n \) be an open and path connected set. A function \( f : X \to \mathbb{R} \) is positive definite with respect to \( x_0 \in X \) if \( f(x_0) = 0 \) and \( f(x) > 0 \) for all \( x \neq x_0 \). It is radially unbounded in \( X \) if there exists a homeomorphism \( \varphi : X \to \mathbb{R}^n \) such that \( f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R} \) is radially unbounded.

Introduce the variables \( \xi \) and \( x_2^* \) verifying the equality

\[
1 - x_2^* \xi = -\tanh(x_1).
\]

An input \( u(t) \) such that

\[
|x_2^*(t) - x_2(t)| \to 0, \quad |\xi(t) - x_2(t)| \to 0, \quad x_2(t) \to 1,
\]

as \( t \to \infty \), renders the equilibrium \((0,1)\) of (1) attractive. To find this control law, consider the Lyapunov function

\[
V(x_1, x_2, \xi) = \frac{1}{2}(x_1^2 + (\xi - x_2)^2 + (x_2 - x_2^*)^2),
\]

which is clearly motivated by our desire to drive \( x_1 \) to zero and by the asymptotic objectives given above. Computing its time-derivative we obtain

\[
\dot{V} = x_1(1 - x_2^2) + (\xi - x_2)(\dot{\xi} - u) + (x_2 - x_2^*)(u - \dot{x}_2).
\]

The right hand side of (2) suggests the choice of the dynamics of \( \xi \) as

\[
\dot{\xi} = u + x_2 - \xi - x_1 x_2^*,
\]

which replaced in (2) yields

\[
\dot{V} = -x_1 \tanh(x_1) - (\xi - x_2)^2 - x_1 x_2 (x_2 - x_2^*) + (x_2 - x_2^*)(u - \dot{x}_2) \]

\[
= -x_1 \tanh(x_1) - (\xi - x_2)^2 - (x_2 - x_2^*)^2,
\]

where the last equation is obtained selecting the control law

\[
u = \dot{x}_2^* + x_1 x_2 - (x_2 - x_2^*).\]

Hence, \( \dot{V} \leq 0 \) and \( \dot{V} = 0 \) when \( x_1 = 0, \xi = x_2 \) and \( x_2^* = x_2 \). Applying La Salle’s invariance principle, when \( x_1 \equiv 0 \) we have \( x_2 \in [-1,1] \). Since \( V \) is radially unbounded in \( \mathbb{R}^2 \times \mathbb{R}^+ \) (and positive definite with respect to \((0,1,1))\), if the initial condition for \( \xi \) is chosen in \( \mathbb{R}^+ \), we have \( \xi(t) \to 1 \) and hence \( x_2(t) \to 1 \).

The first advantage obtained by introducing the variable \( \xi \) is that we can assure that \( x_2 \) tends to 1. Moreover, the choice of the hyperbolic tangent ensures that the input signal is defined for all possible trajectories. Indeed, replacing the definition of \( x_2^* \), that is,

\[
x_2^* = \frac{1 + \tanh(x_1)}{\xi},
\]

in (4), the control signal can be written in the form

\[
u(x_1, x_2, \xi) = \frac{\xi^2 \bar{u}(x_1, x_2, \xi)}{\xi^2 + 1 + \tanh(x_1)},
\]

where \( \bar{u}(x_1, x_2, \xi) \) is well-defined, and the denominator is always positive.

Simulations have been carried out to shown the effectiveness of the method. In Figure 1 the trajectory of the state \((x_1, x_2)\) of system (1) with the control (4) for several initial conditions lying in the circle \( x_1^2 + (x_2 - 1)^2 = 4 \). The initial condition for \( \xi \) is the equilibrium value \( \xi = 1 \). In Figure 2 the time histories of \( x_1, x_2, \xi \) and \( u \) for the initial condition \((x_1(0), x_2(0), \xi(0)) = (2, 1, 1)\) are shown.

In the following section we generalize the previous derivations, and propose a systematic method applicable to a generic planar system in triangular form.

### III. The Two-Dimensional SISO Case

Consider a two-dimensional system in triangular form described by the equations

\[
\dot{x}_1 = f(x_1, x_2), \quad \dot{x}_2 = u,
\]

where \( x_1(t) \in \mathbb{R}, x_2(t) \in \mathbb{R}, u(t) \in U \subset \mathbb{R} \) and \( f : \mathbb{R}^2 \to \mathbb{R} \) is a smooth function. Suppose that \((x_1, x_2) = (0,0)\) is an equilibrium. To formalize the method of the previous example we begin with the following definition.
Definition 3.1: A function \( \sigma : \mathbb{R} \to \mathbb{R} \) has the odd sign property if \( \sigma(x)x > 0 \) for \( x \neq 0 \) and \( \sigma(0) = 0 \).

As shown in the example, the task of rendering the equilibrium attractive can be accomplished if there exists an input \( u(t) \) such that, for some function \( \sigma : \mathbb{R} \to \mathbb{R} \) with the odd sign property,
\[
\lim_{t \to \infty} |f(x_1(t), x_2(t)) + \sigma(x_1(t))| = 0
\]
and \( x_2(t) \to 0 \) when \( t \to \infty \), thus assuring attractiveness of the zero-equilibrium. We now show that this is possible if \( f \) fulfills the condition below.

Assumption 3.1: There exist two function \( k : \mathbb{R} \to \mathbb{R} \) and \( h : \mathbb{R}^2 \to \mathbb{R} \), having the same regularity properties as \( f \), with \( k(0) \geq \epsilon > 0 \), such that \( f \) can be written as
\[
f(x_1, x_2) = k(x_1) - (x_2 + \lambda)h(x_1, x_2),
\]
for some nonzero constant \( \lambda \in \mathbb{R} \).

Note that the requirement \( k(0) \geq \epsilon > 0 \) for all \( x_1 \) implies \( h(0, 0) \neq 0 \). Introduce a new state \( \xi \), the dynamics of which are specified below, and let \( x^*_2 \) be the solution of
\[
(x^*_2 + \lambda)\xi = \sigma(x_1) + k(x_1).
\]

Now, using (7), consider the augmented system
\[
\dot{x}_1 = -(x_2 + \lambda)h(x_1, x_2) + k(x_1),
\]
\[
\dot{x}_2 = u,
\]
\[
\dot{\xi} = \dot{h} + h(x_1, x_2) - \xi - x_1(x^*_2 + \lambda).
\]

Define the point
\[
(x_1, x_2, \xi) = (0, 0, h(0, 0)) := \mathcal{E}_2,
\]
and note that, if \( u \) is a static state feedback verifying \( u(\mathcal{E}_2) = 0 \), then \( \mathcal{E}_2 \) is an equilibrium of (9).

Analogously to the stability result described in the previous section, this equilibrium can be (locally) asymptotically stabilized in such a way that the region of attraction is \( \mathbb{R}^2 \times \mathbb{R}^{sgn(h)} \). The following lemmata are instrumental to establish this result.

Lemma 3.1: The controllability rank condition for system (6) at \((0, 0)\) is
\[
h(0, 0) \neq -\lambda \frac{\partial h}{\partial x_2}(0, 0).
\]

Lemma 3.2: If \( \sigma(x_1) + k(x_1) \neq 0 \) for all \( x_1 \) then the function
\[
V(x_1, x_2, \xi) = \frac{1}{2} \left( x_1^2 + (x_2 - x^*_2)^2 + (\xi - h)^2 \right),
\]
with \( x^*_2 \) and \( \xi \) defined by (8), is radially unbounded in \( \mathbb{R}^2 \times \{ \xi \in \mathbb{R} : \xi h(0, 0) > 0 \} \) if and only if
\[
\lim_{x_2 \to -\infty} h(x_1, x_2) = \infty,
\]
uniformly in \( x_1 \).

Theorem 3.3: Suppose that system (6) verifies Assumption 3.1, the controllability rank condition (10) and the growth condition (12).

There exists a function \( \sigma \) with the odd sign property and a neighborhood \( \mathcal{U} \) of the equilibrium \( \mathcal{E}_2 \) of the closed-loop system (9) such that the control law
\[
u = x^*_2 + x_1 h(x_1, x_2) - (x_2 - x^*_2),
\]
with \( x^*_2 \) given by (8), is well-defined, and asymptotically stabilizes \( \mathcal{E}_2 \) with region of attraction \( \mathcal{U} \).

Remark 3.1: The controllability rank condition in Theorem 3.3 is necessary since the explicit computation of the control law (13) yields, after some manipulations,
\[
u(x_1, x_2, \xi) = \frac{\xi^2 \tilde{u}(x_1, x_2, \xi)}{\xi^2 + (\sigma + k) \frac{\partial h}{\partial x_2}},
\]
where \( \tilde{u} \) is a well-defined function. This allows to give a better description of \( \mathcal{U} \). More precisely, the equation
\[
\xi^2 = -(\sigma(x_1) + k(x_1)) \frac{\partial h}{\partial x_2}
\]
defines a surface in \( \mathbb{R}^3 \), that divides the space \( \mathbb{R}^2 \times \mathbb{R}^+ \) into two sets. Letting \( X \) denote the one containing the equilibrium point, it is easy to see that \( \mathcal{U} \) corresponds to the largest level set of \( V \) contained in \( X \).

Corollary 3.4: If \( \partial h/\partial x_2 > 0 \) for all \( x_1 \) and all \( x_2 \) and if, for some \( \eta \in \mathbb{R}^+ \), \( k(x_1) > \eta \) for all \( x_1 \), then there exists a function \( \sigma \) with the odd sign property such that, with \( u \) given by (13), the equilibrium \( \mathcal{E}_2 \) of (9) is locally asymptotically stable and the region of attraction is \( \mathbb{R}^2 \times \mathbb{R}^{sgn(h)} \) and hence the equilibrium \( x_1 = 0, x_2 = 0 \) is globally asymptotically stable.

\[1\] The notation \( \mathbb{R}^{sgn(h)} \) is a compact form to denote \( \mathbb{R}^+ \) when \( h(0, 0) > 0 \) and \( \mathbb{R}^- \) when \( h(0, 0) < 0 \).
Example 3.1: Consider again the example of Section II. Setting $\hat{x}_2 = x_2 - 1$, system (1) is transformed into
\begin{equation}
\begin{aligned}
\dot{x}_1 &= 1 - (\hat{x}_2 + 1)^2, \\
\dot{x}_2 &= u,
\end{aligned}
\end{equation}
which is of the form (6) with $k(x_1) = 1$, $\lambda = 1$, $h(x_1, \hat{x}_2) = (\hat{x}_2 + 1)$. Moreover, $h(0,0) \neq 0$ and the controllability rank condition holds. Finally, we have $\partial h/\partial \hat{x}_2 = 1 > 0$ and the hypotheses of Corollary 3.4 are fulfilled. Therefore, the function $\sigma(x_1) = \tanh(x_1)$ is such that the control law locally asymptotically stabilizes the equilibrium and the region of attraction is $\mathbb{R}^2 \times \mathbb{R}^n$.

IV. The Three-Dimensional SISO Case

We now extend the result described in the previous section to a system with three states in triangular form. More precisely, consider the system
\begin{equation}
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2), \\
\dot{x}_2 &= f_2(x_1, x_2, x_3), \\
\dot{x}_3 &= u,
\end{aligned}
\end{equation}
for some constants $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, and for some smooth functions $h_1: \mathbb{R}^2 \to \mathbb{R}$ and $h_2: \mathbb{R}^3 \to \mathbb{R}$.

Note that since $f_1(0,0) = 0$ and $f_2(0,0,0) = 0$, we have $h_1(0,0) = 0$ and $h_2(0,0,0) = 0$. Moreover, the requirements on $h_1$ and $h_2$ imply $h_1(0,0) \neq 0$ and $h_2(0,0,0) \neq 0$, respectively. Introduce the new states $\xi_1$ and $\xi_2$, the dynamics of which are specified below, and let $x^*_2$ and $x^*_3$ be defined by
\begin{equation}
\begin{aligned}
(x^*_2 + \lambda_1)\xi_1 &= \sigma(x_1) + k_1(x_1), \\
(x^*_3 + \lambda_2)\xi_2 &= \sigma(x_2 - x^*_2) - x_1 h_1(x_1, x_2) + k_2(x_1, x_2) - \dot{x}_2^*, 
\end{aligned}
\end{equation}
for some function $\sigma$ with the odd sign property. Now, consider the augmented system
\begin{equation}
\begin{aligned}
\dot{x}_1 &= k_1 - (x_2 + \lambda_1)h_1, \\
\dot{x}_2 &= k_2 - (x_3 + \lambda_2)h_2, \\
\dot{x}_3 &= u, \\
\dot{\xi}_1 &= h_1 + (h_1 - \xi_1) - x_1 (x^*_2 + \lambda_1), \\
\dot{\xi}_2 &= h_2 + (h_2 - \xi_2) - (x^*_3 + \lambda_2)(x_2 - x^*_2),
\end{aligned}
\end{equation}
Define the point
$$(x_1, x_2, x_3, \xi_1, \xi_2) = (0, 0, 0, h_1(0,0), h_2(0,0,0)) =: E_3.$$
0 and \( k_{2,i}(0,0) > \epsilon_{2,i} > 0 \), and each component of \( f_1 \) and \( f_2 \) can be written as

\[
\begin{align*}
    f_{1,i}(x_1, x_2) &= k_{1,i}(x_1) - (x_{2,i} + \lambda_{1,i}) h_{1,i}(x_1, x_2), \\
    f_{2,i}(x_1, x_2, x_3) &= k_{2,i}(x_1, x_2) - (x_{3,i} + \lambda_{2,i}) h_{2,i}(x_1, x_2, x_3)
\end{align*}
\]

for some non-zero constants \( \lambda_{1,i} \) and \( \lambda_{2,i} \) and some functions \( h_{1,i} \) and \( h_{2,i} \).

Note that since \( f_1(0) = 0 \) and \( f_2(0) = 0 \), \( k_{1,i}(0) = \lambda_{1,i} h_{1,i}(0) \) and \( k_{2,i}(0) = \lambda_{2,i} h_{2,i}(0) \). Moreover, if Assumption 5.1 holds, then \( h_{1,i}(0) \neq 0 \) and \( h_{2,i}(0) \neq 0 \).

Introduce the variables \( x_{2,i}^*, x_{3,i}^* \), \( \xi_{1,i} \) and \( \xi_{3,i} \) such that

\[
\begin{align*}
    (x_{2,i} + \lambda_{1,i}) \xi_{1,i} &= \sigma(x_{1,i}) + k_{1,i} , \\
    (x_{3,i} + \lambda_{2,i}) \xi_{3,i} &= \sigma(x_{2,i} - x_{2,i}^*) - x_{1,i} h_{1,i} + k_{2,i} - x_{2,i}^* ,
\end{align*}
\]

for some function \( \sigma \) with the odd sign property and consider the augmented system

\[
\begin{align*}
    \dot{x}_{1,i} &= k_{1,i} - (x_{2,i} + \lambda_{1,i}) h_{1,i} , \\
    \dot{x}_{2,i} &= k_{2,i} - (x_{3,i} + \lambda_{2,i}) h_{2,i} , \\
    \dot{\xi}_{1,i} &= \xi_{1,i} h_{1,i} - x_{1,i} (x_{2,i}^* + \lambda_{1,i}) , \\
    \dot{\xi}_{3,i} &= h_{2,i} - x_{3,i} (x_{2,i} - x_{2,i}^*)^2 - (x_{3,i} + \lambda_{2,i}) (x_{2,i} - x_{2,i}^*) ,
\end{align*}
\]

Define the point

\[
(x_1, x_2, x_3, \xi_1, \xi_2) = (0, 0, 0, h_1(0), h_2(0)) := \mathcal{E},
\]

and note that, if \( u \) is a static state feedback verifying \( u(\mathcal{E}) = 0 \), then \( \mathcal{E} \) is an equilibrium of (25).

**Theorem 5.1:** Suppose that system (23) verifies Assumption 5.1, the controllability rank condition at the equilibrium and that \( h_1 \) and \( h_2 \) are such that

\[
\lim_{x_{2,i} \to \infty} h_{1,i}(x_1, x_2) = \infty , \quad \lim_{x_{3,i} \to \infty} h_{2,i}(x_1, x_2, x_3) = \infty
\]

uniformly in \( x_1 \) and \( x_{2,j} \) with \( j \neq i \), for \( h_{1,i} \), and uniformly in \( x_1 \), \( x_2 \) and all \( x_{3,j} \) for \( j \neq i \), for \( h_{2,i} \).

There exist a function \( \sigma \) with the odd sign property and a neighborhood \( \mathcal{U} \) of the equilibrium \( \mathcal{E} \) of the closed-loop system (25), such that the control law

\[
u_i = \dot{u}_i = \dot{x}_{3,i} + h_{2,i}(x_{2,i} - x_{2,i}^*) - (x_{3,i} - x_{3,i}^*),
\]

with \( x_{2,i}^* \) and \( x_{3,i}^* \) given by (24), is well-defined, and locally asymptotically stabilizes \( \mathcal{E} \) with region of attraction \( \mathcal{U} \). \( \square \)

**VI. TRANSIENT STABILIZATION OF POWER SYSTEMS**

We finally come to the practical application that motivated this work, namely the problem of stabilizing the equilibrium of a system of \( N \) power interconnected machines described by the equations [13]

\[
\begin{align*}
    \dot{\delta}_i &= \omega_i , \\
    \omega_i &= -D_i \omega_i + P_i - G_i E_i^2 - \sum_{k=1, k \neq i}^n E_k Y_{ik} \sin(\delta_i - \delta_k + \alpha_{ik}) \\
    \dot{E}_i &= -a_i E_i + b_i \sum_{k=1, k \neq i}^n E_k \cos(\delta_i - \delta_k + \alpha_{ik}) + \frac{1}{r_i} (E_{F,i}^2 + u_i).
\end{align*}
\]

where \( \delta_i, \omega_i \), \( E_i, \) and \( u_i, i = 1, \ldots, n, \) are the states, \( \nu_i, i = 1, \ldots, n, \) are the control inputs, \( D_i, P_i, G_i, a_i, b_i, \tau_i \) and \( E_{F,i}^2 \) are positive constants depending on the physical parameters of the \( i \)-th machine, and \( Y_{ik} \) and \( \alpha_{ik} \) are constants depending on the topology of the connections. Let \((\overline{\delta}, 0, \overline{E})\) be the equilibrium. By setting \( x_1 = (\delta_1, \ldots, \delta_n)^\top - \overline{\delta}, x_2 = (\omega_1, \ldots, \omega_n)^\top, x_3 = (E_1, \ldots, E_n)^\top - \overline{E} \)

\[\nu_i = \tau_i \left( u_i + a_i E_i - b_i \sum_{k=1, k \neq i}^n E_k \cos(\delta_i - \delta_k + \alpha_{ik}) \right) - E_{F,i}, \]

for a new input \( u_i \), we obtain the equations

\[
\begin{align*}
    \dot{x}_1 &= x_2 , \\
    \dot{x}_2 &= -\text{diag}\{D_i\} x_2 + P - \text{diag}\{(x_{3,i} + \overline{E})\}^2 \text{G} - \mathcal{F}(x_1, x_3) (x_3 + \overline{E}) , \\
    \dot{x}_3 &= u ,
\end{align*}
\]

(27)

where \( \mathcal{F}(x_1, x_3) = \text{diag}\{\mathcal{F}_i(x_1, x_3)\} \) with

\[
\mathcal{F}_i = \sum_{k=1, k \neq i}^n (x_{3,k} + \overline{E}) Y_{ik} \sin(x_{1,i} + \overline{\delta}_i - x_{1,k} - \overline{\delta}_k + \alpha_{ik}),
\]

(28)

\[
\text{P} = (P_1, \ldots, P_n)^\top \text{ and G} = (G_1, \ldots, G_n)^\top.
\]

The system (27) satisfies the conditions of Theorem 5.1.

In particular, Assumption 5.1 is verified selecting

\[
\begin{align*}
    k_{1,i} &= 1 , h_{1,i} = -1 , \lambda_{1,i} = -1 , \\
    k_{2,i} &= -D_i x_{2,i} + P_i , \\
    h_{2,i} &= G_i (x_{3,i} + \overline{E_i}) + \mathcal{F}_i \lambda_{2,i} = \overline{E_i} .
\end{align*}
\]

The controller derived in Theorem 5.1 was tested in simulations for the two machine example of [13] with the parameters: \( G_1 = 28.9008, G_2 = 20.3936, D_1 = 1, D_2 = 0.2, P_1 = 52.2556, P_2 = 48.4902, \alpha = 0.5430, \)

\( Y_{12} = 51.2579 \) and \( Y_{21} = 36.6127 \). The initial conditions have been set randomly to \( x_1(0) = (0.0462, 0.0971)^\top, x_2(0) = (0.3235, 0.1948)^\top, x_3(0) = (1.3171, 1.9502)^\top, \)

\( \xi_1(0) = (-1, -1)^\top \) and \( \xi_2(0) = (50.4327, 45.9890)^\top. \)

In Figures 3 and 4 the time-histories of the states of the two machines and of the controller states \( \xi_1 \) and \( \xi_2 \) are depicted. As seen from the figures, the transient is very fast.
The control law

\[ u_i = \dot{x}_{3,i}^a + h_i x_{2,i} - (x_{3,i} - x_{3,i}^a) + \epsilon_i \sigma(\Delta_i) \xi_i \]

globally asymptotically stabilizes the equilibrium of the system.

**Remark 6.1:** Selecting, for instance, \( \sigma(a) = \tanh(a) \), and noting that \( 0 \leq \sigma' \leq 1 \), condition (31) is satisfied selecting

\[ 0 < \epsilon_i < \frac{4D_i}{4 + D_i^2}. \]

VII. CONCLUDING REMARKS

Motivated by the problem of transient stabilization of power systems we have developed a procedure to construct asymptotically stabilizing controllers for systems in triangular form, which are not necessarily globally feedback linearizable. The result may, therefore, be seen as a non-trivial extension of the well-known backstepping procedure.

The application of the technique to power systems solves a longstanding problem of explicit derivation of asymptotically stabilizing controllers with a guaranteed domain of attraction defined by a *bona fide* Lyapunov function. Current research is under way to compare, in realistic multimachine simulations, the performance of both controllers.

**REFERENCES**