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# A systems theoretic analysis of fast varying and state dependent delays

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***Abstract*— The aim of this note is to discuss conditions on delay derivatives, frequently encountered in the literature, from a systems theory point of view. First an overview of potential problems when the delay rate exceeds one is given, including the violation of causality and the violation of the principe of existence and uniqueness of solutions. Second, the required assumptions on the delay variation to overcome these problems are stated, allowing to define a state space in a rigorous way. Finally, it is shown that a stability analysis of systems with a fast varying delay, where no assumptions on the delay rate are made, can be performed in a meaningful and practically relevant way in some cases, but should be interpreted in terms of relaxations of solutions.**

## I. INTRODUCTION

In many works on the robust stability of systems with time-varying delay the following assumption is made on the delay derivative,

$$\dot{\tau}(t) < 1, \tag{1}$$

which includes the often used

$$\dot{\tau}(t) \leq \mu, \quad \mu < 1. \tag{2}$$

In a seminal paper extending the sufficient Riccati type stability condition to systems with time varying delays [1] a candidate Lyapunov-Krasovskii functional included a term of the form

$$\alpha(t) = \int_{t-\tau(t)}^{t} x(s)^T Q x(s)\, ds, \quad Q > 0. \tag{3}$$

Many variants appeared making this assumption, e.g. [2], [3]. Nowadays the equivalent LMI form seems favored over the Riccati-form. The time-derivative of (3) is

$$\dot{\alpha}(t) = x(t)^T Q x(t) - (1-\dot{\tau}(t))\; x(t-\tau)^T Q x(t-\tau).$$

If $\dot{\tau}(t) \geq 1$ then both terms are positive, which implies that including a standard term like (3) in a Lyapunov functional becomes useless. This affects the applicability.

Meanwhile several approaches have been proposed which do not necessarily require assumptions like (1)-(2) in the derivation of the stability criteria, see, e.g. [4], [5], [6], [7], [8] and the references therein. These include input-output approaches, approaches based on Integral Quadratic Constraints, on averaging and on the construction of complete-type Lyapunov Krasovskii functionals. Criteria which do not assume any explicit condition on the delay rate are often referred to in the literature as criteria for systems with fast-varying delays.

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The above discussion mainly concerns *technical aspects* about the derivation of stability criteria, where, depending on the problem or the approach taken, conditions on delay rates may or may not explicitly appear in the derivation or the results. However, the recurrent dichotomy $\dot{\tau} < 1$ / $\dot{\tau} \geq 1$ triggers a thorough analysis of the role of conditions on delay derivatives *from a systems's theory point of view*, to find out whether conditions are purely technical, or more fundamental. This is the main goal of the presented work.

Applications of systems with time-varying delay can be found in the context of cutting and milling processes with variable spindle speed in manufacturing [9], [10], and in the context of continuous-time models for sampled data systems [11]. The time-variation can also be due to uncertainty or due to varying system parameters (see, e.g., [12] for an example from process control). In many cases the time-variation of the delay is implicit, via a dependency of the delay on the state, see, e.g. [13] and the reference therein for continuous-time models (so-called fluid flow models) for the behavior of routers in communication networks. In the latter application the delays are state-dependent as the length of the queues are state variables and the delays depend on queuing times.

The structure of the article is as follows. Section II addresses various system theoretic problems when no restrictions are put on the delay variation, allowing also state-dependence. Section III discusses the required conditions on the delays to eliminate these problems. A practical stability analysis of systems with fast varying delays is outlined in Section IV, and the conclusions are presented in Section V.

Throughout the article it is assumed that the delays are nonnegative and differentiable functions whenever they explicitly depend on time. For $a, b \in \mathbb{R}$, $a < b$, we denote with $\mathscr{C}([a,\ b],\ \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[a,\ b]$ onto $\mathbb{R}^n$ and equipped with the supremum norm,

$$\|\phi\|_s := \sup_{\theta \in [a,b]} \|\phi(\theta)\|_2, \quad \phi \in \mathscr{C}([a,\ b],\ \mathbb{R}^n),$$

with $\|\cdot\|_2$ the Euclidean norm.

## II. SYSTEM THEORETIC PROBLEMS

By means of examples we illustrate several potential problems from a systems theory point of view, which are all induced by delays varying with a rate larger or equal than one.

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### A. Violation of causality

Let us consider the system

$$\dot{x}(t) = Bx(t-\tau(t)),$$

where $B$ is invertible. Let $\varepsilon$ be a small positive number. We have

$$\begin{aligned}&\dot{x}(t)-\dot{x}(t+\varepsilon)\\ &= B\left(x(t-\tau(t)) - x(t+\varepsilon-\tau(t+\varepsilon))\right)\\ &= B\left(x(t-\tau(t)) - x\left(t-\tau(t)+\varepsilon(1-\dot{\tau}(t))+O(\varepsilon^2)\right)\right),\end{aligned}$$

which leads to

$$\begin{aligned}x\left(t-\tau(t)+\varepsilon(1-\dot{\tau}(t))+O(\varepsilon^2)\right) &= \\ x(t-\tau(t)) + B^{-1}\left(\dot{x}(t+\varepsilon)-\dot{x}(t)\right).\end{aligned}$$

If $\dot{\tau}(t) > 1$ we get $(1-\dot{\tau}(t)) < 0$. Thus, retrodiction becomes possible because, given the piece of trajectory, $x(s),\ s\in[t-\tau(t),\ t]$, we can determine the past, $x(s),\ s<t-\tau(t)$, from the future, $x(s),\ s>t$. This violates the principle of causality. The underlying reason of the causality problem is that the function $t\mapsto t-\tau(t)$ is strictly *decreasing* whenever $\dot{\tau}(t)>1$. This issue is further expounded in [14], [15].

### B. Inconsistency

Let us consider the example

$$\dot{x}(t) = x(t-\tau(t)), \tag{4}$$

where

$$\tau(t) = \begin{cases} 0, & t<0,\\ 2t, & t\geq 0.\end{cases} \tag{5}$$

In order to define a forward solution at $t=t_0$, with $t_0<0$, we need as initial data a function segment

$$\phi : (-\infty,\ t_0] \to \mathbb{R}. \tag{6}$$

One may argue that the corresponding solution $x(\phi)$, when considered as a function from $\mathbb{R}\to\mathbb{R}$, is in general inconsistent with (4)-(5) in the sense that it does not satisfy the differential equation for $t\leq t_0$ (note that the equations (4)-(5) can still be 'verified' along $x(\phi)$ if $t\leq t_0$). This situation is clarified in Figure 1 (left). To specify a forward solution at $t=t_0$, with $t_0>0$, the minimal information needed is a function segment

$$\phi_1 : (-\infty,\ -t_0] \to \mathbb{R}, \tag{7}$$

supplemented with the scalar $\phi_2 := x(t_0)$. Now, an additional problem arises, because the function segment $\phi_1$ can be continued forwards using (4)-(5), which may lead to the inconsistency at $t=t_0$, illustrated in Figure 1 (right). The only way to avoid these problem consists of imposing artificial, time-dependent restrictions on the set of initial conditions, which lead to an 'allowable' set of solutions

$$x(t) = \begin{cases} Ce^{t}, & t<0,\\ Ce^{-t}, & t\geq 0,\end{cases} \tag{8}$$

parameterized by the single real constant $C$.

The inconsistency problems illustrated in Figure 1 are due to the fact that, when considering a forward solution starting at $t=t_0$, sufficient data is already available to check the relation described by the differential equation at time instants $t<t_0$. This in turn is due to the violation of the condition $\dot{\tau}(t)<1$. To see this, assume now that $\dot{\tau}(t)<1$ for all $t$, which implies that the function $t-\tau(t)$ is strictly increasing. A forward solution at $t=t_0$ has support on the interval $[t_0-\tau(t_0),\ \infty)$. The relation described by the differential equation can only be checked for $t\geq t_0$ because of the implication

$$(t<t_0) \Rightarrow (t-\tau(t) < t_0-\tau(t_0)).$$

Therefore, there is no issue about imposing restrictions on the initial data and the inconsistency problems do not occur any more.

### C. Violation of existence and uniqueness of solutions

Consider first the following scalar neutral system with state-dependent delay:

$$\dot{x}(t) = 1-\alpha\dot{x}(t-\tau(x(t))), \tag{9}$$

where the variation of the delay is described by

$$\tau(x) = \begin{cases} 1, & x\leq 1,\\ 1-\beta+\beta x, & x\in[1,\ 2],\\ 1+\beta, & x\geq 2.\end{cases} \tag{10}$$

Here $\alpha<1$ and $\beta>0$ are parameters.

Let us define the forward solution at $t_0=0$ with initial condition $x(t)=0$ for $t\in[-1-\beta,\ 0]$. For $t\in[0,1]$ the equation reduces to $\dot{x}=1$, and we get

$$x(t)=t,\quad t\in[0,\ 1].$$

At the time-instant $t=1$ a special situation occurs because $t-\tau(x(t))$ passes through zero, where a jump in the derivative of the solution occurs. To determine the dynamics for $t\in[1,\ 1+\varepsilon]$, with $\varepsilon$ a small positive number, we must take into account two potential situations: one where the delayed argument $t-\tau(x)$ becomes negative (this is called 'Branch I' in what follows) and one where it becomes positive (Branch II).

*Branch I.* The dynamics are still described by $\dot{x}(t)=1$, which yields $x(t)=t$. It follows that

$$t-\tau(x) = t-(1-\beta+\beta t) = (1-\beta)(t-1).$$

If $\beta>1$ then $t-\tau(x)<0$ for $t=1+$, hence the branch exists. On the contrary, the condition $\beta<1$ implies $t-\tau(x)>0$ for $t=1+$ and we arrive at a contradiction, so this type of branch cannot occur.

*Branch II.* The dynamics are determined by $\dot{x}(t)=(1-\alpha)$, thus $x(t)=1+(1-\alpha)(t-1),\ t\in[1,\ 1+\varepsilon]$. Therefore, we get

$$\begin{aligned}&t-\tau(x) =\\ &t-(1-\beta+\beta(1+(1-\alpha)(t-1))) = (1-\beta(1-\alpha))(t-1).\end{aligned}$$

Hence, the branch exists if $1-\beta(1-\alpha)>0$. In the other case it cannot exits.

From the previous we conclude the following results.

1) Under the conditions

$$\beta>1,\quad 1-\beta(1-\alpha)>0, \tag{11}$$

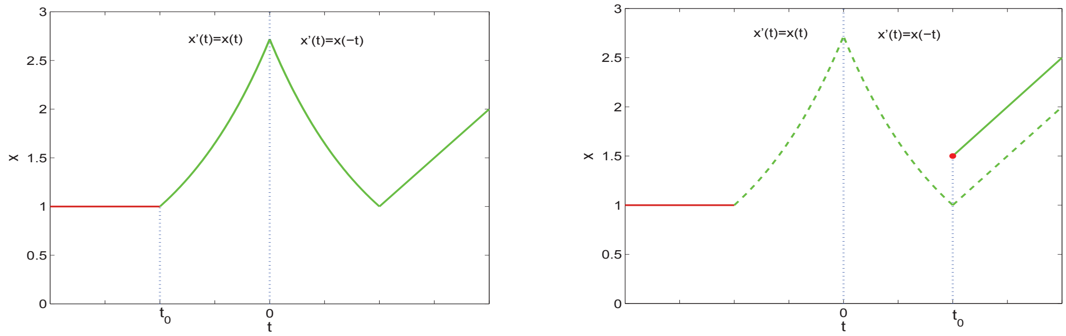


Fig. 1. The definition of a forward solution of (4)-(5) for $t_0 < 0$ (left) and $t_0 > 0$ (right). The initial data is indicated in red, the emanating solution in green. In the right pane the dashed line is obtained by continuing the red function segment.

the solution at $t = 1$ *bifurcates* into two different branches, violating the principle of uniqueness of solutions. This situation, which occurs for instance with the parameter values $\beta = 3/2$ and $\alpha = 1/2$, is illustrated in Figure 2.

2) Under the conditions
$$\beta < 1, \quad 1 - \beta(1-\alpha) < 0, \tag{12}$$
the solution terminates at $t = 1$, violating the principle of existence. This situation occurs for $\beta = 1/2$ and $\alpha = -2$, see Figure 3.
In this case one may argue that the solution can be locally continued in a generalized sense, as a type of "sliding solution", where the delayed time remains pinned at zero. Such a solution is characterized by $t - \tau(x(t)) \equiv 0$, which implies
$$x(t) = \frac{t - 1 + \beta}{\beta}.$$
Note that the conditions (12) imply
$$1 < \beta^{-1} < 1 - \alpha,$$
i.e. the sliding solution (with slope $\beta^{-1}$) lies between Branch I (slope 1) and Branch II (slope $1-\alpha$). It is indicated in Figure 3 with the dashed line.

3) If $(\beta - 1)(1 - \beta(1-\alpha)) < 0$ then the solution can be uniquely continued beyond $t = 1$.

For a detailed analysis of the solutions of neutral delay differential equations with state dependent delays, including existence and uniqueness issues and generalized solutions, we refer to [16].

The special situations described by Case 1 and Case 2 can again be explained by the violation of the condition $\frac{d}{dt}\tau(x(t)) < 1$. In the first case, we have along Branch I,
$$\frac{d}{dt}\tau(x(t)) = \beta > 1.$$
In the second case, we have along Branch II,
$$\frac{d}{dt}\tau(x(t)) = \beta(1-\alpha) > 1.$$

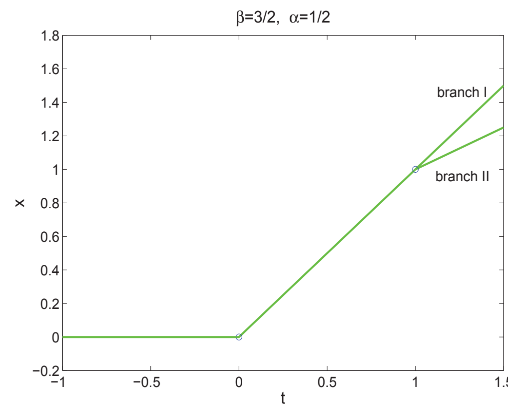


Fig. 2. Forward solution of (9)-(10) at $t_0 = 0$ with zero initial condition, for $\alpha = 1/2$ and $\beta = 3/2$.

We note that if the delays explicitly depend on time (thus, not implicitly via a dependence on the state) it may also happen that a solution ceases to exist. This is illustrated with the example
$$\dot{x}(t) = 1 + \dot{x}(t - \tau(t)), \quad \tau(t) = \begin{cases} 1, & t \le 1, \\ t, & t \in [1,\ 2], \\ 2, & t \ge 2. \end{cases} \tag{13}$$
The solution starting at $t = 0$ with initial condition $x(t) = 1, t \in [-1,\ 0]$, cannot be continued beyond $t = 1$. The derivative of the solution exhibits a discontinuity at $t = 0$. For all $t \in [1,\ 2]$ we have $t - \tau(t) = 0$, hence, no function can satisfy the differential equation almost everywhere on the interval $[1,\ 2]$. It should also be noted that state dependent delay models may give rise to finite escape times.

With the above examples we have isolated the phenomena due to time-varying delays with derivative larger than one. For a detailed discussion of existence and uniqueness issues of functional differential equations, including examples and counter examples, we refer to [17, Chapter I-4.] and [18].

*Remark 1:* The lack of existence and uniqueness can be overcome by imposing additional conditions on the initial

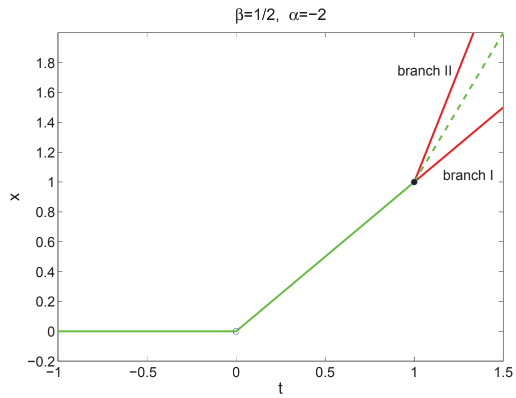


Fig. 3. Forward solution of (9)-(10) at $t_0 = 0$ with zero initial condition, for $\alpha = -2$ and $\beta = 1/2$. At $t = 1$ the solution terminates. The dashed line is the generalized solution characterized by $t - \tau(x(t)) \equiv 0$.

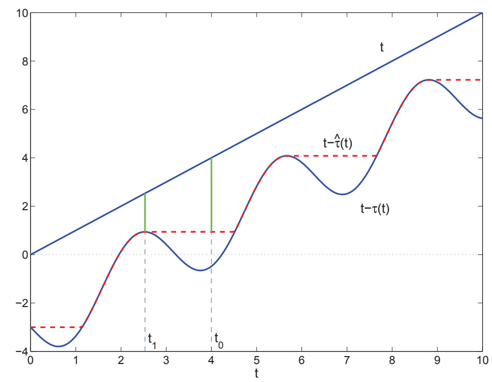


Fig. 4. Principle behind a forgetful causalization. The green segments correspond to the time-interval on which the initial function should be defined at $t = t_1$ and $t = t_0$.

function. For the equation (9), the initial function $\phi$ needs to be differentiable, satisfying

$$\dot{\phi}(0) = 1 - \alpha\dot{\phi}(-\phi(0)), \tag{14}$$

For the equation (13) it needs to be differentiable, satisfying

$$\dot{\phi}(0) = 1 + \dot{\phi}(-1). \tag{15}$$

The conditions (14)-(15) prevent the occurrence and, as a consequence, the propagation of a *discontinuity* in the derivative of the solution at $t = 0$, which is at the basis of the problems discussed above. We refer to [16] and the references therein for more details on time-integration of neutral systems. Note, once again, that such a restriction on the initial conditions is artificial.

### D. Concept of state space

An intuitive way to define initial data for the problem

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t))), \quad x(t) \in \mathbb{R}^n, \tag{16}$$

at time $t = t_0$ would consist of taking a function over the interval $[t_0 - \tau(t_0),\ t_0]$. Similarly one could define the state at time $t$ as a function over the interval $[t - \tau(t),\ t]$, possibly shifted to the interval $[-\tau(t),\ 0]$. With this formulation, there are two problems. First a state-space should naturally have a *stationary* structure, that is, independent of time. Second, if the derivative of the delay is not restricted by one it may be that the initial data turns out to be insufficient since $t - \tau(t)$ can be strictly decreasing. The latter is related to the causality problem discussed in Section II-A.

If the delay is assumed to be uniformly bounded from above, that is,

$$\tau(t) \le \tau_{\max}, \tag{17}$$

for all $t$, one can fix these problems by taking the state space sufficiently large, consisting of functions defined over an interval of length $\tau_{\max}$. This is the standard approach in the literature (see, e.g. [19], [6], [4]). But this renders the state space non-minimal, creating its own problems [15]. Furthermore, the inconsistency problems as described in Section II-B may occur.

In Section III-B we discuss necessary conditions on the delay variation and explain how this allows to define the state space in a mathematical rigorous way.

## III. Conditions on the delay variation

### A. Forgetful causalization

The violation of causality can be avoided by imposing $\dot{\tau}(t) \le 1$. This was the idea behind the so-called *forgetful causalization*, proposed in [20] to regularize problems where $\dot{\tau}(t) > 1$. The underlying idea is explained in Figure 4. The regularization consists of replacing a delay $\tau(t)$ with a delay $\hat{\tau}(t)$, defined by the relation

$$t - \hat{\tau}(t) = \sup_{s \le t} (s - \tau(s)).$$

In this way the missing information is replaced with known information.

Although a forgetful causalization clearly solves the causality problem, the inconsistency problems discussed in §II-B are still present. For instance, the initial data required to define a forward solution at $t = t_0$ (see Figure 4), that is, a function over the interval $[t_0 - \hat{\tau}(t_0),\ t_0]$, already contains sufficient information to define a forward solution at any $t \in [t_1,\ t_0]$, with $t_1$ indicated on the Figure. The latter namely corresponds to a function over the interval

$$[t - \hat{\tau}(t),\ t] \subseteq [t_0 - \hat{\tau}(t_0),\ t_0].$$

Note further that, by definition, a regularization changes the original model. This by itself is not necessarily bad if this original model is unfaithful to the phenomenon one tries to model in the first place.

### B. Natural assumptions on the delay, rigorous definition of state space

One can go a step further and require a strict inequality, $\dot{\tau}(t) < 1$, for all values of $t$. This is still not sufficient to avoid problems, as illustrated with the following example.

*Example 1:* Consider the equation

$$\dot{x}(t) = x + x(t - \tau(t)) \tag{18}$$

and the delay function

$$\tau(t) = t + 1 + e^{-t}, \tag{19}$$

satisfying $\dot{\tau}(t) < 1$ for all $t \in \mathbb{R}$. Defining the initial conditions of (18) and (19) at $t = 0$ as elements of the space $\mathscr{C}([-2,\ 0], \mathbb{R})$ is highly redundant. Given an initial condition $\phi \in \mathscr{C}([-2,\ 0], \mathbb{R})$, the only information required to compute the forward solution is the function segment

$$[-2,\ -1) \ni \theta \mapsto \phi(\theta)$$

and the value $\phi(0)$. This is due to the inequality

$$\lim_{t\to\infty} t - \tau(t) < \infty,$$

or, equivalently, the fact that the function $t \mapsto t - \tau(t)$ is not invertible.

All the above comments and examples eventually lead us to the following natural condition on the delay variation.

*Condition 1:* The function

$$t \mapsto t - \tau(t)$$

from $\mathbb{R}$ to $\mathbb{R}$ is strictly increasing and invertible.

We now come back to the problem of defining a state space, discussed in Section II-D. The relevance of Condition 1 and the property that no additional restrictions are necessary are illustrated by the observation that Condition 1 makes a time-transformation to a system with a *fixed delay* possible, giving the state space a stationary structure. To see this, we reconsider the equation (16). Under Assumption 1 there always exists a function $\mathbb{R} \ni t \mapsto h(t)$ with the following properties.

1) it is strictly increasing, invertible and everywhere differentiable;
2) it satisfies

$$h(t - \tau(t)) = h(t) - 1,\ \forall t \in \mathbb{R}. \tag{20}$$

Such a function can namely be constructed by first considering an arbitrary function segment $\phi :\ [-\tau(0),\ 0] \mapsto \mathbb{R}$, satisfying

$$\begin{array}{l} \phi(-\tau(0)) = \phi(0) - 1, \quad \dot{\phi}(-\tau(0))(1 - \dot{\tau}(0)) = \dot{\phi}(0), \\ \dot{\phi}(t) > 0,\ \forall t \in [-\tau(0),\ 0], \end{array} \tag{21}$$

and, second, extending it to a function over $\mathbb{R}$ by forwards and backward continuation using the equation (20).

We can now apply a transformation of time to (16). By letting $z(\lambda) = x(t)$, where the time-transformation is described by

$$\lambda = h(t), \tag{22}$$

we arrive at the equation with fixed delay

$$\dot{z}(\lambda) = (h^{-1})'(\lambda)\ f\left(h^{-1}(\lambda), z(\lambda), z(\lambda - 1)\right). \tag{23}$$

*Remark 2:* It is important to note that the above construction fails if $\dot{\tau}(t) \geq 1$ for some value of $t$. From the condition (20), which is *necessary* for the transformation to a system with a fixed delay, we namely get

$$\dot{h}(t - \tau(t))\ (1 - \dot{\tau}(t)) = \dot{h}(t).$$

If $\dot{\tau}(t) \geq 1$ we get a contradiction with the other requirement, $\dot{h}(t) > 0$ for all $t \in \mathbb{R}$.

A natural state space for the equation (23) with fixed delay is the space $\mathscr{C}([-1\ 0],\ \mathbb{R}^n)$, see [21], [19]. A solution $x(\phi, \lambda_0)$ is uniquely defined by the starting time $\lambda_0$ and a function $\phi \in \mathscr{C}([-1\ 0], \mathbb{R}^n)$. The state at time $\lambda > \lambda_0$ is given by the function segment $x_\lambda \in \mathscr{C}([-1\ 0],\ \mathbb{R}^n)$, defined as

$$x_\lambda \equiv x(\phi, \lambda_0)(t + \theta),\ \theta \in [-1,\ 0].$$

The corresponding initial condition and states for equation (16) can be obtained via the time-transformation (22).

*Remark 3:* For linear control systems with piecewise constant input, as they appear in the context of digital control, the closed-loop system can often be interpreted as a linear time-delay system with time-varying delay, where the delay function has the form of a sawtooth, with derivative equal to one a.e., see for instance [11]. In this case the function $t \mapsto t - \tau(t)$ is piecewise constant and not invertible. Therefore, a time-transformation to a (continuous time) linear system with constant delay is not possible. However, the system states, sampled at the sample instants, satisfy a (discrete time) linear equation, see [22].

## IV. A PRACTICAL STABILITY ANALYSIS BASED ON A RELAXATION OF SOLUTIONS

Although situations where the delay derivatives exceed one may lead to problems from a system theoretic point of view (see Section II), in many cases these problems do no hinder to uniquely define forward solutions in a *relaxed sense*, which make a practical stability analysis possible and meaningful. This is illustrated in what follows.

Consider a retarded equation of the form (16) and assume the bound (17). When specifying an initial condition by the initial time $t_0$ and an arbitrary element $\phi$ from the (possibly highly redundant) space

$$\mathscr{C}([-\tau_{\max},\ 0],\ \mathbb{R}^n),$$

the corresponding forward solution $x(\phi, t_0)$ for $t \geq t_0$ is uniquely defined, *provided* that one ignores the problems of inconsistency and non-minimality. Stating this requirement in a different way, it corresponds to defining a forward solution $x(\phi, t_0)$ as a continuous function from the interval $[t_0,\ +\infty)$ (instead of $[t_0 - \tau_{\max},\ +\infty)$) to $\mathbb{R}^n$ satisfying[1]

$$\begin{array}{l} x(t_0) = \phi(0), \\ \dot{x}(t) = f(t, x(t), \tilde{x}(t - \tau(t))) \quad \text{a.e.}\ , \end{array}$$

where

$$\tilde{x}(t) = \begin{cases} x(t), & t \geq t_0 \\ \phi(t - t_0) & t \in [t_0 - \tau_{\max},\ t_0]. \end{cases}$$

[1]We suppress the dependence of $\phi$ and $t_0$ in the notation

This definition of forward solutions, albeit in a relaxed sense, is sufficient to investigate the solutions's growth rate and stability properties. For instance, if $f(t,0,0) \equiv 0$ in (16), uniform asymptotic stability of the null solution can be defined as follows,

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall \phi \in \mathscr{C}([-\tau_{\max},\ 0], \mathbb{R}^n)\ \forall t_0 \in \mathbb{R}$$
$$(\|\phi\|_s < \delta) \Rightarrow (\forall t \geq t_0\ \|x(\phi, t_0)(t)\|_2 < \varepsilon), \quad (24)$$
$$\forall \phi \in \mathscr{C}([-\tau_{\max},\ 0], \mathrm{R}^n)\ \forall t_0 \in \mathbb{R}\ \lim_{t \to +\infty} \|x(\phi, t_0)(t)\|_2 = 0. \quad (25)$$

Note that in the implications in (24)-(25) the norm of a vector, $\|x(\phi, t_0)(t)\|_2$, is used. In this way the problem of a rigorous definition of the state or state space is avoided.

The available results on stability of time-delay systems with varying delays, without assumptions on the delay variation like Condition 1 (e.g. [6], [4], [7]), can be interpreted in the context sketched above.

## V. Conclusions

The goal of this note was to point out that the condition $\dot{\tau}(t) < 1$, appearing in many stability criteria for time-delay systems, is not merely a technical condition. We have illustrated that several problems and inconsistencies from a system's theory point of view may occur if the condition is violated. Next we have shown that the Condition 1 on the delay variation is necessary and sufficient. Its violation however does not exclude a practical stability analysis, which relies on a relaxation of the concept of solutions.

It is easy to see that the problems of violation of causality and inconsistency, illustrated in Section II-A and Section II-B, also occur for the discrete time system

$$x(k+1) = f(x(k), x(k - \tau(k))),\ \ \tau(k) \in \mathbb{N},$$

if the function $k \mapsto \tau(k)$ is allowed to be increasing. Moreover, requiring the invertibility of the function $k \mapsto k - \tau(k)$ and requiring a stationary structure and minimality of the state space leads to the necessity of a constant delay. Of course, based on relaxations of solutions stability like notions can also be defined for the case of bounded, time-varying delays, similarly to the analysis in Section IV.

Because the violation of causality has been an important issue in the analysis, we would like to end with a quote:

> *"One cannot remember what one has already forgotten".*


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