Robust Design of PID Controllers for Arbitrary-order LTI Systems with Time Delay
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Abstract—For a given arbitrary-order LTI (linear time-invariant) plant with time delay, we propose a directly parametric design method of the $H_\infty$ PID controller in an analytical manner. The design problem of the PID controllers satisfying the $H_\infty$ norm requirement is first cast into simultaneous stabilization problem of a family of complex quasipolynomials and the characteristic equation. Then, the linear programming characterization of the PID controllers that can ensure the stability of the complex quasipolynomials are developed on the basis of the extended Hermite-Bieler Theorem. Finally, the admissible set of the $H_\infty$ PID controllers is presented by combining such the results with the stabilizing set of the PID controllers. The results reveal that the set of the integral and derivative gains is a union of convex sets for a fixed proportional gain. The proposed scheme works without any approximation and enable one to find a set of the PID parameters satisfying the $H_\infty$ norm requirement conveniently.

I. INTRODUCTION

Time delay often occurs in systems of engineering, biology and ecology, especially in process control. The existence of time delay can degrade the achievable performance and even induce the system instability. Since the time-delay terms cause an infinite number of roots of the characteristic equation, they make the system difficult to design with the classical methods. Thus, such problems are often solved indirectly by using rational approximation. However, the rational approximation constitutes a limitation in accuracy and can lead to the instability of the actual system. Although the controllers can be designed for the system with time delay by using the Lyapunov framework or algebraic Riccati equations, these methods require complex formulations, and lead to conservative results and possibly redundant control. More seriously, these control strategies for systems with time delay cannot present the simple low-order controllers.

It is well known that the simple low-order controllers, especially PID controllers, are the most widely-used control strategy in various engineering application fields. Due to this, a lot of efforts have been made to design PID controllers for the system with time delay. Most of these methods are based on the first-order or second-order plus dead-time model. For high-order plants with time delay, some design methods of low-order controllers rely on model reduction methods or controller reduction methods, which will inevitably lead to degradation of system performance. Hence, it is desired to develop an effective approach for PID controller design to satisfy these requirements simultaneously: 1) It is applicable to a broad set of plants with time delay; 2) It is simple to understand and easy to implement; 3) It can deal with robustness under the unavoidable exogenous disturbances in practice. Such a design problem has not been well resolved in current literature.

The controller design for robustness can be casted into the computation and minimization of $H_\infty$ norm of a prescribed transfer function of the system, and the $H_\infty$ control theory has been developed to the control system synthesis. Unfortunately, the $H_\infty$ optimal controller cannot be directly applied to the systems with time delay. Even if the system with time delay is converted to the approximated one with rational transfer function, the order of the resultant $H_\infty$ controller is always larger or equal than the order of the plant. Recently, the analytical design methods of the $H_\infty$ PID and first-order controllers were developed for systems without time delay, but the design is not applicable to the systems with time delay. To the author’s knowledge, only a few literatures present the low-order $H_\infty$ controller design method for arbitrary-order systems with time delay. A fixed-order $H_\infty$ controllers for a class of time-delay systems was proposed based on a non-smooth, non-convex optimization method. The method needs largely numerical computation. For a given derivative or integral gain, the graphically design method of the robust PID controllers was proposed based on the D-decomposition technique. Therefore, it remains an open question to design a $H_\infty$ PID controller directly in an analytical way.

For a given arbitrary-order LTI plant with time delay, we proposed an algorithm to determine the complete stabilizing set of PID controllers, which can be viewed as the first step for PID controller design. Based on the result, this paper presents a direct parametric design method in an analytical manner for $H_\infty$ PID controllers. The controller design problem is first translated into simultaneous stabilization problem of a family of complex quasipolynomials and then the entire set of all PID gains that can guarantee the stability of these quasipolynomials and the closed-loop characteristic equation is found. The proposed method shows that the set of the integral and derivative gains is a union of convex sets for...
a fixed proportional gain, which is similar to the results for the plant free of time delay. The method is applicable to arbitrary-order LTI systems with time delay.

Fig. 1. Block diagram of the unity feedback control system

II. PROBLEM STATEMENT

The SISO LTI plant with time delay can be described as

\[ G(s) = \frac{N(s)}{D(s)} e^{-\theta s}, \]

where \( \theta \) is the time delay, and \( N(s) \) and \( D(s) \) are coprime polynomials in \( s \), defined as

\[ N(s) = v_m s^m + v_{m-1} s^{m-1} + \cdots + v_0 \]
\[ D(s) = s^n + u_{n-1} s^{n-1} + \cdots + u_0 \]

Here, \( v_0, v_1, \ldots, v_m \) and \( u_0, u_1, \ldots, u_n \) are real numbers, and \( n > m \). Consider the feedback control system shown in Fig. 1, where \( G(s) \) is a plant with the transfer function (1) and \( C(s) \) is a PID controller with the form

\[ C(s) = k_p + \frac{k_i}{s} + k_d s \]  \hspace{1cm} (2)

The goal of the paper is to determine the set of \((k_p, k_i, k_d)\) for which the closed-loop system is stable and satisfies the following \( H_\infty \) performance index

\[ \|W(s)T(s, k_p, k_i, k_d)\|_\infty < \gamma \]

for a given number \( \gamma > 0 \), where \( T(s, k_p, k_i, k_d) \) can be sensitivity function, complementary sensitivity function or input sensitivity function and \( W(s) \) is a stable weighting function to specify the performance requirements of the closed-loop system. Since the closed-loop system is stable, the following \( \|W(s)T(s, k_p, k_i, k_d)\|_\infty \) for a given number \( \gamma > 0 \), where \( T(s, k_p, k_i, k_d) \) is the complementary sensitivity function, i.e.

\[ T(s, k_p, k_i, k_d) = \frac{C(s)G(s)}{1 + C(s)G(s)} \]  \hspace{1cm} (3)

Let the weighting function \( W(s) = W_f(s) W_r(s) \) where \( W_f(s) \) and \( W_r(s) \) are coprime polynomials and \( W_r(s) \) is stable. Substituting (1) and (2) into (3) and then multiply (3) by \( W(s) \), we have

\[ W(s)T(s, k_p, k_i, k_d) = \frac{W_f(s)(k_d s^2 + k_i s + k_p)N(s)}{s W_f(s)D(s) e^{-\theta s} + W_r(s)(k_d s^2 + k_i s + k_p)N(s)} \]  \hspace{1cm} (4)

In order to find \( k_p, k_i, k_d \) values satisfying \( \|W(s)T(s, k_p, k_i, k_d)\|_\infty < \gamma \) in an analytical way, a feasible approach is to convert the synthesis problem of \( H_\infty \) controllers to the quasipolynomial stabilization problem first and then determine the \( k_p, k_i, k_d \) values by solving the stabilization problem. According to the result for the proper rational function, the following lemma is given:

**Lemma 1** Assume that

\[ F(s) = \frac{A(s)}{B(s)e^{\omega s} + E(s)} \]

is stable where \( A(s), B(s) \) and \( E(s) \) are respectively the polynomials with \( \deg(A(s)) = q \), \( \deg(B(s)) = p \) and \( \deg(E(s)) = r \), \( q \leq p \), \( q \leq r \), \( b_i, e_i \), and \( a_i \) are the highest-order coefficient of \( B(s), E(s) \) and \( A(s) \), respectively.

The inequality \( \|F(s)\| < 1 \) holds if and only if

(1) \[ b_i > a_i \] if \( p > r \), \( e_i > a_i \) if \( p < r \), or \( b_i + e_i > a_i \) if \( p = r \);

(2) \[ |B(s)e^{\omega s} + E(s)| + e^{\omega s}A(s) \] is stable for all \( \omega \) in \([0, 2\pi]\).

Proof of Lemma 1 is similar to the case for the rational transfer function in [16] and is omitted here due to the space limit.

The closed-loop characteristic function is given by

\[ \delta(s, k_p, k_i, k_d) = s D(s) e^{\omega s} + (k_d s^2 + k_i s + k_p)N(s) \]

and the quasipolynomial \( \upsilon(s, k_p, k_i, k_d, \varphi) \) is defined as

\[ \upsilon(s, k_p, k_i, k_d, \varphi) = s D(s) W_f(s) e^{\omega s} + (k_d s^2 + k_i s + k_p)N(s) \left[W_f(s) + e^{\omega s}W_r(s)/\gamma\right] \]

From Lemma 1, it is seen that for a given number \( \gamma > 0 \), the necessary and sufficient conditions that the PID gains satisfy \( \|W(s)T(s, k_p, k_i, k_d)\|_\infty < \gamma \) are presented as follows:

(1) \( \delta(s, k_p, k_i, k_d) \) in (6) is stable;

(2) \( \upsilon(s, k_p, k_i, k_d, \varphi) \) in (7) is stable for all \( \varphi \) in \([0, 2\pi]\);

(3) \( \|W(s)T(\omega, k_p, k_i, k_d)\|_\infty \) < \( \gamma \).

As a result, the synthesis of \( H_\infty \) PID controllers is cast into simultaneous quasipolynomial stabilization problem. The set of \((k_p, k_i, k_d)\) for which \( \delta(s, k_p, k_i, k_d) \) is stable can be determined using the result in [14]. However, such a result is not applicable to \( \upsilon(s, k_p, k_i, k_d, \varphi) \) since \( \upsilon(s, k_p, k_i, k_d, \varphi) \) is a complex quasipolynomial. In the following sections, the necessary and sufficient conditions for the stabilization of the complex quasipolynomial will be first given in a simple manner in terms of the extended Hermite-Biehler theorem; based on the result the approach to determine the values of \((k_p, k_i, k_d)\) for which \( \upsilon(s, k_p, k_i, k_d, \varphi) \) is stable is given.

III. PRELIMINARY KNOWLEDGE FOR STABILITY OF THE COMPLEX QUASIPOLYNOMIAL

We first state the extended Hermite-Biehler Theorem. Consider the quasipolynomial

\[ f(s) = \sum_{k=0}^{M} \sum_{j=0}^{N} a_{k,j} s^j e^{\theta_j s}, \]

where \( a_{k,j} \neq 0 \), \( M \) and \( N \) are positive integers, \( a_{k,j} \) is real or complex number, and \( \theta_0, \theta_1, \ldots, \theta_M \) are real numbers satisfying \( 0 < \theta_0 < \theta_1 < \cdots < \theta_M \). The term \( a_{k,j} s^j e^{\theta_j s} \) is called the principle term. For the stability of quasipolynomial \( f(s) \) in (8), the extended Hermite-Biehler Theorem is described as follows:

**Theorem 1**[17,18] \( f(s) \) in (8) is stable if and only if

(i) \( f_j(\omega) \) and \( f_j(\omega) \) have only real zeros and these zeros interlace;

(ii) \( f_j(\omega)f_j'(\omega) > 0 \) for some \( \omega \in (-\infty, +\infty) \).
Here, \( f_{1}(\omega) \), \( f_{2}(\omega) \), \( f'_{1}(\omega) \) and \( f'_{2}(\omega) \) denote the real and imaginary parts of \( f(j\omega) \) and their first derivatives with respect to \( \omega \), respectively.

To ascertain that \( f(\omega) \) and \( f'(\omega) \) have only real zeros, the following theorem is given:

**Theorem 2** Assume that all \( \Theta_{k} \)’s values in (8) are the integers and let \( \eta \) be a constant so that the coefficients of the highest degree terms in \( f_{1}(\omega) \) and \( f_{2}(\omega) \) do not vanish at \( \omega = \eta \). The necessary and sufficient condition under which \( f_{1}(\omega) = 0 \) or \( f_{2}(\omega) = 0 \) has only real roots is that, in the interval \( -2\ell'\pi + \eta \leq \omega \leq 2\ell'\pi + \eta \), \( f_{1}(\omega) \) or \( f_{2}(\omega) \) has exactly \( 4\ell' \Theta_{d} + N \) real zeros starting with a sufficiently large integer \( \ell' \), respectively.

\[ v(s, k_{p}, k_{i}, k_{d}, \phi) \] in (7) is a complex quasipolynomial with the following general form:

\[ H(s) = d(s)e^{\theta s} + n(s) \] (9)

where \( n(s) \) and \( d(s) \) are the polynomials given by

\[ n(s) = (a_{0} + j\beta_{0})s^{k_{0}} + (a_{1} + j\beta_{1})s^{k_{1}} + \cdots + (a_{0} + j\beta_{0}) \]

\[ d(s) = s^{l_{0}} + d_{l_{0}}s^{l_{1}} + \cdots + d_{l_{0}}s + d_{l_{0}} \]

Here, \( a_{0}, a_{1}, \ldots, a_{l_{0}}, \beta_{0}, \beta_{1}, \ldots, \beta_{l_{0}} \) and \( d_{0}, d_{1}, \ldots, d_{l_{0}} \) are the real numbers and \( h > g \).

Based on the preceding theorems, we can obtain the following necessary and sufficient condition for the stability of the complex quasipolynomial \( H(s) \).

**Theorem 3** If \( h > g \), \( H(s) \) in (9) is stable if and only if (i) \( H_{1}(z) \) has exactly \( 4\ell' + h \) real zeros in \([-2\ell'\pi - \xi, 2\ell'\pi - \xi] \);

(ii) All the zeros of \( H_{1}(z) \) in \([-2\ell'\pi - \xi, 2\ell'\pi - \xi] \) interlace with those of \( H_{2}(z) \).

Here \( z = \theta \omega, \ell' \) is a sufficiently large integer, \( H_{1}(z) \) and \( H_{2}(z) \) are, respectively, the real and imaginary parts of \( H(zj/\theta) \), and \( \xi \) is given by

\[ \xi = \begin{cases} \pi/2 & \text{if } h \text{ even and } \alpha_{l_{0}}\beta_{0} = 0 \\ 0 & \text{if } h \text{ odd and } \alpha_{l_{0}}\beta_{0} = 0 \\ \arctan(-\beta_{l_{0}}/\alpha_{l_{0}}) & \text{if } h \text{ even and } \alpha_{l_{0}}\beta_{0} = 0 \\ \arctan(\alpha_{l_{0}}/\beta_{l_{0}}) & \text{if } h \text{ odd and } \alpha_{l_{0}}\beta_{0} = 0 \end{cases} \]

\[ \tau = \begin{cases} \pi/2 & \text{if } e \text{ even and } a_{l_{0}}b_{0} = 0 \\ 0 & \text{if } e \text{ odd and } a_{l_{0}}b_{0} = 0 \\ \arctan(-b_{l_{0}}/a_{l_{0}}) & \text{if } e+f \text{ even and } b_{l_{0}}a_{l_{0}} = 0 \\ \arctan(\alpha_{l_{0}}/a_{l_{0}}) & \text{if } e+f \text{ odd and } b_{l_{0}}a_{l_{0}} = 0 \end{cases} \]

\[ \xi_{i} = \begin{cases} \arctan(-b_{i}/a_{i}) & \text{if } e+f \text{ even and } b_{i}a_{i} = 0 \\ \arctan(\alpha_{i}/a_{i}) & \text{if } e+f \text{ odd and } b_{i}a_{i} = 0 \end{cases} \]

For a given value of \( k_{p} \), let \( z_{i} < z_{i+1} \cdots < z_{i-1} \) be the real and distinct zeros of \( p(z, k_{p}) \) in (13) in the interval \((Z, \bar{Z})\), and assume \( z_{0} = \bar{Z} \) and \( z_{1} = \bar{Z} \). Denote \( \xi_{i} = a_{i} + jb_{i} \) as the leading coefficient of \( M(s) \) and define \( i_{i} \) as follows:

\[ i_{i} = \text{sgn}[p(z, k_{p}, k_{i})] = \begin{cases} 0 & \text{if } M(-jz_{i}/\theta) = 0 \\ -1 & \text{if } M(-jz_{i}/\theta) = 0 \end{cases} \]

where \( t = 0, 1, 2, \ldots \).

**Definition 2** Let \( I = \{i_{0}, i_{1}, \ldots, i_{n} \} \) or \( I = \{i_{1}, i_{2}, \ldots, i_{n} \} \). Then, the signature \( \sigma(I) \) is denoted by
According to Theorem 3, the set of $(k_d, k)$ forming the necessary and sufficient condition for the stability of $\nu(s, k, k_d, k)$ is equivalent to the condition that the net phase angle $\Delta_{z}^{\nu}$ of $\nu(z)$ is sufficiently large, where $\Omega$ represents a very small approximation error. The value of $|\Omega|$ is sufficiently small if $\nu$ is a sufficiently large value. Then, from (18) and (22), it is seen that the necessary and sufficient condition for the stability of $\nu(s, k, k_d, k)$ is equivalent to the condition that the net phase angle $\Delta_{z}^{\nu}$ of $\nu(z)$ is sufficiently large. The foregoing necessary and sufficient condition for the stability of $\nu(s, k, k_d, k)$ is equivalent to the condition that the net phase angle $\Delta_{z}^{\nu}$ of $\nu(z)$ is sufficiently large.

### Theorem 4

Let $l(M)$ and $r(M)$ denote the numbers of left-half plane and right-half plane zeros of $M(s)$, respectively. For a fixed $k_p$, if there exists one string $I$ satisfying

$$\sigma(I) = \left\{ \begin{array}{ll}
\frac{1}{2} \left[ i \cdot \sum_{\nu=1}^{\nu-1} \nu \cdot \left( -1 \right)^{\nu} \cdot \left( -1 \right)^{\nu} \cdot \text{sgn} \left[ q(z_{\nu-1}) \right] \right] \\
\text{if } f \text{ is odd and } \nu \text{ is not purely imaginary} \\
\frac{1}{2} \left[ 2 \cdot \sum_{\nu=1}^{\nu-1} \nu \cdot \left( -1 \right)^{\nu} \cdot \left( -1 \right)^{\nu} \cdot \text{sgn} \left[ q(z_{\nu-1}) \right] \right] \\
\text{if } f \text{ is even and } \nu \text{ is not purely real} \\
\end{array} \right. \tag{16}$$

the set of $(k_d, k)$ ensuring the stability of $\nu(s, k, k_d, k_d, k)$ is the intersection of the following inequalities:

$$\left[ k_i - A(z_i)k_d + B(z_i) \right] i \neq 0 \text{ for } \forall i \in I \text{ and } i \neq 0 \text{ .} \tag{17}$$

Here, $A(z_i)z_i^{-\theta}$ and $B(z_i) = p_i(z_i)[M_z^z(z_i) + M_z^z(z_i)]$. If the strings $I_1, I_2, \ldots, I_n$ all satisfy (16), then the set of $(k_d, k)$ is the union of the regions of $(k_d, k)$ satisfying (17) for $I_1, I_2, \ldots, I_n$.

**Proof:** We first present the condition for which $\nu(z)$ is stable. According to Theorem 3, $\nu(s, k, k_d, k_d, k)$ is stable if and only if $\nu(x)$ has $4\nu + e$ real zeros in $(z, \hat{z})$ and the zeros of $\nu(x)$ interlace with those of the real part $\nu(x)$. The foregoing necessary and sufficient condition for the stability of $\nu(s, k, k_d, k_d, k)$ is equivalent to the condition that the net phase angle $\Delta_{z}^{\nu}$ of $\nu(z)$ is sufficiently large.

The net phase angles of $\nu(z)$ for $z$ changing from $z_1$ to $z_{n-1}$ satisfies

$$\Delta_{z_{1}}^{\nu} = \pi(4\nu + e - 1), \tag{18}$$

The net phase angles of $\nu(z)$ for $z$ changing from $z_{1}$ to $z_{n}$ are, respectively, given to be

$$\Delta_{z_{1}}^{\nu} = \left\{ \begin{array}{ll}
\pi - \tan^{-1} \frac{\nu_1(Z)}{\nu_1(Z)} \\
\tan^{-1} \frac{\nu_1(Z)}{\nu_1(Z)} \\
\end{array} \right. \text{ if } \nu_1(z) < 0 \\
\text{ if } \nu_1(z) > 0 \tag{19}$$

$$\Delta_{z_{1}}^{\nu} = \left\{ \begin{array}{ll}
\pi - \tan^{-1} \frac{\nu_1(Z)}{\nu_1(Z)} \\
\tan^{-1} \frac{\nu_1(Z)}{\nu_1(Z)} \\
\end{array} \right. \text{ if } \nu_1(z) < 0 \\
\text{ if } \nu_1(z) > 0 \tag{20}$$

When $z \to \infty$ and $e > f + 2$, it can be readily be obtained that

$$\nu_1(z) = \sqrt{L_1^z(z) + L_1^z(z) \sin[z + \phi(z)]}$$
$$\nu_1(z) = \sqrt{L_1^z(z) + L_1^z(z) \cos[z + \phi(z)]}$$

where

$$\phi(z) = \left\{ \begin{array}{ll}
0 \text{ or } \pi & \text{for } e \text{ even} \\
\pi/2 \text{ or } -\pi/2 & \text{for } e \text{ odd} \end{array} \right.$$  

In each case for $\phi(z) = 0, \pi, \pi/2$ and $-\pi/2$, the inequality $\nu_1(z) < 0$ always holds due to the periodicity property of $\nu_1(z)$ and $\nu_1(z)$ when $z \to \infty$. Furthermore, $\nu_1(Z) = \nu_1(Z) > 0$ and $\tan^{-1} \frac{\nu_1(Z)}{\nu_1(Z)} = -\arctan \frac{\nu_1(Z)}{\nu_1(Z)}$.
from Definition 1 that $Z = 2\pi \rho$ and $\bar{Z} = 2\pi \rho$ for $e$ odd and $Z = 2\pi \rho + \pi/2$ and $\bar{Z} = 2\pi \rho + \pi/2$ for $e$ even. Thus, as $z \to \infty$, $Z$ and $\bar{Z}$ can be regarded as the zeros of $q(z, k_p)$ for $f$ odd, while for $f$ even, $Z$ and $\bar{Z}$ can be regarded as the zeros of $p(z, k_p) $.

2) $b_f \neq 0$ and $a_f = 0$

If $b_f \neq 0$ and $a_f = 0$, the leading coefficient of $M(s)$ is purely imaginary. Following the similar lines as that in Case 1), it can be obtained that as $z \to \infty$, $Z$ and $\bar{Z}$ are the approximate zeros of $q(z, k_p) $ for $f$ even, and otherwise, they are the approximate zeros of $p(z, k_p, k_d) $.

3) $a_f \neq 0$ and $b_f \neq 0$

If $a_f \neq 0$ and $b_f \neq 0$, the leading coefficient of $M(s)$ is complex. From (26)-(28), it is seen that the zeros of $q(z, k_p)$ tend to those of $\sin(\pi z) = 0$ as $z \to \infty$. Hence, $Z = 2\rho - \pi$ and $\bar{Z} = 2\pi - \pi$ are presented in Definition 1 in the case $a_f \neq 0$ and $b_f \neq 0$, where

$$\epsilon = \left\{ \begin{array}{ll}
\pi \left( \text{sgn}(a_r) \right) + \pi \left( \text{sgn}(a_f) \right) & \text{if } e + f \text{ even} \\
\pi \left( \text{sgn}(a_r) \right) & \text{if } e + f \text{ odd}
\end{array} \right. \quad (30)
$$

Thus, $Z$ and $\bar{Z}$ can be regarded as the zeros of $q(z, k_p)$.

If $Z$ and $\bar{Z}$ are approximate zeros of $q(z, k_p)$, then from the result in [17], it can be derived that the net phase angle $\Delta \theta$ of $\psi(zj/\theta, k_p, k_i, k_d, \varphi)M(-zj/\theta)$ is

$$\Delta \theta = \sum_{i=0}^{\infty} \pi \left( \sum_{i=0}^{\infty} \text{sgn}(p(z, k_j)) \right) \quad (31)
$$

If $Z$ and $\bar{Z}$ are approximate zeros of $p(z, k_p, k_d)$, $\Delta \theta$ is

$$\Delta \theta = \frac{\pi}{2} \left( \sum_{i=0}^{\infty} \text{sgn}(p(z, k_i)) \right) \quad (32)
$$

Combining (23), (31) and (32), taking $\text{sgn}(p(z, k_i)) = i$, for $t = 1, 2, \cdots, c$, and using Definition 2, we have

$$T(s) = \frac{(k_i s^2 + k_i s + k_i)(s + 2)}{(s^3 + 5s^2 + 7s + 3)e^{0.5\pi} + (k_i s^2 + k_i s + k_i)(s + 2)} \quad (33)
$$

A sufficiently large value of $\epsilon$ can always be found to make $|\psi(0, O)| < \pi$ . Since $4\epsilon + e^{-[\psi(0, M) - r(M)]} - \sigma(I) = (O - O)/T$ can be derived. This means that $\psi(s, k_p, k_i, k_d, \varphi)$ is stable if and only if (16) holds.

Finally, for a fixed $k_z$, we determine the set of $(k_p, k_i)$ for which $\psi(zj/\theta, k_p, k_i, k_d, \varphi)$ is stable. Each feasible string $I$ can be found by using (16). As in the proof of the main results in [14], if $M(zj/\theta) = 0$, the corresponding $i$ belonging to the feasible string $I$ is pre-determined and the inequality (17) can be obtained. Thus, the stabilizing set of $(k_p, k_i)$ is the union of stabilizing regions of $(k_p, k_i)$ satisfying (17) for all feasible strings $I_1, I_2, \cdots, I_c$.

V. ALGORITHM FOR SYNTHESIS OF $H_\infty$ PID CONTROLLER

Based on the results given in Section II and [14], and Theorem 4, the entire set of the PID control parameters that guarantee $\|W(s)T(s, k_p, k_i, k_d)\| < \gamma$ can be derived. The detailed algorithm is presented as follows:

Step 1: Determine the allowable range of $k_p$ over which the partitioning needs to be carried out by using Theorem 6 in [14].

Step 2: Pick up a grid point of $k_p$ in the resultant range.

Step 3: Determine the stabilizing set of $(k_p, k_i)$ based on Theorem 7 in [14] and denote it as $S_{(k_p, k_i)}$.

Step 4: For a fixed value of $\varphi$, set $L(s) = \text{sd}(s)W(s)$ and $M(s) = N(s)[W(s) + e^{\varphi}W'(s)/\gamma]$. By using Theorem 4 and solving a linear programming problem, present the set of $(k_p, k_i)$ for which $\psi(s, k_p, k_i, k_d, \varphi)$ is stable and denote it as $S_{(k_p, k_i)}$. By sweeping over $\varphi \in [0, \pi]$, determine the set of $(k_p, k_i)$ such that each $\psi(s, k_p, k_i, k_d, \varphi)$ is stable and let this set be defined as $S_{(k_p, k_i)} = \bigcap S_{(k_p, k_i)}$.

Step 5: Determine the admissible set of $(k_p, k_i)$ for which $\|W(s)T(s, k_p, k_i, k_d)\| < \gamma$ and define it as $S_{(k_p, k_i)}$.

Step 6: Determine the entire set of $(k_p, k_i)$ for which $\|W(s)T(s, k_p, k_i, k_d)\| < \gamma$ by finding the intersection of $S_{(k_p, k_i)}$ and $S_{(k_p, k_i)}$.

Step 7: Go to Step 2 with another grid point of $k_p$ till all the grid points are considered.

In order to check the validity of the above-mentioned algorithm, the following example is given.

**Example 1** Consider the time-delayed plant

$$G(s) = \frac{s + 2}{s^3 + 5s^2 + 7s + 3} e^{-0.5\pi}$$

The weight $W(s)$ is chosen as $W(s) = (s + 0.1)/\gamma$, which is the same as that in [10]. The problem is to determine the set of PID control parameters so that $\|W(s)T(s)\| < 1$.

The complementary sensitivity function is given as

$$T(s) = \frac{(k_i s^2 + k_i s + k_i)(s + 2)}{(s^3 + 5s^2 + 7s + 3)e^{0.5\pi} + (k_i s^2 + k_i s + k_i)(s + 2)} \quad \gamma = 1$$

and $\gamma = 1$. From (6) and (7), we have

$$\psi(s, k_p, k_i, k_d, \varphi) = (s^3 + 5s^2 + 7s + 3)e^{0.5\pi} + (k_i s^2 + k_i s + k_i)(s + 2) \quad \psi(s, k_p, k_i, k_d, \varphi) = 0.1$$

It is known that the PID gains satisfy $\|W(s)T(s)\| < 1$ if and only if the following three conditions hold:

1) $(s, k_p, k_i, k_d)$ is stable;

2) $\psi(s, k_p, k_i, k_d, \varphi)$ is stable for all $\varphi \in [0, 2\pi]$;

3) $\|W(s)T(s)\| < 1$.

Since Condition (3) always holds, we only need consider Conditions (1) and (2). By using Theorem 6 in [14], it is obtained that the allowable range of $k_p$ is from -1.364 to 3.782. Then, the stabilizing region of $(k_p, k_i)$ is determined for a fixed $k_p \in (-1.364, 3.782)$, for example, $k_p = 1$. When $k_p = 1$, the stabilizing set $S_{(1, 1)}$ is derived based on Theorem 7 in [14]. By Setting $L(s) = s(s^3 + 5s^2 + 7s + 3)(s + 1)$ and $M(s) = (s + 2)[s + 1 + e^{0.5\pi}(s + 0.1)]$, the admissible set $S_{(1, 1)}$ can be presented on the
basis of Theorem 4. Thus, for \( k_p = 1 \), the set of \((k_d, k_i)\) for which \( \| W(s) \| < 1 \) is the intersection of \( S \) and \( S_2 \), which is sketched in Fig. 2. The values of \( \| W(s) \| \) corresponding to different \((k_d, k_i)\) values are presented in Table I, which shows that all the values of \( \| W(s) \| \) are less than 1 for the \((k_d, k_i)\) values inside the region in Fig. 2, while all of them are larger than 1 for those outside the region. By repeatedly using Steps 3, 4, 5 and 6 for each \( k_p \) value in \((-1.364, 3.782)\), we can obtain the set of \((k_d, k_i, k_p)\) for which \( \| W(s) \| \) < 1. The admissible set of is shown \((k_d, k_i, k_p)\) in Fig. 3.

![Figure 2](image_url)

**Fig. 2.** The set of \((k_d, k_i)\) satisfying \( \| W(s) \| < 1 \) for \( k_p = 1 \)

**Table I**

<table>
<thead>
<tr>
<th>((k_d, k_i)) values outside the set</th>
<th>((k_d, k_i)) values inside the set</th>
<th>( | W(s) | )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.5, 2)</td>
<td>(0.5, 0.5)</td>
<td>1.687</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>(1.5, 1)</td>
<td>1.51</td>
</tr>
<tr>
<td>(1, 2.5)</td>
<td>(0.5, 0.6)</td>
<td>2.011</td>
</tr>
</tbody>
</table>

![Figure 3](image_url)

**Fig. 3.** The set of \((k_d, k_i)\) for which \( \| W(s) \| < 1 \)

**VI. CONCLUSION**

In this paper, we propose a parametric design method of the \( H_\infty \) PID controller for the general LTI plant with time delay. It is shown that the design problem to the \( H_\infty \) PID controller can be converted to simultaneous stabilization problem of the complex quasipolynomials and the characteristic equation, and thus, the characterization of the PID gain values for which the complex quasipolynomial is stable is presented in terms of the extended Hermite-Biehler Theorem. Based on the result, the algorithm to determine the set of the PID controller is provided both to meet \( H_\infty \) norm requirement and to ensure the stability of the system. The characterization for \( H_\infty \) PID controllers involves a procedure of solving a linear programming problem, which allows the simple and effective computation. The proposed method is a systematic method and applicable to an arbitrary-order system with time delay. Given the frequency occurrence of time delay and the widespread use of PID controllers in the industrial practice, it is expected that the results of this paper will contribute to the development of the practical control system design.

**REFERENCES**