Finite time control design for contPNs according to piecewise constant control actions

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Abstract—Advanced control design for continuous Petri nets requires the characterization of positive invariant and attractive regions in marking space. Based on such characterizations, piecewise constant control design is investigated. Sufficient conditions for control design in finite time are proposed. An algorithm for piecewise constant control design in minimal time is proposed as a consequence.

I. INTRODUCTION

DEPENDENT control for manufacturing systems leads to advanced methods and models. In this context, timed continuous Petri nets (contPNs) can be used in order to provide accurate representations of the behaviors of discrete event systems and to take benefit from the main advances in continuous systems control theory [6,13]. Controllability and control actions for contPNs have been considered by adapting linear control theory to contPNs in order to take into account their specific properties [10,19]. In particular conditions so that a contPN is controllable with bounded control actions, initial marking, steady state and neighborhood of a desired value are proposed. An algorithm moves the marking from a given initial value to the attractive regions in marking space. Based on such characterizations, piecewise constant control design in minimal time is proposed as a consequence.

II. CONPNs WITH CONTROL ACTIONS

A. Petri nets

A Petri net (PN) is defined as \(<P,T,W_{PB},W_{PO}>>\) where \(P\) = \({P_i}\) is a set of \(n\) places and \(T = \{T_j\}\) is a set of \(q\) transitions and \(W = W_{PO} − W_{PB} \in (Z)^{n×q}\) is the incidence matrix. \(<PN, M_0>\) is a marked PN, \(M_i\) is the PN initial marking, \(M = (m_j) \in (Z)^q\) denotes the current marking vector and \(R(PN, M_f)\) is the PN reachability set (i.e. the set of markings that are reachable from initial marking \(M_0\)) [6]. A Petri net \(b\)-bounded is a marking \(m_i\) of each place \(P_i\) is bounded by \(b\) (i.e. for all \(M \in R(W_{PB},W_{PO},M_b)\) and for all \(P_i \in P\), \(m_i \leq b\)). Each transition \(T_j\) fires according to the integer part of its enabling degree \(enab(M)\):

\[
enab(M) = \min \{m_k / w_{P_j k} : P_k \in \sigma T_j\}
\]

where \(\sigma T_j\) stands for the set of \(T_j\) upstream places. A firing sequence \(\sigma\) is defined as an ordered series of transitions that successively fire from initial marking \(M_0\) to reaching \(M\) (i.e. \(M_0 \in \sigma > M\)). Such a sequence may be represented by the PN firing count vector \(\sigma = (\sigma) \in (Z)^q\), where \(\sigma\) is the number of \(T_j\) firings. PN may have P-semiflows. A P-semiflow \(\gamma\) in \((Z)^q\) is a non-zero solution of equation \(\gamma^T W = 0\). Let us define \(\gamma \in (Z)^q \times \text{rank}(\gamma) = \gamma^T \gamma = (y_1 \ldots y_{\text{rank}(\gamma)})\) as the matrix obtained according to all minimal P-semiflows \(\gamma_i\), \(i = 1,\ldots,\text{rank}(\gamma)\). \(Y^T M = C\) with \(C = Y^T M\).

B. Controlled timed continuous Petri nets

Timed continuous PNs under infinite server semantic (contPNs) have been developed in order to provide continuous approximations of the discrete behaviors of PNs [6,13]. The marking of each place is a continuous non negative real valued function of time and \(M(t, M_j) \in (R)^q\), \(t \geq 0\) is the continuous marking trajectory that starts with \(M_j\) at \(t = 0\). \(X_{max} = \text{diag}(x_{max}) \in (R)^{n \times q}\) is the diagonal matrix of maximal firing speeds \(x_{max \{j\}, j = 1,\ldots,q}\) and \(X(t, M_j) = (x_1(t, M_j) \ldots x_q(t, M_j)) \in (R)^{q \times q}\) is the firing speeds vector at time \(t\) in free regime that depends continuously on the marking of the places. The flow through the transition \(T_j\) is defined by (2):

\[
x_j(t, M_j) = x_{max \{j\}},enab(M(t, M_j) )
\]

Control actions may be introduced according to a reduction of the flow through the transitions [9]. Transitions in which a control action can be applied are called controllable and \(T_c\) is defined as the set of controllable transitions. Similarly \(T_{nc}\) is the set of uncontrollable transitions. For a contPN with \(q_c\) controllable transitions, let us assume, without any loss of generality that \(T_c = \{T_1,\ldots, T_{q_c}\}\) (\(T_c\) can eventually be empty) and \(T_{nc} = \{T_{q_c+1},\ldots, T_q\}\) and define \(Q_c = (I_{q_c} | 0_{q_c \times (q-q_c)}) \in (Z)^{q \times q}\) and \(Q_{nc} = (0_{(q-q_c) \times q_c} | I_{q-q_c}) \in (Z)^{q \times q}\). The control actions are summed up in...
control vector \( U(t) = (u_j(t)) \in (R^+)^q \). As a consequence, the marking variation of contPNs is given by (3):

\[
dM(t, M_j) / dt = W_c X(t, M_j) - U(t) 
\]

with \( 0 \leq Q_c U(t) \leq Q_c X(t, M_j) \) and \( Q_{nc} U(t) = 0 \).

A control sequence \( U(t) \), \( t \geq 0 \) that satisfies the preceding conditions for a given marking trajectory is named an bounded input control and the set \( \text{BIC}(T_c, M_j) \) of admissible control sequences \{\( U(t) \), \( t \geq 0 \)\} for initial marking \( M_j \) is defined as a consequence:

\[
\text{BIC}(T_c, M_j) = \{ \{ U(t), t \geq 0 \}, \text{such that} \ 0 \leq Q_c U(t) \leq Q_c X(t, M_j) \text{ and } Q_{nc} U(t) = 0 \}
\]

C. Regions for contPNs

For a contPN with P-semiflows represented by matrix \( Y \), any reachable marking \( M(t, M_j) \in (R^+)^q \) satisfies \( Y^T M(t, M_j) = C \). So, linear dependencies between marking variables appear and the reachable set of contPNs is defined as a consequence. The reachable set, \( R(\text{contPN}, M_j) \subset (R^+)^q \), of a marked contPN, \( \langle PN, X_{max}, M_j \rangle \), is defined as the set of all reachable markings \( M(t, M_j), t \geq 0 \), from a given initial marking \( M_j \) and for all matrices \( X_{max} \in (R^+)^{q \times q} \) of maximal firing speeds.

Switches occur in contPNs according to the function \( \text{"min\((),\)"}\) in the expression of the enabling degree (1). Let us define the critical place(s) for transition \( T_j \) at time \( t \) as the place(s) \( P_i \) such that \( i = \text{argmin} \{m_i(M_j) / w^P_{P_i}, P_i \in \bigcirc T_j\} \). \( R(\text{contPN}, M_j) \) can be partitioned in \( K \) reachable regions (r-regions) with \( K \leq \Pi \{^\circ T_j \}, j = 1,...,q\} : R(\text{contPN}, M_j) = A_1 \cup \ldots \cup A_k \). In the interior of any r-region \( A_k \), each transition has a single critical place. PN configurations [14] are used to define the r-regions. A configuration is a cover of \( T \) by its input arcs and assigns to each transition a single input place: \( \text{config}(k) = \{P_{i(k,j)}, T_j\}, j = 1,...,q\}, k = 1,...,\Pi \{^\circ T_j \}, j = 1,...,q\}, \) where \( P_{i(k,j)} \in ^\circ T_j \) is the single input place of transition \( T_j \) in configuration \( k \) (\( i(k, j) \) stands for the index of the concerned place). R-regions of marked contPNs are defined as:

**Definition:** The reachable region (r-region) \( A_k \subset R(\text{contPN}, M_j), k = 1,..., K \) of a marked contPN, \( \langle PN, X_{max}, M_j \rangle \), is defined for a given configuration \( \text{config}(k) \), and for all matrices \( X_{max} \in (R^+)^{q \times q} \) as the set of all reachable markings \( M(t, M_j), t \geq 0 \), that satisfy (1) \( Y^T M(t, M_j) = C \). (2) \( \forall T_j \in T \), \( P_{i(k,j)} \) is the critical place of transition \( T_j \) for marking \( M(t, M_j) \).

Each r-region \( A_k \) is characterized by a constraint matrix \( A_k = (a_{ij}^k) \in (R^+)^{n \times n} \), \( k = 1,..., K, i = 1,..., q \) and \( j = 1,..., n \):  

- \( a_{i(k,j)}^k = 1/w^{P_{i(k,j)}} \) for all \( T_j \in T \), 
- \( a_{ij}^k = 0 \) otherwise.

The constraint matrices \( A_k \) lead to a linear matrix inequality (LMI) that characterizes the r-regions according to proposition 1.

**Proposition 1** [12]: Let us consider a contPN with \( K \) r-regions \( A_k \). Each r-region \( A_k \) is a polyhedral set characterized by the LMI \( H_k M \leq h_k \) with:

\[
H_k = \begin{pmatrix}
-A(k) \\
A(k)
\end{pmatrix}
\]

\[
h_k = \begin{pmatrix}
0 \\
C
\end{pmatrix}
\]

and \( A(k) = \begin{pmatrix}
A_k - A_1 \\
A_k - A_{k-1} \\
\vdots \\
A_k - A_{K+1} \\
A_k - A_K
\end{pmatrix}
\]

and \( I_n \) is the identity matrix of size \( n \).

The LMI \( H_k M \leq h_k \) is composed of \( n + q(K-1) + 2hp \) inequalities to be satisfied by \( n \) variables. Let us notice that the set of inequalities \( A(k) \leq 0 \) include numerous trivial inequalities (i.e. \( 0 \leq 0 \)) and also several identical inequalities that can be removed for rapidity.

D. Piecewise constant control actions

The marking variation (3) of contPNs can be rewritten in each r-region \( A_k \). For all \( M(t, M_j) \in A_k \), equation (5) holds:

\[
dM(t, M_j)/dt = W_c X_{max} A_k M(t, M_j) - U(t)
\]

with \( 0 \leq Q_c U(t) \leq Q_c X_{max} A_k M(t, M_j) \) and \( Q_{nc} U(t) = 0 \).

If the control actions are constant in r-region \( A_k \) (i.e. \( U(t) = U_k \) for all \( M(t, M_j) \in A_k \)), contPNs are piecewise-affine hybrid systems and for all \( M(t, M_j) \in A_k \), equation (5) leads to (6):

\[
dM(t, M_j)/dt = W_c X_{max} A_k M(t, M_j) - W U_k
\]

with \( 0 \leq Q_c U_k \leq Q_c X_{max} A_k M(t, M_j) \) and \( Q_{nc} U_k = 0 \).

A constant control vector \( U_k \) that satisfies the preceding conditions for a given marking trajectory included in \( A_k \) is named an admissible constant bounded input control and the sets \( \text{CBIC}(A_k, T_c, M_j) \) of admissible constant control vectors for initial marking \( M_j \in A_k \) and \( \text{CBIC}(A_k, T_c) \) of admissible constant control vectors in \( A_k \) for all initial marking \( M_j \in A_k \) are defined as a consequence:

\[
\text{CBIC}(A_k, T_c, M_j) = \{ U_k \in (R^+)^q : \text{such that} \ 0 \leq Q_c U_k \leq Q_c X_{max} A_k M(t, M_j), t \geq 0 \text{ and } Q_{nc} U_k = 0 \}
\]

\[
\text{CBIC}(A_k, T_c) = \cup \{ \text{CBIC}(A_k, T_c, M_j) : M_j \in A_k \}
\]

For numerical issues, discrete time approximations of the continuous trajectories of contPNs will be used. First order approximations of equation (6) in discrete time with sampling period \( \Delta t \) result from direct or retrograde schemes. If the control actions are constant in r-region \( A_k \) and \( U_k \in \text{CBIC}(A_k, T_c, M_j) \), the direct schema is given by (7):

\[
M(t, M_j) = (A_{Dk})^j M_j - \Sigma_k(t) U_k
\]

with \( A_{Dk} = W_c X_{max} A_k \Delta t + I_n \) and \( \Sigma_k(t) = (\Sigma_j j=1 (A_{Dk})^{j-1}) \Delta t \).
III. POSITIVE INVARIANCE AND ATTRACTION IN FINITE TIME

Positive invariance and attraction of polyhedral regions for linear and non-linear systems have been investigated according to the computation of LMI's and linear programming problems (LPP)\(^1\) [3, 4, 5].

A. Positive invariance in finite time

Let us consider any region \(A \subseteq A_k\). \(A\) is positive invariant for the contPN (3) in free of forced regime if and only if \(M_f \in A\) implies \(M(t, M_f) \in A\), for all \(t \geq 0\). From a practical point of view, \(\tau\)-positive invariance is introduced.

**Definition:** A region \(A \subseteq A_k\) is \(\tau\)-positive invariant for the contPN (3) in free of forced regime if and only if \(M_f \in A\) implies \(M(t, M_f) \in A\), for all \(t = 0, \ldots, \tau\).

LMI's can be used to check the \(\tau\)-positive invariance of a given region \(A \subseteq A_k\).

**Proposition 2:** Let us consider a contPN with \(T_c = \{T_i, \ldots, T_{qc}\}\) and \(T_{nc} = \{T_{qc+1}, \ldots, T_q\}\) and \(K r\)-regions \(A_k\). Let us also consider an arbitrary region \(A \subseteq A_k\) defined by \(H.M \leq h\). The region \(A\) is \(\tau\)-positive invariant for initial marking \(M_f \in A\) and system (7) under constant control actions \(U_i\) if and only \(M_f\) and \(U_i\) satisfy LMI (8):

\[
\begin{pmatrix}
Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & M_f \\
0 & -Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & \cdots & M_f \\
: & : & \cdots & : \\
0 & 0 & \cdots & 0 & M_f
\end{pmatrix}
\leq
\begin{pmatrix}
Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & M_f \\
0 & -Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & \cdots & M_f \\
: & : & \cdots & : \\
0 & 0 & \cdots & 0 & M_f
\end{pmatrix}
\]

**Proof:** The proof is a direct consequence of \(r\)-regions and polyhedral sets characterization (proposition 1), CBIC and \(\tau\) - positive invariance definitions. The first group of inequalities (rows 1 to 3 of (8)) ensures that \(M(t, M_f) \in A, 0 \leq t < \tau\). The second group of inequalities (rows 4 to 8 of (8)) ensures that the constant control actions are admissible CBIC according to \(T_c\) and \(T_{nc}\).

Let us notice that proposition 2 can also be used to work out the set of CBIC control actions and also the set of initial markings such that the region \(A\) is \(\tau\)-positive invariant. LPPs are used for that purpose.

B. Attraction in finite time

The region \(A \subset R^2\) is attractive for the contPN (3) with initial marking \(M_f\) if and only if \(\exists \Delta > 0, \forall \varepsilon > 0\), there is \(\tau > 0\) such that \(\inf\|M_f - M'\| : M' \in A\) < \(\Delta\) implies \(\inf\|M(t, M_f) - M'\| : M' \in A\) < \(\varepsilon\), \(t > \tau\). For engineering needs, it is more important to ensure attraction with finite attraction time \(\tau\) [8]. A region \(A \subset R^2\) is attractive with finite attraction time \(\tau\) for the contPN (3) if and only if \(\exists \Delta > 0, \forall M_f\) such that \(\inf\|M_f - M'\| : M' \in A\) < \(\Delta\) implies \(\inf\|M(t, M_f) - M'\| : M' \in A\) = 0, \(t \geq \tau\). The time \(\tau\) is called the finite attraction time. For practical issues (\(\tau, \tau'\)) – attractivity is introduced.

**Definition:** The region \(A \subset (R')^n\) is \((\tau, \tau')\) – attractive for the contPN (3) if and only if \(\exists \Delta > 0, \forall M_f\) such that \(\inf\|M_f - M'\| : M' \in A\) < \(\Delta\) implies \(\inf\|M(t, M_f) - M'\| : M' \in A\) = 0 for \(\tau < t < \tau + \tau' + 1\).

LMI's can be used to check the \((\tau, \tau')\) – attractivity of a given region \(A \subseteq A_k\).

**Proposition 3:** Let us consider a contPN with \(T_c = \{T_i, \ldots, T_{qc}\}\), \(T_{nc} = \{T_{qc+1}, \ldots, T_q\}\) and \(K r\)-regions \(A_k\). Let us also consider an arbitrary region \(A \subseteq A_k\) defined by \(H.M \leq h\). The region \(A\) is \((\tau, \tau')\) – attractive for system (7) under constant control actions \(U_i\) if there exists a non empty set of initial markings \(M_f \in A\) such that \(M_f\) and \(U_i\) satisfy the LMI (9):

\[
\begin{pmatrix}
Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & M_f \\
0 & -Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & \cdots & M_f \\
: & : & \cdots & : \\
0 & 0 & \cdots & 0 & M_f
\end{pmatrix}
\leq
\begin{pmatrix}
-Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & M_f \\
0 & -Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & \cdots & M_f \\
: & : & \cdots & : \\
0 & 0 & \cdots & 0 & M_f
\end{pmatrix}
\]

**Proof:** The proof is a direct consequence of the definition of \((\tau, \tau')\) – attractivity. The first group of inequalities (rows 1 to 3 of (9)) ensures that \(M(t, M_f) \in A, 0 \leq t < \tau\). The second group of inequalities (rows 4 to 6 of (9)) ensures that \(M(t, M_f) \in A, \tau < t < \tau + \tau' + 1\). The last group of inequalities (rows 7 to 11 of (9)) ensures that the constant control actions are admissible CBIC according to \(T_c\) and \(T_{nc}\). Proposition 3 holds as a consequence.

Let us notice that proposition 3 can also be considered as a LPPs in order to work out the set of CBIC control actions such that the region \(A\) is \((\tau, \tau')\) - attractive.

C. Control for \(r\)-regions switching in finite time

Proposition 3 and LMI's can be used to provide conditions on control actions such that the marking switches from \(r\)-region \(A_i\) to \(r\)-region \(A_j\) in a specific number \(\tau\) of steps. Such a characterization is equivalent to \((\tau, 0)\) – attractivity of region \(A_j\).

**Proposition 4:** Let us consider a contPN with \(T_c = \{T_i, \ldots, T_{qc}\}\), \(T_{nc} = \{T_{qc+1}, \ldots, T_q\}\) and \(K r\)-regions \(A_k\). The region \(A_j\) is \((\tau, 0)\) - attractive for the system (7) from initial markings \(M_f \in A_i\) and for control actions \(U_i\) if and only if \(M_f\) and \(U_i\) satisfy the LMI (10):

\[
\begin{pmatrix}
Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & M_f \\
0 & -Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & \cdots & M_f \\
: & : & \cdots & : \\
0 & 0 & \cdots & 0 & M_f
\end{pmatrix}
\leq
\begin{pmatrix}
-Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & M_f \\
0 & -Hc(A_0) + Hc(A_1) + \cdots + Hc(A_t) + H. \Sigma(0) & \cdots & M_f \\
: & : & \cdots & : \\
0 & 0 & \cdots & 0 & M_f
\end{pmatrix}
\]
Proof: the proof is also a consequence of the definition of \((\tau, r') -\) attractivity. The first group of inequalities (rows 1 to 3 of (10)) ensures that \(M(t, M_d(\tau)) \in \mathbf{A}_i, 1 \leq t < \tau\). The second group of inequalities (row 4 of (10)) ensures that \(M(\tau, M_d(\tau)) \in \mathbf{A}_i\) (i.e. the marking switches from region \(A_i\) to \(A_j\)). The last group of inequalities (rows 5 to 7 of (10)) ensures that the constant control actions are admissible CBIC according to \(T_e\) and \(T_{nc}\). Proposition 4 holds as a consequence.

Let us notice that proposition 4 can also be considered as a LPPs in order to work out the set of CBIC control actions and the set of initial markings such that the region \(A_j\) is \((\tau, 0) -\) attractive.

IV. PIECEWISE CONSTANT CONTROLS ACTIONS FOR FINITE TIME CONTROL DESIGN

The results of the section III can be used in order to design piecewise constant control actions suitable to drive the marking vector from an initial marking \(M_i\) to the neighborhood of a desired marking \(M_d\) according to a sequence \(S = s(1)\ldots s(L)\) of \(L \leq K\) \(r\)-regions to be crossed in finite times \(\tau_1, \tau_2, \ldots, \tau_L\). The \(i^{th}\) \(r\)-region in sequence \(S\) will be referred as \(s(i)\). It is assumed that \(S\) satisfies:

(1) \(M_i \in s(1)\) and \(M_d \in s(L)\)
(2) two successive \(r\)-regions in \(S\) have nonempty intersections: \(s(i) \cap s(i+1) \neq \emptyset\) for \(i = 1, \ldots, L-1\),
(3) each \(r\)-region does not appear more than once in \(S\): \(s(i) \neq s(j)\) for \(i \neq j\).

For any initial marking \(M_i\) and desired marking \(M_d\), such sequences \(S\) exist as long as the \(R(\text{contPN}, M_d)\) is a convex set that is partitioned according to the \(r\)-regions. Without any loss of generality one can label the \(r\)-regions according to the sequence \(S: S = A_1 \ldots A_L\) and \(s(i) = A_i\), \(i = 1, \ldots, L\). Let us also consider a polyhedral neighborhood \(A \subset A_i\) of the desired marking \(M_d\). The piecewise constant control actions that drive the marking vector from \(M_i\) to \(A\) in finite time and according to \(S\) can be worked out by proposition 5:

**Proposition 5:** Let us consider a contPN with \(T_e = \{T_1, \ldots, T_{w_e}\}, T_{nc} = \{T_{w+1}, \ldots, T_{w+T_{nc}}\}\), \(K\) \(r\)-regions \(A_i\), an initial marking \(M_i\) and a desired marking \(M_d\). Let us also consider a sequence of \(r\)-regions \(S = s(1)\ldots s(L) = A_1 \ldots A_L\), a set of finite times \(\tau = \{\tau_1, \tau_2, \ldots, \tau_L\}\) and a region \(A \subset A_i\) defined by \(H.M \leq h\). There exists a set of piecewise constant control actions \(\{U_j, i = 1, \ldots, L\}\) that drive the marking of the system (7) from \(M_i\) to \(A\) in finite time \(\tau_1 + \tau_2 + \ldots + \tau_L\) if \(U_j, i = 1, \ldots, L\), satisfy the LMI (11):

\[
\left(\begin{array}{c}
t(1,1) \\
T(2,1)
\end{array}\right) \cdot \begin{array}{c}
u_1 \\
\nu_2
\end{array} \leq \begin{array}{c}
y_1 \\
y_2
\end{array}
\]

Let us also consider a polyhedral neighborhood \(A \subset A_i\) of the desired marking \(M_d\). The piecewise constant control actions that drive the marking vector from \(M_i\) to \(A\) in finite time and according to \(S\) can be worked out by proposition 5:

**Proof:** The marking trajectory is detailed in any region \(s(i) = A_i\), \(i = 1, \ldots, L\), according to the following equation:

\[
M(t) = (A_d)^{t-i} \cdot M_i,
\]

The marking trajectory is detailed in any region \(s(i) = A_i\), \(i = 1, \ldots, L\), according to the following equation:

\[
M(t) = (A_d)^{t-i} \cdot M_i,
\]

Then, the proof results directly from the application of the results established for regions switching and \((\tau, r') -\) attractivity.

**Remarks:**

1. The computation of equation (11) provides all sequences of CBIC control actions that drive the marking vector from \(M_i\) to \(A\) according to the sequence \(S\) and to the set \(\tau\). The solutions are described as a set of \(L\) regions in \((\mathbf{R}^d)^n\).

2. Let us notice that the proposition 5 can also be used to work out simultaneously the region of initial marking in

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region \( s(I) = A_1 \) and the sequences of CBIC control actions that drive the marking vector \( A \). In that case the LMI (12) is considered:

\[
\begin{pmatrix}
T(1,0) & T(1,1) & \cdots & 0 \\
T(2,0) & T(2,1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
T(L,0) & T(L,1) & \cdots & T(L,L) \\
T(L+1,0) & T(L+1,1) & \cdots & T(L+1,L)
\end{pmatrix}
\begin{pmatrix}
M_1 \\
M_2 \\
\vdots \\
M_L
\end{pmatrix}
\leq
\begin{pmatrix}
Y'_1 \\
Y'_2 \\
\vdots \\
Y'_{L+1}
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
\vdots \\
U_L
\end{pmatrix}
\]

(12)

with \( T(L+1,0) = H_i(A_{D_i})^3, \pi(i, 0), Y_{L+1} = h, \) and:

\[
T(i,0) = \begin{pmatrix}
H_V(A_D)^{\pi(i,0)} & \pi(i,0) \\
H_V(A_D)^{\pi(i,0)} & \pi(i,0) \\
\vdots & \vdots \\
H_V(A_D)^{\pi(i,0)} & \pi(i,0)
\end{pmatrix}
\begin{pmatrix}
h_1 \\
h_i \\
\vdots \\
h_L
\end{pmatrix}

\begin{pmatrix}
y'_1 \\
y'_2 \\
\vdots \\
y'_{L+1}
\end{pmatrix}
\]

(13)

3. The proposition 5 can be combined with an optimization procedure in order to work out the control actions that drive the marking vector from \( M_I \) to \( A \) according to the sequence \( S \) in minimal time (figure 4). The idea is to start the computation by considering only the first line of equation (11) with \( \tau_1 = 1 \). Then \( \tau_1 \) increases as long as no solution is found to reach \( A_1 \) from \( M_I \) in finite time \( \tau_1 \). When a non empty region of solution is found, the restriction LMI(2) of equation (11), given by (13) is considered with \( \pi(2) = \{ \tau_1, 1 \} \), and the computation starts again.

\[
\text{LMI}(2): \begin{pmatrix}
T(1,1) & 0 & \cdots & 0 \\
T(2,1) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
T(L,1) & 0 & \cdots & 0 \\
T(L+1,1) & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
\vdots \\
U_L
\end{pmatrix}
\leq
\begin{pmatrix}
Y'_1 \\
Y'_2 \\
\vdots \\
Y'_{L+1}
\end{pmatrix}
\]

(14)

The same optimization continues with the restrictions LMI(2), LMI(3),..., LMI(L) and sets of finite times \( \pi(2) = \{ \tau_1, 1 \} \), \( \pi(3) = \{ \tau_1, \tau_2, 1 \}, \ldots \), \( \pi(L) = \{ \tau_1, \tau_2, \ldots, 1 \} \). A failing test \( \tau_i \leq \tau_{\text{max}} \) can be also considered to limit the computation time. In case the algorithm fails in region \( A_i \) (i.e. \( \tau_i > \tau_{\text{max}} \)), \( \tau_{i+1} \) is increased and LMI(i-1) is considered again. This algorithm leads to all sequences of piecewise constant control actions that drive the marking vector from \( M_I \) to \( A \) according to the sequence \( S \) in minimal time.

V. EXAMPLE

Let us consider the marked contPN in figure 1 with 2 P-semiflows \( Y = ((0 0 0 0 1 1)^T, (1 1 2 1 0)^T) \in (Z^5)^{3 \times 2}, C = (4 5)^T \), maximal firing rates \( \mu = (1, 1, 1, 1)^T \), and \( T_e = \{ T_1, T_2, T_3, T_4 \} \). Three r-regions \( A_1 \) to \( A_3 \) exist for this contPN.

Let us also consider \( X_{\text{max}} = \text{diag}(\mu) \) with \( \mu = (1, 1, 1, 1)^T \), \( \Delta = 0.05, M_I = (4 0.3 0.1 0.5 3.5)^T \in A_1 \) and \( M_D = (2 0.5 0.75 1 3)^T \in A_1 \). In that simple case the sequence \( S = A_1 \) is considered. Let also consider a neighborhood of \( M_D \) according to (14) with \( \varepsilon = 0.5 \). The computation of the LMI given by equation (11) with the set of finite times \( \tau = \{ \tau_1, \tau_2, \tau_3 \} = \{ 10, 80, 20 \} \) leads to a non empty region of
solutions. Considering the particular constant action $U_j = (1.54, 0.40, 1.88, 0.89)^T$ in $A_1$, $U_j = (2.43, 0.53, 5, 2)^T$ in $A_2$, and $U_j = (1.07, 0.31, 0.09, 0.06)^T$ in $A_3$, the marking trajectory in figure 4 is obtained. The application of the LMI given by equation (12) also provides the set of initial markings in $r$-region $A_1$ so that $A$ is reached according to $S$ and $r$. This region (figure 4, in grey color) coincides with $A_1$.

IV. CONCLUSIONS

Positive invariant and attractive regions in finite time have been characterized for contPNs with piecewise constant control actions. Linear matrix inequalities are used to that purpose. The computation of such LMIs with LPP not only checks the invariance or attractivity of a given polyhedral region, but also provides the set of admissible, bounded input, constant control actions. Piecewise constant controls in finite and minimal time result as a consequence.

In future works we will continue the investigation of control problems for contPNs and also the approximation of stochastic Petri nets by contPNs with piecewise-constant maximal firing speeds.

![Figure 4: Control design in finite time with sequence $S = A_1, A_2, A_3$: marking trajectory in plan $(m_1, m_2)$ to drive $M(t, M_3)$ from $M_0$ to $M_1$ in $A_3$.](image)

**REFERENCES**


