Preservation of piecewise-linear Lyapunov function under Padé discretization

Francesco Rossi, Patrizio Colaneri, Robert Shorten.

Abstract—In this paper we show that certain piecewise-linear Lyapunov functions are preserved for LTI systems under Padé approximations. In particular, we present a simple method to find a piecewise-linear Lyapunov function that is so preserved under the Padé discretization of any order and sampling time. This result may be of interest in the discretisation of switched linear systems for both simulation and control design.

I. INTRODUCTION

The investigation of the properties of control systems when passing from the continuous-time analysis to the discrete-time one has been subject of particular attention in the literature of control theory. The general framework is the following: given a time-continuous control system
\[ \dot{x} = f_c(x, u), \]
we look for a discrete-time control system
\[ x_{k+1} = f^d(x_k, u_k) \]
that shares some properties with the original one (e.g. behavior of trajectories, or stability), and such that \( f^d \) is easily computable.

This general goal is almost completely understood for the investigation of stability of linear time-invariant (LTI) systems \( \dot{x} = A_c x \). In this context, the natural choice for the discretization is to find a sampling time \( h > 0 \) and define \( x_{k+1} = A_d x_k \) with \( A_d = e^{A_c h} \). Since the exponential of matrices is hard to compute (see [1]), it can be replaced by its diagonal Padé approximation of a given order \( p \). The choice of the Padé approximation is very common in engineering. For example, the Tustin or bilinear approximation is a particular Padé approximation, and even the \( \text{expm} \) function in MATLAB is realized by Padé approximation. It is also intensively studied from the numerical viewpoint, see [2], [3].

Our goal is to study the more general problem of good discretization of switched linear systems (SLS). This problem is new but is emerging as a topic of increasing interest in the control and simulation communities; see for example [4], [5], [6], [7]. We recall that SLS are particular cases of hybrid systems in which the dynamics \( f \) changes (i.e. it switches) between different possible linear laws \( \{ A_1, \ldots, A_m \} \), that are fixed a priori. The set of rules that orchestrate the switching among is the set of all possible time-dependent laws. The continuous-time case is
\[ \dot{x} = A_c^\sigma(t) x \quad \text{where } \sigma : [0, T] \to \{1, \ldots, m\} \text{ measurable}, \]
while the discrete-time case is
\[ x_{k+1} = A_d^{\sigma_k} x_k \quad \text{where } \sigma : \{0, \ldots, K\} \to \{1, \ldots, m\}. \]

Then, the problem of good discretization of SLS can be restated as following: find a rule for time discretization \([0, T] \to \{0, \ldots, K\}\) and a method to compute \( A_d^\sigma \) from \( A_c^\sigma \). The first idea, coming from LTI system, is to fix a sampling time \( h > 0 \) and to compute each \( A_d^\sigma \) as the diagonal Padé approximation of \( A_c^\sigma \). Surprisingly, this discretization method fails to preserve stability of SLS. Examples can be found in [8], [4].

Even though this negative result is known, it is still unclear in which cases stability is preserved when discretization is computed via the diagonal Padé approximation. Our contribution is a first step in this direction. We thus focus our attention on the preservation of piecewise-linear Lyapunov function under Padé discretization. Our contribution is to show that, given a stable LTI system, it is always possible to find a particular piecewise-linear Lyapunov that is preserved for all kind of Padé approximations, regardless to the order \( p \) and the sampling time \( h \).

Beside being interesting directly in the context of LTI systems, the existence of such a piecewise-linear Lyapunov function can be a starting point to investigate the stability of SLS.

II. DEFINITIONS AND KNOWN RESULTS

A. Padé discretization

Consider a linear autonomous system
\[ \dot{x}(t) = A_c x(t) \] (1)
where \( x(t) \in \mathbb{R}^n \) and assume that the system is asymptotically stable, i.e. matrix \( A_c \) is Hurwitz (all eigenvalues in the open left half of the complex plane). It is well known that the motion of the state, associated with an initial state \( x(0) = x_0 \), can be written as \( x(t) = e^{A_c t} x_0 \). The exponential matrix \( e^{A_c t} \) can be numerically approximated in a variety of different ways. In this paper we focus on the most popular one, that is diagonal Padé approximation of \( p^h \) order, see e.g. [2], [3]. This operator is well known to engineers and is commonly used by
both the control and signal processing community. To be precise, taking a sampling time \( h \), the \( p \)th order Padé discretization of \( e^{A_c h} \) is defined as

\[
A_d = Z(A_c h)Z(-A_c h)^{-1}
\]

and

\[
Z(X) = \sum_{i=0}^{p} c_i X^i, \quad c_i = \frac{(2p-i)!p!}{(2p)!i!(p-i)!}
\]

Hence, it is possible to associate with system (1), its discrete approximation

\[
x_{k+1} = A_d x_k
\]

where \( x_k \) approximates \( x(kh) = (e^{A_c h})x_0 \). It is well known that the Padé discretization preserves the stability properties. As a matter of fact, \( A_c \) is Hurwitz if and only if \( A_d \) is Schur stable (all eigenvalues inside the open unit disc), for any given sampling times \( h > 0 \). Moreover, the eigenvalues of \( A_c \) and \( A_d \) are related by the same transformation induced by (2), and the eigenstructure is preserved. If \( \lambda \) is an eigenvalue of \( A_c \) associated with an eigenvector \( \tilde{x} \), then \( z = Z(\lambda h)Z(-\lambda h)^{-1} \) is an eigenvalue of \( A_d \) associated with the same eigenvector \( \tilde{x} \). Even more, the transformation is basis independent, i.e.,

\[
Z(TA_c T^{-1}h)Z(-(TA_c T^{-1}h)^{-1}) = TZ(A_c h)Z(-A_c h)^{-1} = T A_d T^{-1}
\]

Finally, if \( TA_c T^{-1} \) is a Jordan form for \( A_c \), then \( TA_d T^{-1} \) is a Jordan form for \( A_d \). A particular Padé transformation is the celebrated bilinear transformation (or Tustin transformation), that is given by (2) with \( p = 1 \), i.e.,

\[
A_d = (I + \frac{h}{2} A_c)(I - \frac{h}{2} A_c)^{-1}
\]

B. Piecewise-linear Lyapunov functions

Consider system (1) and its Padé discretization (3). Assume that \( A_c \) is Hurwitz stable, so that \( A_d \) is Schur stable for each \( h > 0 \). Also, Let \( X_{ij} \) be the entries of a square matrix \( X \). We define the \( \infty \)-measure as

\[
\mu_{\infty}(X) = \max_i \left( X_{ii} + \sum_{j \neq i} |X_{ij}| \right)
\]

and the \( \infty \)-norm as

\[
\|X\|_{\infty} = \max_j \sum_i |X_{ij}|
\]

The main known results about stability of LTI systems that we will use are recalled below, see e.g. [10]. The same reference recalls results about existence of quadratic Lyapunov functions.

**Lemma 2.1:**

(i) \( A_c \) is Hurwitz stable if and only if there exists a full column rank matrix \( W_c \in \mathbb{R}^{N \times n}, N \geq n \), and \( Q_c \) such that

\[
W_c A_c = Q_c W_c, \quad \mu_{\infty}(Q_c) < 0.
\]

(ii) \( A_d \) is Schur stable if and only if there exists a full column rank matrix \( W_d \in \mathbb{R}^{N \times n}, N \geq n \), and \( Q_d \) such that

\[
W_d A_d = Q_d W_d, \quad \|Q_d\|_{\infty} < 1.
\]

**Remark 2.2:** Notice that \( W_c A_c = Q_c W_c \), always imply that \( W_d A_d = Q_d W_d \) with \( W_d = W_c \) and \( Q_d = Z(Q_c)h \). However it is not true in general that \( \mu_{\infty}(Q_c) < 0 \) implies \( \|Q_d\|_{\infty} < 1 \), unless \( h \) is small. A counterexample is given in [9].

The Lyapunov functions associated with Lemma 2.1 are piecewise-linear, i.e. \( ||W_c x_c||_{\infty} \) resp. \( ||W_d x_d||_{\infty} \), where the number of vertices of the polyhedra is \( 2N \). In general \( N > n \), and the minimal \( N \) for which (5), (6) are verified depends on the location of the eigenvalues of \( A_c \), \( A_d \) in the complex plane. It can be proved, see [11], that for a matrix \( A_c \) with distinct eigenvalues a necessary and sufficient condition for \( N = n \) is that the complex eigenvalues \( \lambda = -\alpha + j\beta, \alpha > 0 \), belong to the sector \( |\beta/\alpha| < 1 \).

C. Computation of piecewise-linear Lyapunov function

In this section we look for matrices \( W_c \) satisfying (5), and matrices \( W_d \) satisfying (6). We recall two results available in the literature, [12] for the continuous-time and [13] for the discrete-time case. They have been shown to be valid also in case of multiple eigenvalues. However, for the sake of simplicity, we assume that the eigenvalues are distinct.

We start with the continuous-time setting. Given a stable matrix \( A_c \), we provide a method to compute a particular \( W_c \).

**Lemma 2.3 (Existence for continuous-time LTI):**

Consider a Hurwitz stable matrix \( A_c \), with distinct eigenvalues, with \( n_c \) real and \( 2n_c \) complex eigenvalues. For each pair of conjugate complex eigenvalue \( \lambda_i = \alpha_i \pm j\beta_i \), \( i = 1, 2, \cdots, n_c \), take an integer \( m_i \) such that \( \lambda_i \) lies in the sector \( S_c(m_i) \), where

\[
S_c(m) = \{ \lambda = -\alpha + j\beta : \alpha > 0, |\beta| < \frac{\sin(\frac{\pi}{m})}{1 - \cos(\frac{\pi}{m})} \alpha \}.
\]

Then there exist \( W_c \in \mathbb{R}^{N \times n} \) and \( Q_c \in \mathbb{R}^{N \times n} \), with \( N = \sum_{i=1}^{n_c} m_i + n_r \), satisfying (5).

In Figure 1, the sectors \( S_c(m) \) are drawn for \( m = 2 \) (angle \( \pi/4 \)), \( m = 3 \) (angle \( \pi/3 \)), \( m = 4 \) (angle \( 3\pi/8 \)) and \( m = 5 \) (angle \( 4\pi/10 \)). One can remark that the minimum number \( m \) such that an eigenvalue \( \lambda \) lies in \( S_m \) is increasing for \( \lambda \) approaching the imaginary axis. Nevertheless, a finite \( m \) exists for each \( \lambda \) with negative real part.

**Lemma 2.4 (Computation for continuous-time LTI):**

Consider a Hurwitz stable matrix \( A_c \), as in the previous lemma. Take \( T_c \) the state-space transformation that puts \( A_c \) in its real Jordan form, i.e.,

\[
T_c A_c T_c^{-1} = \begin{bmatrix}
H_{c1} & 0 & \cdots & 0 & 0 \\
0 & H_{c2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & H_{cn_c} & 0 \\
0 & 0 & 0 & 0 & R_c
\end{bmatrix}
\]
Fig. 1. The sectors $S_c(m)$ for $m = 2$ (angle $\pi / 4$), $m = 3$ (angle $\pi / 3$), $m = 4$ (angle $3\pi / 8$) and $m = 5$ (angle $4\pi / 10$).

In Figure 2 the polygons $P_{cd}(m)$ are depicted for $m = 2$ (square), $m = 3$ (hexagon), $m = 4$ (octagon), $m = 5$ (decagon).

**Lemma 2.6 (Computation for discrete-time LTI):**
Consider a Schur stable matrix $A_d$ as in the previous lemma. Take $T_d$ the state-space transformation that puts $A_d$ in its real Jordan form, i.e.

$$T_d A_d T_d^{-1} = \begin{bmatrix} H_{d1} & 0 & \cdots & 0 & 0 \\ 0 & H_{d2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & H_{dn_c} & 0 \\ 0 & 0 & 0 & 0 & R_d \end{bmatrix}$$

where

$$H_{di} = \begin{bmatrix} -\alpha_i & \beta_i \\ -\beta_i & -\alpha_i \end{bmatrix}$$

and $R_d$ is a $n_r \times n_r$ diagonal matrix accounting for the real eigenvalues. Define

$$W_d := \tilde{W}_d T_d$$

and $W_d$ defines a Lyapunov function $F(x) = \|W_d x\|_\infty$ for the system $\dot{x} = A_d x$.

We now state the corresponding results for the discrete-time case.

**Lemma 2.5 (Existence for discrete-time LTI):**
Consider a Schur stable matrix $A_d$, with distinct eigenvalues, with $n_r$ real and $2n_c$ complex eigenvalues. For each pair of conjugate complex eigenvalue $\lambda_i = \sigma_i \pm j\omega_i$, $i = 1, 2, \cdots, n_c$, take an integer $m_i$ such that $\lambda_i$ lies in the interior of the regular polygon $P_{cd}(m_i)$, where

$$P_{cd}(m) = \text{int conv} \left\{ e^{j\frac{2\pi p}{m}} \right\}_{p=0}^{2m-1}. \quad (8)$$

Then there exists $W_d \in \mathbb{R}^{N \times n}$ and $Q_d \in \mathbb{R}^{N \times N}$, with $N = \sum_{i=1}^{k} m_i + n_r$, satisfying (6).
these proofs. Given $A_c$, pass to the real Jordan form via $T_c$. For each submatrix $H_{ci}$, find $W_{ci}$ and $Q_{ci}$. Then find the global $W_c$ and $Q_c$ by applying the inverse change of coordinate $T_c^{-1}$ (being careful about covariant and contravariant matrices). The same idea is applied in the discrete-time setting, for which it is a bit harder to find $Q_d$.

III. MAIN RESULT

In this section we state the main result of the paper. Given a matrix $A_c$ and its diagonal Padé approximation $A_d$ of a given order $p$ and sampling time $h$, the function $F$ computed in Lemma 2.4 is a Lyapunov function both for $A_c$ and $A_d$. As already stated, this implies that $F$ is a Lyapunov function for $A_c$ and all its Padé approximations.

We first prove that, given an integer $m$, the image of $\mathcal{S}_c(m)$ under a Padé approximation is contained in $\mathcal{P}_{ol}(m)$. We only give a graphical idea of the proof, since it is rather technical. The complete proof is given in [14].

**Lemma 3.1:** Let $m$ be a positive integer number, $\mathcal{S}_c(m)$ defined in (7) and $\mathcal{P}_{ol}(m)$ defined in (8). Fix a sampling time $h > 0$ and consider $\mathcal{S}_d(m, h)$ the image of $\mathcal{S}_c(m)$ under the Padé transformation (2), i.e.

$$\mathcal{S}_d(m, h) = \{z = Z(\lambda h)Z(-\lambda h)^{-1}, \lambda \in \mathcal{S}_c(m)\}$$

Then $\mathcal{S}_d(m, h) \subseteq \mathcal{P}_{ol}(m)$.

**Sketch of the proof.** Take the two half-lines $\Lambda_m, \Lambda_m^-$ that are the boundaries of $\mathcal{S}_c(m)$. Their precise expression is

$$\Lambda_m = \left\{-\alpha + j\beta \mid \beta = \frac{\sin\left(\frac{\pi}{m}\right)}{1 - \cos\left(\frac{\pi}{m}\right)}\alpha\right\}$$

and $\Lambda_m^- = \{z \in \Lambda_m\}$. First prove that $\mathcal{S}_d(m, h) \subseteq \mathcal{P}_{ol}(m)$ if and only if the image of $\Lambda_m, \Lambda_m^-$ under the Padé approximation is contained in $\mathcal{P}_{ol}(m)$. Due to invariance of $\mathcal{P}_{ol}(m)$ under complex conjugation, it is equivalent to prove that $Z((\Lambda_m h)Z(-(\Lambda_m h)^{-1}) \in \mathcal{P}_{ol}(m)$. Due to invariance of $\Lambda_m$ with respect to rescaling, it is equivalent to prove that $Z((\Lambda_m h)Z(-(\Lambda_m h)^{-1}) \in \mathcal{P}_{ol}(m)$. This result being technical, we only show some images of $\Lambda_2$ under Padé approximations of order $p = 1, 2, 3$ and images of $\Lambda_2, \Lambda_3, \Lambda_4$ under Padé approximation of order $p = 2$. For a complete proof, see [14].

We now state precisely the main result of the paper.

**Theorem 3.2:** Consider a Hurwitz stable matrix $A_c$ of dimension $n$ and its Padé discretization $A_d$ of order $p$ and sampling time $h > 0$. Let $n_c$ be the number of real negative eigenvalues, and $2n_c$ be the number of pairs of complex eigenvalues $-\alpha_i \pm j\beta_i$, $i = 1, 2, \cdots, n_c$. For each pair of complex eigenvalues, let $m_i$ be an integer greater than one such that $-\alpha_i \pm j\beta_i$ belongs to the sector

$$|\beta_i| < \frac{\sin\left(\frac{\pi}{m_i}\right)}{1 - \cos\left(\frac{\pi}{m_i}\right)}\alpha_i.$$  

Then there exist $W = W_c = W_d \in \mathcal{R}^{N \times n}$, with $N = \sum_{i=1}^{n_r} m_i + n_r$ such that $F(x) = \|Wx\|_\infty$ is a Lyapunov function both for $\dot{x} = A_c x$ and $x_{k+1} = A_d x_k$. Moreover, $W$ can be computed as in Lemma 2.4.

**Proof.** First recall that the Padé transformation preserves the Jordan form of $A_c$ and $A_d$. Now take $T$ such that both $J_c = TA_c T^{-1}$ and $TA_d T^{-1}$ are in the real Jordan form. Compute $W_c$ as in Lemma 2.4. For each pair of complex eigenvalue $\lambda_i = -\alpha_i \pm \beta_i$, the expression of $W_{ci}$ is uniquely determined by $m_i$ such that $-\alpha_i \pm \beta_i \in S_c(m_i)$.

Now consider the expression of $W_d$, computed as in Lemma 2.6. Applying Lemma 3.1, we have that $\mu_i$ lies in the interior of the regular polygon $P_{ol}(m_i)$, with the same $m_i$ of the eigenvalue $\lambda_i$ of the continuous system. As a consequence, the expression of $W_{di}$ can be chosen to be identical to $W_{ci}$. Thus $W_d = W_c T^{-1} T_d$. Since $T_c = T_d$, we have the conclusion.

**Comment:** The result above says that there always exists a common piecewise-linear Lyapunov function $\|Wx\|_\infty$ for $A_c$ and $A_d$. Moreover, the construction of $W$ is based on the matrix $A_c$ only, and the previous theorem...
shows that $\|Wx\|_\infty$ is a piecewise-linear Lyapunov function for $A_d$ computed with a Padé approximation of any order $p$. As a consequence, the piecewise-linear Lyapunov function is common to $A_c$ and all Padé approximants, of any order $p$ and with any sampling time $h$. In all these cases, it is the matrix $Q_d$ that changes its expression.

### IV. Examples

In this section, we give two examples that highlight some implications of our main result. The first, positive result, comes from [9].

**Example 1:** Consider the Hurwitz matrix

$$A_c = \begin{bmatrix} -1 & 0 \\ -2.4 & -3 \end{bmatrix}$$

In [9], the authors show that a piecewise linear Lyapunov function is given by choosing $W_c = I$, with $Q_c = A_c$. Now, take $A_d$ given by the $1^{st}$ order Padé approximation with $h = 2$, namely

$$A_d = (I + A_c)(I - A_c)^{-1} = \begin{bmatrix} 0 & 0 \\ -0.6 & -0.5 \end{bmatrix}$$

and notice that $Q_d = (I + Q_c)(I - Q_c)^{-1}$ satisfies $A_d W_d = W_d Q_d$, with $W_d = W_c$. However, $\|Q_d\|_\infty > 1$. From this the authors in [9] concluded that $W_c = I$ is not preserved.

Nevertheless, using our result, it is possible to find another $W = W_c = W_d$ that is preserved. A choice is given by

$$W = \begin{bmatrix} -1 & 0 \\ 1.2 & 1 \end{bmatrix},$$

$$Q_c = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, Q_d = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix}.$$  

Remark that $W$ has been computed as follows: first compute $A_c = T_c A_c T_c^{-1}$ the real Jordan form of $A_c$. Then compute the minimum $m$ such that $S_m$ contains the eigenvalues of $A_c$ (in this case, $m = 2$). Finally, define $W$ via Lemma 2.4 and $W = WT_c$.

Our first example highlights the fact that some polyhedral Lyapunov functions are preserved by Padé transformations. This result is interesting as it says that Padé transformations preserve all quadratic functions, and some polyhedral Lyapunov functions. This observation is most interesting in the context of switched systems as it implies that Padé methods will preserve the stability of certain switched systems even if they are not quadratically stable to begin with. That much work remains to be done is illustrated by the following example. We start from an example given by [8], with the parameter $\alpha = 7$.

**Example 2:** The switching system is given by two matrices

$$A_{c1} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, A_{c2} = \begin{pmatrix} -1 & 1/7 \\ -7 & -1 \end{pmatrix}.$$  

As already stated in [8], the switching system is asymptotically stable. One can compute explicitly the Padé approximation of order $1$ of $A_{ci}$ as a function of the sampling time $h$, that are

$$A_{d1} = \begin{pmatrix} \frac{2 - h^2}{h^2 + 2h + 2} & \frac{2h}{h^2 + 2h + 2} \\ \frac{-h^2 + 2h + 2}{h^2 + 2h + 2} & \frac{-h^2 - 2h + 2}{h^2 + 2h + 2} \end{pmatrix},$$

$$A_{d2} = \begin{pmatrix} \frac{2 - h^2}{h^2 + 2h + 2} & \frac{2h}{h^2 + 2h + 2} \\ \frac{-h^2 + 2h + 2}{h^2 + 2h + 2} & \frac{-h^2 - 2h + 2}{h^2 + 2h + 2} \end{pmatrix}.$$  

One can observe that for small $h$ the system is stable, while for $h = 1$ we have instability, since one of the eigenvalues of $A_{d2} A_{d1}$ is

$$-93 + 16\sqrt{29} \begin{pmatrix} -1 \end{pmatrix}.$$  

To study stability for $h \in [0, 1]$, one can look for a piecewise linear Lyapunov function for the continuous-time system $A_{ci}$ that is preserved under Padé approximation. Since the eigenvalues of both $A_{c1}$ and $A_{c2}$ lie in $S_r(3)$, our method provides the following matrices $W_{ci}$, one for each matrix $A_{ci}$:

$$W_{c1} = \begin{pmatrix} 1 & 0 & \sqrt{\alpha} \\ \frac{1}{\sqrt{\alpha}} & \frac{\sqrt{\alpha}}{\alpha} \\ -\frac{\sqrt{\alpha}}{\alpha} & \frac{1}{\sqrt{\alpha}} \end{pmatrix},$$

$$W_{c2} = \begin{pmatrix} 7 & -7\sqrt{\alpha} & 1 \\ -\frac{7\sqrt{\alpha}}{\alpha} & \frac{1}{\alpha} + \frac{\sqrt{\alpha}}{\alpha} \\ -\frac{1}{\alpha} + \frac{\sqrt{\alpha}}{\alpha} \end{pmatrix}.$$  

One can then check if one of the two, say $W_{c1}$, defines a Lyapunov function for the other system, say $A_{c2}$. It is easy to see that it is not the case. Then one can check for a linear combination of the two, but also in this case we are unable to find a common piecewise linear Lyapunov function. Then, since $S_r(3) \subset S_r(4) \subset \ldots$, one can increase the dimension of $W_{ci}$ and use the method to find other candidate piecewise linear Lyapunov functions. We do not go further in this direction, since the method becomes increasingly hard from the computational point of view.

It is immediately clear from the above example that the fact that some Lyapunov functions are preserved is not enough to guarantee that the discrete time system will be stable. Thus, Padé, while being well suited to discretising LTI systems, is somewhat lacking when applied to switched systems. Future work will look at the problem of developing discretisation methods that are suited for the discretisation of switched linear systems.

### V. Conclusions

In this paper we present a method to compute a piecewise-linear Lyapunov function $F$ for a continuous-time LTI system $\dot{x} = A_c x$. $F$ is moreover preserved under Padé approximation of $A_c$ of any order and sampling time. This result is a first step in the context of switching linear systems. Some examples show applications of the method, as well as some negative results.
REFERENCES