On the Distinguishability of Discrete Linear Time-Invariant Dynamic Systems

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Abstract—This paper introduces the notion of absolutely distinguishable discrete dynamic systems, with particular applicability to linear time-invariant (LTI) systems. The motivation for this novel type of distinguishability stems, in particular, from the stability and performance requirements of worst-case adaptive control methodologies. The main results presented herein show that, in most practical cases, a persistence of excitation type of condition and a minimum number of iterations are required to properly distinguish two dynamic systems. We also demonstrate that the former constraint can be written as a lower bound on the intensity of the exogenous disturbances. The applicability of the developed theory is illustrated with a set of examples.

I. INTRODUCTION

The identifiability of dynamic systems plays an important role in certain areas, where it is fundamental to ensure that the model of the plant can be inferred from input/output data. In particular, model estimation methods require the system to be identifiable. Otherwise, the estimation problem may not be well-posed and lead to erroneous results. Such methods also require, in general, a persistence of excitation condition in the exogenous inputs, in order to avoid the issues related to the indistinguishability due to the small amplitude of the disturbances.

However, in applications such as fault detection [1], [2], [3], [4], [5], identification of hybrid systems [6], [7], and multiple-model adaptive control [8], [9], [10], it suffices to guarantee that we can identify the family of systems to which the true plant belongs, among a finite set of families of dynamic systems. In such cases, we say that the families of systems are distinguishable. This notion was introduced in [11], where in fact the authors relate the concept of identifiability with that of distinguishability.

Indeed, these two concepts (see, for instance, [11], [12], [13]) are naturally related to each other. If a system can be identified by using a certain input signal, then it is obvious that it can be distinguished from any other system with the same structure. A discussion on this topic can be found in [14]. The interested reader is also referred to [15], [16], [17].

However, most of the results on system identifiability are not enough for system absolute distinguishability. In particular, for two absolutely distinguishable systems we require, for all the admissible input signals, the corresponding outputs to be different from one another, unlike the most common definition of (structurally) identifiable system, which only demands that the unknown parameters of the plant can be estimated for some input signal.

The most common notion of (in)distinguishability [11], [14] states that two systems are indistinguishable only if there are neither initial conditions nor exogenous inputs that can generate different outputs for the systems. This definition, however, may not be enough in applications where the distinguishability of the systems must be guaranteed from a worst-case perspective.

Therefore, this paper introduces a novel definition of system distinguishability, referred to as absolute distinguishability, as detailed in the sequel. We also provide necessary and sufficient conditions for the absolute distinguishability of two dynamic LTI systems under different scenarios, and show how to design observers that can be used to distinguish among dynamic systems. These observers are used for model falsification, i.e., to (in)validate models of the system, based upon the input/output data. Although this can be achieved by using methods such as in [18], [19], the solution adopted herein provides an iterative algorithm to solve the problem of model falsification, which is suitable for run time applications like multiple model adaptive control.

The remainder of this paper is organized as follows: in Section II, the notation and some of the basic concepts used throughout the paper are introduced; in Section III, a series of properties is derived for absolutely distinguishable LTI systems; Section IV is devoted to the introduction of the set-valued observers, used for system distinguishability; the applicability of the distinguishability theory is illustrated in a series of examples, in Section V; finally, in Section VI, some conclusions regarding the results obtained are highlighted.

II. PRELIMINARIES AND NOTATION

In this paper, we are going to start by considering the broad class of discrete time-varying dynamic systems described by

\[ y(k) = F_k(x_0, \phi_0, \phi_1, \phi_2, \ldots, \phi_k, p), \]

where \( F_k : \mathbb{R}^n \times \Phi \times \cdots \times \Phi \times \Omega \to \mathbb{R}^n, \) \( \phi_i \in W_d \times U =: \Phi \subseteq \mathbb{R}^{n_u + n_d} \) for \( i = 0, 1, \ldots, k, \) and \( p \in \Omega \subseteq \mathbb{R}^p \) is a vector of parameters. The sequence \( (\phi_0, \phi_1, \ldots, \phi_k) \), where \( \phi_i = [d_i^T, u_i^T]^T \), denotes the exogenous disturbances, \( d_i \in W_d \subseteq \mathbb{R}^{n_d}, \) and control input signals, \( u_i \in U \subseteq \mathbb{R}^{n_u}, \) at time instant \( i, \) and \( y(k) \) is the output of the system at time \( k. \) The initial state is represented by \( x(0) \in X(0) \subseteq \mathbb{R}^n. \)

The notion of distinguishability in [11], [14] states that two realizations of \( F_k, \) parametrized by the pair of parameter vectors \( (p_A, p_B), \) are indistinguishable in \( N \) sampling times only if there are neither initial conditions nor exogenous disturbances...
inputs that can generate different outputs for the systems parametrized by \( p_A \) and \( p_B \). Hence, this definition is not useful in some applications, namely to provide guarantees that, regardless the input signals, the pair of parameter values is distinguishable.

Thus, a new definition of distinguishability, referred to as \( U \)-input distinguishability, was presented in [7]. Consider the dynamic systems \( S_A \) and \( S_B \), which correspond to realizations of (1), for \( p = p_A \) and \( p = p_B \) and with outputs at sampling time \( k \) denoted by \( y_A(k) \) and \( y_B(k) \), respectively. The definition in [7] states that, unless all the initial conditions and inputs are zero, if two systems are \( U \)-input distinguishable in \( N \) sampling times, then the corresponding outputs must be different at some time instant \( k \) smaller than or equal to \( N \). This guarantees that we can distinguish two such systems in, at most, \( N \) sampling times, just by measuring the outputs. Nevertheless, exogenous disturbances, \( d(k) \) \( \in \) \( W_d \), and measurement noise, \( n(k) \) \( \in \) \( W_n \), are not taken into account in this definition of distinguishability.

Therefore, this motivated the introduction of the following definitions, which are going to be used extensively throughout this article:

**Definition 1:** Systems \( S_A \) and \( S_B \) are said to be absolutely distinguishable in \( N \) measurements if, for any non-zero
\[
(x_o^A, x_o^B, d_{0,N-1}^A, d_{0,N-1}^B, n_{0,N}^A, n_{0,N}^B, u_{0,N-1}) \in \mathbb{R}^{2n} \times \mathbb{R}^{2Nn_d} \times \mathbb{R}^{2(N+1)n_n} \times \mathbb{R}^{2Nn_u},
\]
there exists a \( k \in \{0,1,\ldots,N\} \) such that
\[
y_A(k) \neq y_B(k).
\]

Moreover, two systems are said to be absolutely distinguishable if there exists \( N \geq 0 \) such that they are absolutely distinguishable in \( N \) measurements. \( \diamond \)

In the definition, we used the short-hand notation \( v_{0,N} \) to denote a concatenation of a sequence of vectors
\[
v_{0,N} := [v_0^T, \ldots, v_N^T]^T.
\]

**Definition 2:** Systems \( S_A \) and \( S_B \) are said to be absolutely \((X_o, U, W)\)-input distinguishable in \( N \) measurements if, for any non-zero
\[
(x_o^A, x_o^B, d_{0,N-1}^A, d_{0,N-1}^B, n_{0,N}^A, n_{0,N}^B, u_{0,N-1}) \in X_o^A \times W_d^{2N} \times W_n^{2(N+1)} \times U^N
\]
where \( W := W_n \times W_d \), there exists a \( k \in \{0,1,\ldots,N\} \) such that
\[
y_A(k) \neq y_B(k).
\]

Moreover, two systems are said to be absolutely \((X_o, U, W)\)-input distinguishable if there exists \( N \geq 0 \) such that they are absolutely \((X_o, U, W)\)-input distinguishable in \( N \) measurements. \( \diamond \)

These two definitions are important when we want to guarantee that, regardless of the input signals, two systems can be distinguished in a given number of iterations. This fact is going to be further stressed in the sequel.

Due to space limitations, the proofs of the results presented in this paper are omitted. A 7-pages version of this paper, with the complete set of proofs, is available at www.arxiv.org.

III. ABSOLUTE INPUT-DISTINGUISHABILITY OF LTI SYSTEMS

In this section, we are going to specialize the concept of absolute input-distinguishability for linear time-invariant (LTI) models and discuss some of the properties of this a class of dynamic systems.

Let \( S_i \) be a discrete LTI dynamic system described by
\[
\begin{align*}
x_i(k+1) &= A_i x_i(k) + B_i u_i(k) + L_i d_i(k), \\
y_i(k) &= C_i x_i(k) + N_i n_i(k),
\end{align*}
\]
where \( x_i(0) = x_i^0, \) \( x_i(k) \in \mathbb{R}^n, \) \( u_i(k) \in U \subseteq \mathbb{R}^{n_u}, \) \( d_i(k) \in W_d \subseteq \mathbb{R}^{n_d}, \) \( y_i(k) \in \mathbb{R}^{n_y} \) and \( n_i(k) \in W_n \subseteq \mathbb{R}^{n_n}. \)

Notice that, according to Definition II, two systems are absolutely \((X_o, U, W)\)-input distinguishable in \( N \) sampling times if, for all the input signals in \( U \), the corresponding outputs are different at least at some time instant \( k \leq N \). This is obviously a stronger constraint than simply saying that the two systems are distinguishable whenever the corresponding outputs are different for a particular input sequence, as in [11], [14].

**Remark 1:** In the remainder of this paper, we are going to use the terms distinguishable and absolutely distinguishable interchangeably. \( \diamond \)

For the sake of simplicity, let us consider, for the time being, that \( U = \mathbb{R}^{n_u}, \) \( W_d = \mathbb{R}^{n_d} \) and \( W_n = \mathbb{R}^{n_n} \). The following theorem can be used to test whether or not two systems are distinguishable (in the sense of Definition II).

**Theorem 1:** Let
\[
M_N = \begin{bmatrix}
C_A & -C_B \\
C_{AA} & -C_{BA} \\
C_{A^2} & -C_{B^2} \\
\vdots & \vdots \\
C_{A^N} & -C_{B^N}
\end{bmatrix} \tilde{Q} \tilde{R} \tilde{J},
\]
where
\[
\tilde{Q} = \text{diag}(Q, Q, \ldots, Q), \quad Q = [N_A \quad -N_B],
\]
\[
\tilde{R} = \begin{bmatrix}
R_{11} & 0 & 0 & \ldots & 0 \\
0 & R_{21} & 0 & \ldots & 0 \\
0 & 0 & R_{22} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & R_{NN}
\end{bmatrix}, \quad \tilde{J} = \begin{bmatrix}
J_{11} & 0 & 0 & \ldots & 0 \\
0 & J_{21} & 0 & \ldots & 0 \\
0 & 0 & J_{22} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & J_{NN}
\end{bmatrix},
\]
\[
R_k^k = [C_{A^{k-1}}B_A \ -C_{B^{k-1}}B_B],
\]
\[
J_k^k = [C_{A^{k-1}}B_A \ -C_{B^{k-1}}B_B].
\]

Systems \( S_A \) and \( S_B \) are absolutely distinguishable in \( N \) sampling times if and only if there does not exist a non-zero vector \( v \in \mathbb{R}^{2n+2(N+1)n_n+n(n_u+2n_d)} \) such that
\[
M_N v = 0,
\]
i.e., if and only if,
\[
\text{rank}(M_N) = 2n + 2(N+1)n_n + N(n_u+2n_d).
\]

The importance of this theorem is twofold: on the one hand, it provides necessary and sufficient conditions for the
absolute distinguishability of two systems; on the other hand, it allows the use of set-valued observers [20] for system distinguishability, as shown in the sequel.

As remarked in [7] for continuous-time systems, the distinguishability of two systems implies their observability. In fact, we are going to obtain a similar conclusion for discrete-time systems, but using an alternative path.

**Corollary 1:** Let $S_A$ and $S_B$ be two absolutely distinguishable systems. Then, $S_A$ and $S_B$ are both observable.

**Remark 2:** The concept of absolute distinguishability of two systems requires the corresponding outputs to be different from each other, for every non-zero initial condition. However, if a system is non-observable, then there are (non-zero) initial conditions for the state which do not impact on the output of the plant.

Corollary 1 provides necessary conditions for the distinguishability of two discrete LTI dynamic systems. For the sake of completeness, we recall that there are also results in the literature providing sufficient conditions for absolute distinguishability. For instance, the authors of [21] show that, if $J$ has full column rank and the sets $X_0$ and $W$ are compact, then the systems can be distinguished for sufficiently large input control signals, regardless of their direction.

**IV. DISTINGUISHING SYSTEMS USING SET-VALUED OBSERVERS**

The results presented in the previous section give us necessary and sufficient conditions for absolute distinguishability. They do not provide, however, a systematic way of distinguishing two models, say $S_A$ and $S_B$. In this section, we will show how to use set-valued observers (SVOs) to (in)validate models of systems, i.e., to distinguish among them. This type of observers was applied to linear time-varying systems excited by bounded-but-unknown disturbances in [20], and were used for model falsification in [10]. At each iteration, these observers compute the set-valued estimate of the state, by taking into account the model of the plant and the most recent measurements. Since, no conservatism to the solution is added for LTI systems (see [20]) we conclude that an SVO can be used to falsify (invalidate) a given model of the plant using the following reasoning:

- if the set-valued estimate of the state, $X(k)$, at iteration $k$ is empty, then the model of the plant is falsified, i.e., the input/output data invalidates the model of the plant;

- if the set-valued estimate of the state, $X(k)$, at iteration $k$ is not empty, then the model of the plant is not falsified, i.e., the input/output data is compatible with the model of the plant.

In order to use the SVOs for model falsification, let us pose the following assumption:

**Assumption 1:** The set of admissible initial states, $X_0$, i.e., the minimum set containing all the possible initial states $x_0^*$ of the systems, is convex and compact.

Under Assumption 1, the set of initial states can be represented by

$$X_0 = \text{Set}(M_{X_0}, m_{X_0}) := \{ x : M_{X_0}x \leq m_{X_0} \}. $$

Analogously, suppose that:

**Assumption 2:** The set of admissible exogenous disturbances, $W_d$, is convex and compact, i.e.,

$$W_d = \text{Set}(M_d, m_d).$$

**Assumption 3:** The set of admissible measurement noise, $W_n$, is convex and compact, i.e.,

$$W_n = \text{Set}(M_n, m_n).$$

Then, we can state the following theorem, which is an extension of Theorem 1 for convex and compact $X_0$ and $W$:

**Theorem 2:** Suppose that Assumptions 1–3 are satisfied. Then, systems $S_A$ and $S_B$ are absolutely distinguishable if and only if

$$\forall v \in \mathcal{G}, v \neq 0 \Rightarrow \begin{bmatrix} M_N & -M_N \\ M_{X_0} & M_W \end{bmatrix} v \preceq \begin{bmatrix} 0 \\ 0 \\ m_{X_0} \end{bmatrix},$$

where

$$\hat{M}_{X_0} = \text{diag}(M_{X_0}, M_{X_0}) \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad m_{X_0} = \begin{bmatrix} m_{X_0} \end{bmatrix},$$

$$\hat{M}_W = \begin{bmatrix} 0 & \text{diag}(M_d, \cdots, M_d) & 0 \\ 0 & \text{diag}(M_d, \cdots, M_d) & 0 \\ m_{W_0} & \cdots & m_{W_0} \end{bmatrix},$$

and $\mathcal{G} = \mathbb{R}^{2n} \times \mathbb{R}^{2n(n+1)} \times \mathbb{R}^{2n_dN} \times \mathbb{R}^{n_nN}$.

Although Theorem 2 provides necessary and sufficient conditions for distinguishability of systems with polytopic constraints on the disturbances and initial states, the condition in (3) is seldom satisfied in practice. Therefore, we add the following constraint on the disturbances intensity, which can be interpreted as a persistence of excitation, like condition

$$\frac{1}{N} \sum_{k=1}^{N} ||d(k)||^2 \geq \gamma.$$ (4)

Such a condition can be easily merged with (3), as explained in the sequel.

We start by introducing the Fourier-Motzkin elimination method, described in [22]. This method can be used to project polyhedral convex sets on to subspaces. Indeed, we are interested in projecting a polytope described by $\{ x \in \mathbb{R}^{n_x} : Ax \leq b \}$ on to $\mathbb{R}^{n_x}$, where $\tilde{n}_x < n_x$.

Let

$$\left( A_{LFM}, b_{LFM} \right) := \text{LFM}(A, b, n),$$

and

$$\left( A_{RFM}, b_{RFM} \right) := \text{RFM}(A, b, n),$$

where LFM and RFM stand for the left- and right-Fourier Motzkin elimination methods, $n = n_x - \tilde{n}_x > 0$, and where $A_{LFM}, b_{LFM}, A_{RFM}$ and $b_{RFM}$, satisfy, for all $\tilde{x} \in \mathbb{R}^{\tilde{n}_x}$,

$$A_{LFM}\tilde{x} \leq b_{LFM} \iff \exists x \in \mathbb{R}^{n_x} : A \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \leq b,$$
and
\[ A_{\text{RFM}} \bar{x} \leq b_{\text{RFM}} \Leftrightarrow \exists x \in \mathbb{R}^n : A \begin{bmatrix} \bar{x} \\ x \end{bmatrix} \leq b. \]

At this point, we can state the following theorem:

**Theorem 3:** Suppose that Assumptions 1–3 are satisfied and let
\[ (A_N, b_N) = \text{RFM} \left\{ \text{LFM} \begin{pmatrix} M_N \\ -M_N \\ M_{X_n} \\ M_W \end{pmatrix}, \begin{bmatrix} 0 \\ 0 \\ \bar{m}_{X_n} \\ m_W \end{bmatrix}, 2n \right\}, \]

Further define
\[ P_A = \frac{1}{N} \text{diag}(0_{n_a}, \ldots, 0_{n_a}, \bar{P}, \ldots, \bar{P}, 0_{n_a}), \]
and
\[ P_B = \frac{1}{N} \text{diag}(0_{n_a}, \ldots, 0_{n_a}, 0_{n_d}, \bar{P}, \ldots, 0_{n_d}), \]

and let \( \gamma_{\min} \geq 0 \) be such that
\[ \gamma_{\min} \geq \max_{A_Nx \leq b_N} x^T P_A x \quad \text{and} \quad \gamma_{\min} \geq \max_{A_Nx \leq b_N} x^T P_B x. \]

Then, the systems \( S_A \) and \( S_B \) are \( (X_o, \mathbb{R}^{r_u}, W) \)-input distinguishable in \( N \) measurements if
\[ \frac{1}{N} \sum_{k=0}^{N} \| \bar{P}d_A(k) \|^2 > \gamma_{\min} \quad \text{and} \quad \frac{1}{N} \sum_{k=0}^{N} \| \bar{P}d_B(k) \|^2 > \gamma_{\min}. \]

(5)

(6)

**Remark 3:** Notice that 5 can be interpreted as a concave quadratic programming problem, which can be solved, for instance, by testing the solution at the vertices of the polytope \( S = \{ x : Ax \leq b \} \) (cf. [23]).

**Remark 4:** The Fourier-Motzkin elimination method removes the dependence of the distinguishability of systems \( S_A \) and \( S_B \) on the initial state and control inputs. Hence, it reduces the number of variables in the optimization procedure.

The constraint in (4) can be replaced by a similar condition on the intensity of the output, \( y(.) \). To see this, we rewrite the output sequence as
\[
y(k) = Cx(k) + Nu(k), \quad k = 0, 1, \ldots, N
\]
\[
= \begin{bmatrix} C & N \end{bmatrix} \begin{bmatrix} x(k) \\ n(k) \end{bmatrix}, \quad k = 0, 1, \ldots, N
\]
\[
= C\bar{x}(k), \quad k = 0, 1, \ldots, N,
\]

where
\[
x(1) = Ax(0) + Bu(0) + Ld(0),
\]
\[
\vdots
\]
\[
x(N) = A^N x(0) + A^{N-1} Bu(0) + \ldots + A Bu(N-1) + Ld(N-1).
\]

Hence,
\[
\bar{x}(0) = \tilde{I}v_N,
\]
\[
\bar{x}(1) = \bar{A}(0)v_N,
\]
\[
\vdots
\]
\[
\bar{x}(N) = \bar{A}(N-1)v_N,
\]

where
\[
v_N^T = [x^T(0), n^T(0), \ldots, n^T(N), d^T(0), \ldots, d^T(N-1), u^T(0), \ldots, u^T(N-1)]^T
\]

and
\[
\tilde{I} = \begin{bmatrix} I \vdots 0 \vdots 0 \vdots 0 \vdots 0 \vdots 0 \vdots 0 \vdots 0 \vdots 0 \end{bmatrix},
\]
\[
\bar{A}(0) = \begin{bmatrix} A & 0 & \ldots & 0 & L & 0 & \ldots & 0 \vdots & B & 0 & \ldots & 0 \vdots & 0 & \ldots & 0 \end{bmatrix},
\]
\[
\bar{A}(N-1) = \begin{bmatrix} A^N & 0 & \ldots & 0 & A^{N-1}L & \ldots & L \vdots & A^{N-1}B & \ldots & 0 \vdots & 0 & \ldots & 0 \end{bmatrix}.
\]

Then, we have that
\[
\theta := \frac{1}{N} \sum_{k=0}^{N} \| y(k) \|^2
\]
\[
= \frac{1}{N} \left[ \bar{I}^T \overline{C} \bar{I} + \bar{I}^T (1) \overline{C} \bar{I} (1) + \cdots + \bar{I}^T (N) \overline{C} \bar{I} (N) \right]
\]
\[
= \frac{1}{N} \left( V_N \right)^T \overline{C} \left( V_N \right) v_n
\]

where
\[
\tilde{V}_N = \begin{bmatrix} \tilde{I} \\ \bar{A}(0) \\ \bar{A}(1) \\ \vdots \\ \bar{A}(N-1) \end{bmatrix},
\]

and
\[
\overline{C}(N) = \text{diag}(\overline{C}^T \overline{C}, \ldots, \overline{C}^T \overline{C}).
\]

Therefore, we are now in conditions of stating the following theorem:

**Theorem 4:** Define \( \tilde{P}_B = \frac{1}{N} \tilde{C}_B^N \), and let \( \theta_{\min} \geq 0 \) be such that
\[
\theta_{\min} \geq \max_{A_Nx \leq b_N} x^T \tilde{P}_B x,
\]

(7)

where
\[
(\bar{A}_N, \bar{b}_N) = \text{RFM} \left\{ \text{LFM} \begin{pmatrix} M_N^N \\ -M_N^N \end{pmatrix}, \begin{bmatrix} 0 \\ \bar{m}_{X_n} \\ m_W \end{bmatrix}, n + (N+1)n_n + N(n_d + n_u) \right\}, \]

\[
\tilde{C}_B^N = \left( \tilde{V}_N \right)^T \begin{bmatrix} \tilde{C}_B^N \tilde{C}_B & 0 & \cdots & 0 \\ 0 & \tilde{C}_B^N \tilde{C}_B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{C}_B^N \tilde{C}_B \end{bmatrix} \tilde{V}_N.
\]
\[
\tilde{V}_B^N = \begin{bmatrix}
\bar{I} \\
A_B^0 \\
\vdots \\
A_B^{N-1}
\end{bmatrix},
\]

\[
M_{x_o}^* = \begin{bmatrix}
M_{x_o} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{x_o} & 0 & 0 & 0
\end{bmatrix},
\]

\[
M_{\tilde{y}} = \begin{bmatrix}
0 & D_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D_d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_d & 0 & 0
\end{bmatrix},
\]

\[D_n = \text{diag}(M_n, \ldots, M_n), \quad D_d = \text{diag}(M_d, \ldots, M_d),\]

and, for \(i \in \{A, B\},\)

\[
M_i^N = \begin{bmatrix}
C_i & C_iA_i & \ldots & C_iA_i^{N-1} \\
\text{diag}(N_i) & \bar{L}_i^N & \bar{B}_i^N
\end{bmatrix},
\]

\[
\tilde{L}_i^N = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
C_iL_i & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_iA_i^{N-1}L_i & C_iA_i^{N-2}L_i & \ldots & C_iL_i
\end{bmatrix},
\]

\[
\tilde{B}_i^N = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
C_iB_i & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_iA_i^{N-1}B_i & C_iA_i^{N-2}B_i & \ldots & C_iB_i
\end{bmatrix}.
\]

If an SVO is designed for system \(S_A\) (denoted SVO\(_A\)) and the true plant is \(S_B\), then SVO\(_A\) is able to invalidate model \(S_A\) in less than \(N\) iterations if

\[
1 - \frac{1}{N} \sum_{k=1}^{N} \|y(k)\|^2 > \theta_{\text{min}}.
\]

V. EXAMPLES

In this section, we illustrate the concepts introduced in this paper with a couple of examples.

A. Example I

Consider the LTI discrete-time systems described by

\[
S_A\left\{ \begin{array}{l}
x(k+1) = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(k), \\
y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + n(k),
\end{array} \right.
\]

\[
S_B\left\{ \begin{array}{l}
x(k+1) = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(k), \\
y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix} x(k) + n(k),
\end{array} \right.
\]

where \(|d| \leq 1\) and \(|n| \leq 0.001\). Notice that system \(S_B\) can be seen as system \(S_A\) with a sensor failure.

Suppose we design an SVO for system \(S_A\) (denoted SVO\(_A\)), and that \(|x(0)| \leq 1\). Further suppose that we simulate SVO\(_A\) with the outputs coming from \(S_B\). Then, we might want to ask: What is the minimum intensity of the disturbances such that the SVO falsifies model \(S_A\) with probability \(\gamma\)? The answer to this question can be obtained using Theorem 3 and is illustrated in Fig. 1. Notice that in this case we only considered the disturbances intensity, while no assumptions have been posed regarding the intensity of the measurement noise.

\[
\text{Fig. 1. Minimum disturbances intensity that guarantee the distinguishability between systems } S_A \text{ and } S_B \text{ in Example I.}
\]

Indeed, let us consider that we simulate system \(S_B\) with SVO\(_A\). The results for 1000 Monte-Carlo runs are summarized in Table I. The green cells represent the combinations of \(\gamma_{\text{min}}\) and \(N\) such that the falsification of system \(S_A\) are guaranteed. Despite the fact that, in most of the cases analyzed in Table I, system \(S_A\) is falsified in more than 90% of the simulation runs, there are, however, worst-case disturbances that can help concealing the differences between the two systems, and thus it may happen that SVO\(_A\) does not falsify \(S_A\). Hence, although such disturbances may have a small probability of occurrence, they cannot be disregarded in worst-case approaches, such as the one presented herein.

\[
\begin{array}{cccccccc}
\gamma_{\text{min}} & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0.001 & 3.7 & 30 & 51.4 & 66.9 & 76.8 & 82.9 \\
0.005 & 78.2 & 95.2 & 99 & 99.6 & 99.9 & 100 \\
0.01 & 99.9 & 99.9 & 99.9 & 100 & 100 & 100 \\
0.1 & 100 & 100 & 100 & 100 & 100 & 100 \\
0.2 & 100 & 100 & 100 & 100 & 100 & 100 \\
0.5 & 100 & 100 & 100 & 100 & 100 & 100
\end{array}
\]

TABLE I

\[
\text{PERCENTAGE OF SIMULATION RUNS THAT FALSIFIED SYSTEM } S_A \text{ IN EXAMPLE I.}
\]

B. Example II

In this second example we consider a harder distinguishability problem. Indeed, let us define \(S_A\) and \(S_B\) as in Example I, but with the second element of the diagonal of the \(C\) matrix of system \(S_B\) being 0.9, rather than 0.6. In
comparison with Example I, the $C$ matrix of systems $S_A$ and $S_B$ are much similar. In fact, while Example I can illustrate a sensor loss-of-effectiveness of 40%, Example II can represent a sensor loss-of-effectiveness of only 10%. Repeating the design procedures as in Example I, we obtain the minimum intensity of the disturbances, depicted in Fig. 2, such that the SVO falsifies model $S_A$, with probability 1.

![Figure 2](image-url)

As expected, for the same intensity of the disturbances, we require a higher number of iterations to guarantee the distinguishability between the two systems, when compared to Example I. Alternatively, for the same number of iterations, we require a higher intensity of the disturbances to guarantee the distinguishability.

These results are further sustained by the Monte-Carlo simulations, summarized in Table II.

<table>
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<th>$\gamma_{\min}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
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<td>0.0001</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.0005</td>
<td>0</td>
<td>0</td>
<td>0.9</td>
<td>2.3</td>
<td>4.4</td>
<td>6.2</td>
</tr>
<tr>
<td>0.001</td>
<td>0</td>
<td>6.5</td>
<td>19.8</td>
<td>32.3</td>
<td>42.9</td>
<td>50.8</td>
</tr>
<tr>
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<td>96.8</td>
<td>99</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
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<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
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<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
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</tr>
<tr>
<td>0.5</td>
<td>100</td>
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</tr>
</tbody>
</table>

VI. CONCLUSIONS

In this paper, we introduced the concept of absolutely distinguishable discrete dynamic systems, highlighting its applicability to linear time-invariant (LTI) models. We further demonstrated that, in general, a persistence of excitation condition is required on the exogenous disturbances. It turned out that this condition can be written as a lower bound on the intensity of the perturbations. Necessary and sufficient conditions for the distinguishability of two systems were derived, showing that, under mild assumptions, set-valued observers are adequate for model falsification of LTI systems. The theory was illustrated with a set of examples that demonstrate the applicability of the results presented.

REFERENCES