Optimal Terminal Wealth under Partial Information: Both the Drift and the Volatility Driven by a Discrete Time Markov Chain

Michael Taksar and Xudong Zeng

Abstract—We consider a multi-stock market model. The stock price process satisfies a stochastic differential equation where both the drift and the volatility are driven by a discrete-time Markov chain of finite states. Not only the underlying Brownian motion but also the Markov chain in the stochastic differential equation are assumed to be unobservable. Investors can observe the stock price process only. The main result of this paper is that we derive the approximation of the optimal trading strategy and the corresponding optimal expected utility function from terminal wealth.

I. INTRODUCTION

We consider an incomplete market model in which stocks are driven by an m-dimensional Geometric Brownian motion the same as in the Black-Scholes model. However, the drift and the diffusion coefficients of this process depend on a discrete time Markov chain since when the Markov changes from one state to another, the market is considered to change from one scheme to another. Investors observe the stock price only. They only have ‘partial information’ since we assume the state of the Markov chain is not observable. The objective is to optimize trading strategy by maximizing a utility function from terminal wealth.

There are quite a few papers devoted to studies of the problem of maximizing the expected utility function from terminal wealth under partial information. Pham and Quenez [15] considered a stochastic volatility model. They solved the portfolio optimization problem under partial information by stochastic filtering techniques and adapting martingale duality methods. For more literature on partial information and stochastic volatility problems, we refer to Lakner [12],[13], Frey [8], Runggaldier [16] and Frey and Runggaldier [9].

Sass and Haussman [18] considered a multi-stock market model in continuous time. The drift is a continuous time, finite state Markov chain, and the volatility matrix is constant and nonsingular. They used Malliavin calculus and Hidden Markov Chain theory to derive an explicit expression for the optimal portfolio selection. However, their method can not be extended to the case in which the volatility is driven by a Markov chain, because the EM algorithm they used to estimate the drift does not work for the volatility due to the fact that the measures involved in their method are not equivalent if the volatility is driven by a Markov chain.

In this paper, we consider a discrete time multi-stock market model where both the drift and the volatility are driven by a Markov chain. In our paper, we use the method of estimating volatility studied by Elliott[4], Elliott, et al. [5], developed for the discrete time setting. The algorithm enables us to estimate the states of the Markov chain and its transition matrix. We solve the problem of optimizing the expected utility from terminal wealth, and using dynamic programming we construct the optimal strategy in terms of the filter of the price process. The proofs of the results given and further analysis can be found in Taksar and Zeng[20].

II. MODELS AND PRELIMINARY RESULTS

A. Regime Switching Model: Continuous-Time

Consider an m-dimensional stock price process whose dynamics is given by a geometric Brownian motion equations:
\[
d S(t) = diag(S_t)(\mu(Y(t))dt + \hat{\sigma}(Y(t))dW(t)), \quad 0 \leq t \leq T.
\]
(1)

Here \( S_t = (S_t^{(1)}, S_t^{(2)}, ..., S_t^{(m)}) \), the column vector \( W_t \) is a m-dimensional standard Brownian motion. \( Y(t) \) is a finite state, homogeneous Markov chain with a generator \( Q = (q_{ij})_{d \times d} \), independent of \( W(t) \). The distribution of \( Y(0) \) is known. \( Y(t) \) has a state space \( \mathcal{M} = \{e_1, ..., e_d\} \), where \( e_i, i = 1, 2, ..., d \) is the unit vector in \( \mathbb{R}^d \).

\[ Y(t) \in \mathcal{M} := \{e_1, ..., e_d\} \]

There are different values for the drift and different matrices for the volatility corresponding to states of the Markov chain \( Y(t) \). Thus \( \mu(\cdot) \) (resp. \( \sigma(\cdot) \)) is a mapping of \( \mathcal{M} \) (resp. \( \mathcal{N} := \{B := (b_{ij})_{1 \leq i,j \leq m} \text{ is invertible } b_{ij} \in \mathbb{R}^+\} \) into \( \mathbb{R}^m \) (resp. into \( \mathbb{R}^{m \times m} \)).

Assume the short interest rate \( r \) is constant for simplicity. Then (1) may be written as follows.
\[
d \log(e^{-rt}S(t)) = (\mu(Y(t)) - r1_m - diag(\hat{\sigma}_{n-1}\hat{\sigma}_{n-1}))dt + \hat{\sigma}(Y(t))dW(t),
\]
(2)

where \( 0 \leq t \leq T, 1_m = (1, 1, ..., 1) \in \mathbb{R}^{m \times 1} \), where we use the following convention:
\[
\log((x_1, x_2, ..., x_n)^T) = (\log(x_1), ..., \log(x_n))^T.
\]

B. Regime Switching Model: Discrete-Time

In this paper, we will consider a discrete approximation to the continuous time model (2).
Let \( \Delta t = \frac{T}{N}, Y_n = Y(n\Delta t) \),
\[
\mu_n = \mu(Y(n\Delta t)), \hat{\sigma}_n = \hat{\sigma}(Y(n\Delta t)), S_n = S(n\Delta t),
\]
where \( n = 0,1,...,N \). Then the equation (2.2) becomes
\[
y_n := \log(S_n e^{-r \Delta t}) - \log(S_{n-1}) \\
= (\mu_{n-1} - r_1 m - \text{diag}(\hat{\sigma}_{n-1} \hat{\sigma}_n'))/2 \Delta t + \hat{\sigma}_{n-1}(W_n - W_{n-1}),
\]
for \( n=1,2,3,...,N \).

Let
\[
\begin{align*}
g_n & := (\mu_n - r_1 m - \text{diag}(\hat{\sigma}_n \hat{\sigma}_n'))/2 \Delta t \\
& = (\mu(Y_n) - r - \text{diag}(\hat{\sigma}(Y_n)\hat{\sigma}(Y_n')))\Delta t, \\
\sigma_n & := \sigma_n \sqrt{\Delta t} = \sigma(Y_n)\sqrt{\Delta t}
\end{align*}
\]
Then
\[
y_n = g_n + \sigma_n Z_n, n = 1,2,...,N,
\]
where \( Z_n = (W_n - W_{n-1})/\sqrt{\Delta t}, n = 1,2,...,N \) is a sequence of standard normal i.i.d. random variables.

Note that \( g_n \) and \( \sigma_n \) are functions of \( Y_n \) and can be written as \( G(Y_n) \) and \( H(Y_n) \) respectively which obviously satisfy
\[
\begin{align*}
\frac{G_{\epsilon_i}}{\Delta t} &= \mu(e_i) - r_1 m - \text{diag}(\hat{\sigma}(e_i)\hat{\sigma}(e_i')) / 2, \\
\frac{H_{\epsilon_i}}{\Delta t} &= \hat{\sigma}(e_i)
\end{align*}
\]

In this paper, we assume only the price of stock \( S_n \) or \( y_n \) can be observed. Denote the filtration generated by \( S_n \) by \( \{\mathcal{F}_n\} \). We will study an optimization of the utility function from terminal wealth in the discrete time model (5).

C. Preliminary Results

We present some preliminary results which will be used in the proofs in the subsequent. By the definition of \( y_n \) (3), we know that for each \( i=1,2,...,m \),
\[
G^{(i)} = S^{(i)}_{n-1} e^{y^{(i)}_n} e^{\Delta t}.
\]
For \( k = 1,2,...,d \), denote
\[
b_k := G(e_k) \in \mathbb{R}^{m \times 1}, f_k := H(e_k) \in \mathbb{R}^{m \times m}.
\]
Let \( b_k^{(i)} \) stands for the \( i^{th} \) component of \( b_k \), let \( f_k^{(i)} \) stands for the \( i^{th} \) row of \( f_k \). Then we have
\[
Pr(y_n^{(i)} \leq t|\mathcal{F}_{n-1}) = \sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1}) \int_{-\infty}^{t-b_k^{(i)}} \phi_k(x)dx,
\]
where \( \phi_k(x) = \frac{1}{\sqrt{2\pi}f_k(x)/f_k(x')}) e^{-\frac{x^2}{2}} \).

Proof: of (7) \( Pr(y_n^{(i)} \leq t|\mathcal{F}_{n-1}) = Pr(g_n^{(i)} + \sigma_n^{(i)} Z_n \leq t|\mathcal{F}_{n-1}) \)
\[
= \sum_{k=1}^d Pr(b_k^{(i)} + f_k^{(i)}Z_n \leq t, Y_{n-1} = e_k|\mathcal{F}_{n-1})
\]
\[
= \sum_{k=1}^d Pr(b_k^{(i)} + f_k^{(i)}Z_n \leq t)Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1})
\]
\[
= \sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1}) \int_{-\infty}^{t-b_k^{(i)}} \phi_k(x)dx.
\]
Similarly, for the multi-dimensional case, \( x \in \mathbb{R}^{m \times 1} \), we have
\[
Pr(y_n \leq x|\mathcal{F}_{n-1}) = \sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1}) \int_{-\infty}^{t-b_k^{(i)}} \phi_k(x)dz,
\]
where \( \phi_k(x) = (2\pi|f_k^{(i)}|)^{-\frac{1}{2}} e^{-\frac{x^2}{2} f_k^{(i)}x'}\frac{1}{2} f_k^{(i)}x'^2, x \in \mathbb{R}^{m \times 1} \).

And we have a recursive filter :
\[
Pr(Y_n = e_k|\mathcal{F}_n) = \sum_{i=1}^d Pr(Y_{n-1} = e_i|\mathcal{F}_{n-1})\phi_k(y_{n-1} - b_k),
\]
where \( p_{ki} \) is the (k,i) entry of the transition matrix \( P \).

In the sequel, we will use the notation: \( E_n[\zeta] := E[\zeta|\mathcal{F}_n] \).

Let \( \alpha < 1 \). For \( i = 1,2,...,m \), we have
\[
(i) \quad |E_n(\epsilon^{h(i)} - 1)| = O(\Delta t),
\]
\[
(ii) \quad |E_n((\epsilon^{h(i)} - 1)^2)| = O(\Delta t),
\]
\[
(iii) \quad |E_n(\epsilon^{h(i)} - 1)^3)| = O(\Delta t)^2,
\]
\[
(iv) \quad |E_n(1 - \epsilon^{h(i)} - 1)^3)| = O(\Delta t)^2,
\]
\[
(v) \quad |E_n(1 - \epsilon^{h(i)} - 1)^3)| = O(\Delta t)^2.
\]

III. DEFINITION AND OPTIMIZATION PROBLEM

A. WEALTH PROCESS AND ADMISSIBLE STRATEGIES

In this section we describe a discrete time optimization model which approximates the original continuous time model (2).

Definition \( h_{n-1} \in \mathcal{F}_{n-1}, n = 1,...,N \) are column vectors \( \in \mathbb{R}^{m \times 1} \). A wealth process \( \{X^{h_{n-1}}\}_{n=1,2,...,N}, X_0 = x_0 \) is defined as
\[
X^{h_{n-1}} = X^{h_{n-1}} e^{\Delta t} \left( \sum_{i=1}^m h_{n-1}^i \right) + \sum_{i=1}^m h_{n-1}^i \phi_{n-1}^i S_{n-1}^{(i)} e^{\Delta t},
\]
where \( h_{n-1}^i \) or \( S_{n-1}^{(i)} \) denotes the \( i^{th} \) component of the vector \( h_{n-1} \) or \( S_{n-1} \). Using the notations defined in Section II, the wealth process has a simpler expression:
\[
X^{h_{n-1}} = X^{h_{n-1}} e^{\Delta t} (1 + h_0 \cdot (e^{h_{n-1}} - 1)),
\]
where "\cdot" stands for the vectors or matrices multiplication. Generally, we have
\[
X^{h_{n-1}} = X_0 e^{\Delta t} (1 + h_0 \cdot (e^{h_{n-1}} - 1)),
\]
where \( n=1,...,N \).

Definition A vector sequence \( h = \{h_{i}\}_{i=0}^N \) is an admissible strategy if \( Pr(X^{h_{n-1}} > 0, \text{ for all } n=1,2,...,N) = 1 \).

We use \( \mathcal{H} \) to denote the set of all admissible strategies.

One can check that if \( h \) is admissible, then
\[
\|h_{n-1}\|_1 = 1, 1 \geq h_{n-1}^i \geq 0,
\]
for each \( i \in \{1,2,...,m\} \).

The inequalities above imply no short stock as well as no money borrowing in our model. Rogers also
mentioned such a restriction on the portfolio in his h-investor model([17]).

**B. HARA Utility Functions**

**Definition** A function \( u : (x_u, \infty) \rightarrow \mathcal{R}, x_u \in \mathcal{R} \cup \{-\infty\} \), is called a utility function, if \( u \) is strictly increasing, strictly concave, twice continuously differentiable on \( (x_u, \infty) \), satisfies \( \lim_{x \to \infty} u'(x) = 0 \) and \( \lim_{x \to x_u^+} u'(x) = \infty \).

Next we define the coefficient of absolute risk aversion,

\[
R_u(x) = -\frac{u''(x)}{u'(x)}.
\]

**Definition** If \( R_u^{-1}(x) \) is a linear function, i.e. \( R_u^{-1}(x) = ax + bx \), then we say that \( u(x) \) is of the hyperbolic absolute risk aversion (HARA) class.

The most popular HARA utility functions are Constant Relative Risk Aversion (CRRA): \( u(x) = x^{\beta} / \beta \), where \( \beta < 1 \).

Constant Absolute Risk Aversion (CARA): \( u(x) = -e^{-\beta x} / \beta \), where \( \beta > 0 \).

**C. Optimization Problem**

Let \( u(x) \) be a utility function, and \( \{X^N_n\} \) be the wealth process. The objective is to calculate

\[
V^* = \sup_{h \in \mathcal{H}} \{ E[u(X^N_n)] \}
\]

and to find an admissible trading strategy \( h^* \) s.t. \( E[u(X^N_n)] = V^* \).

**IV. DYNAMIC PROGRAMMING**

Define

\[
U_n(x) = \sup_{h \in \mathcal{H}} E[u(X^N_n)|\mathcal{F}_n], \quad n = 0, 1, ..., N.
\]

From this definition \( U_0 = V^* \) as defined in the optimization problem of Section III. The dynamic programming equation for the sequence \( u_0, u_1, ..., u_N \) is

\[
\begin{cases}
U_n(x) = \sup_{h \in \mathcal{H}} E[U_{n+1}(xe^{r\Delta t} + xe^{r\Delta t}h_n \cdot (e^{yn+1} - 1))|\mathcal{F}_n], \\
U_N(x) = u(x).
\end{cases}
\]

(13)

In what follows we derive a numerical scheme for the solution of these dynamic programming equations and approximations for the optimal strategies for the original optimization problem.

For each \( \eta_n \in \mathcal{F}_n, n = 0, 1, ..., N-1 \) define

\[
A^{-1}_{n-1, \eta_n} = E^{-1}_{n-1}[\eta_n(e^{yn} - 1) \cdot (e^{yn} - 1)].
\]

**Proposition 4.1:** Let \( u(x) = x^{\alpha}/\alpha, 0 \neq \alpha < 1 \). Let

\[
\begin{align*}
\lambda_N &:= 1, \quad \lambda_{n} := E_n[\prod_{k=n}^N \lambda_k], \\
\lambda_{n-1} &:= (1 + \frac{1}{\alpha}(e^{yn} - 1)^{\alpha} A^{-1}_{n-1, \eta_n} E_{n-1}[\eta_n e^{yn} - 1])^{\alpha}, \quad n=1, ..., N.
\end{align*}
\]

(i) \( \eta_n \in \mathcal{F}_n \) is bounded,

(ii) \( A^{-1}_{n-1, \eta_n} \) is invertible and there exists a constant \( C \), such that \( \Delta t||A^{-1}_{n-1, \eta_n}|| < C \) for all \( n=1,2, ..., N \).

(iii) \( |E_{n-1}[\eta_n e^{yn} - 1]| = o(\sqrt{\Delta t}) \).

The proof of this proposition follows from Takas and Zeng[20] Theorem D.1 (iii), (ii) and Theorem D.2 in Appendix D.

For each \( \eta_n \in \mathcal{F}_n, n = 0, 1, ..., N-1 \), define

\[
W_n(x_n, h_n) := E[\eta_n u(x_n)|\mathcal{F}_n] = E[\eta_{n+1}u(x_n e^{r\Delta t} + xe^{r\Delta t}h_n \cdot (e^{yn+1} - 1))|\mathcal{F}_n],
\]

\[
V_n(x_n, h_n) := E[\eta_{n+1}(u(x_n e^{r\Delta t}) + u'(x_n e^{r\Delta t})xe^{r\Delta t}h_n \cdot (e^{yn+1} - 1)) + \frac{1}{2} u''(x_n e^{r\Delta t})xe^{r\Delta t}h_n \cdot (e^{yn+1} - 1))^{2})|\mathcal{F}_n],
\]

(14)

Assume for any \( x_{n-1} \geq 0 \), there exists \( h^*_{n-1}, h^{**}_{n-1} \in [0, 1]^m \subset \mathcal{R}^m, s.t.

\[
W_{n-1}(x_{n-1}, h^*_{n-1}) = \sup_{h_{n-1}} W_{n-1}(x_{n-1}, h_{n-1}),
\]

(15)

and

\[
V_{n-1}(x_{n-1}, h^{**}_{n-1}) = \sup_{h_{n-1}} V_{n-1}(x_{n-1}, h_{n-1}).
\]

(16)

As is seen from (14), \( V_n \) is obtained from \( W_n \) by taking a Taylor expansion up to the second term. Thus the function \( V_n \) approximates \( W_n \) when \( \Delta t \) is small (and as a result \( y_n \) is close to 0).

**Lemma 4.2:** Let \( u(x) = x^{\alpha}/\alpha, 0 \neq \alpha < 1 \). Let \( \eta_n \) be defined as in Prop. 4.1. Then

\[
|W_{n-1}(x_{n-1}, h^{**}_{n-1}) - W_{n-1}(x_{n-1}, h^*_{n-1})| = x^{\alpha} o(\Delta t),
\]

(17)

where

\[
h^{**}_{n-1} = \frac{1}{1 - \alpha} A^{-1}_{n-1, \eta_n} \cdot E_{n-1}[\eta_n e^{yn} - 1].
\]

(18)

Using the same route we can prove that Lemma 4.2 holds for any \( \eta_n \) satisfying the three conditions in Proposition 1. The motivation to define \( W_n \) and \( V_n \) can be seen as follows. Let \( n = N-1 \). Then in (13) we have

\[
U_{N-1}(x) = \sup_{h_{N-1}} E[u(x e^{r\Delta t} + xe^{r\Delta t}h_{N-1} \cdot (e^{YN} - 1))|\mathcal{F}_{N-1}].
\]

Clearly, \( U_{N-1}(x_{N-1}) \) coincides with \( W_{N-1}(x_{N-1}, h^{**}_{N-1}) \) defined in (15) with \( \eta_N = 1 \). Below, we write \( \mathcal{E}_{n-1}(x_{n-1}) := W_{n-1}(x_{n-1}, h^*_{n-1}) - W_{n-1}(x_{n-1}, h^{**}_{n-1}) \) for convenience.

In other words,

\[
U_{N-1}(x_{N-1}) = W_{N-1}(x_{N-1}, h^{**}_{N-1}) = W_{N-1}(x_{N-1}, h^{*}_{N-1}) + \mathcal{E}_{N-1}
\]

\[
\approx W_{N-1}(x_{N-1}, h^{**}_{N-1}) = E_{N-1}[u(x_{N-1} e^{r\Delta t} (1 + h^{*}_{N-1} \cdot (e^{YN} - 1)))]
\]

since \( \mathcal{E}_{N-1} \) is small by Lemma 4.2.
As a result, one can find an approximation for the optimal strategy in a recursive way. Moreover the expected utility of the terminal wealth associated with $h^{**} = \{h_{k}^{**}\}_{k=0}^{N-1}$ is an approximation for the value function $V^{*}$ in the optimization problem. The next two theorems show that in a limit as $\Delta t$ tends to zero the value of $V^{*}$ converges to that of $V^{*}$. Thus $h^{**}$ can serve as an approximation for the optimal strategy.

Hence Lemma 4.2 shows that the difference between the wealth associated with optimal portfolio $h^{*}$ and the portfolio $h^{**}$ is small when $\Delta t$ is small.

**Theorem 4.3:** Let $u(x) = \frac{x^{\alpha}}{\alpha}, 0 \neq \alpha < 1$. Define

$$
\begin{align*}
\lambda_{N} :&= 1, \eta_{N} = 1, \\
\lambda_{n-1} :&= \frac{1}{n} A_{n-1, \eta_{n}} \cdot E_{n-1}[(e^{y_{n}} - 1)\eta_{n}], \\
\lambda_{n} :&= e^{r\Delta t(1 + \frac{1}{n}h^{**}_{n-1} \cdot (e^{y_{n}} - 1))}, \\
\eta_{n} :&= E_{n}[\prod_{i=n}^{N} \lambda_{i}], 1 \leq n \leq N.
\end{align*}
$$

Then

$$
\|U_{n}^{**}(x_{n}) - U_{n}(x_{n})\| = o(1),
$$

where

$$
U_{n}^{**}(x_{n}) := E_{n}[u(x_{n}^{**})], \text{ and } x_{n,k}^{**} := x_{n}^{e^{(k-n)r\Delta t}} \cdot \prod_{i=n}^{k-1} (1 + h_{i}^{**} \cdot (e^{y_{i+1}} - 1)).
$$

The case of a logarithmic utility function can be treated as the same way as the power utility function. The same results hold, although with a higher rate of convergence.

**Lemma 4.4:** Let $u(x) = \log(x)$, choose $\eta_{n} = 1, n = 1, 2, \ldots, N$. Then we have

$$
\|W_{n-1}(x_{n-1}, h^{**}_{n-1}) - W_{n-1}(x_{n-1}, h^{*}_{n-1})\| = O(\Delta t)^{2},
$$

where

$$
h^{**}_{n-1} = A_{n-1, 1} \cdot E_{n-1}[e^{y_{n}} - 1].
$$

The proof is the same as that of Lemma 4.2. Moreover when we repeat the proof, we see that the resulting convergence rate is $O(\Delta t)^{2}$, which is higher than $O(\Delta t)$ obtained in the case of the power utility function.

**Theorem 4.5:** Let $u(x) = \log(x)$, choose $\eta_{n} = 1, n = 1, 2, \ldots, N$. Then

$$
\|U_{n}(x_{n}) - U_{n}^{**}(x_{n})\| = O(\Delta t),
$$

where

$$
U_{n}^{**}(x_{n}) := E_{n}[u(x_{n}^{**})], \\
x_{n,k}^{**} := x_{n}^{e^{(k-n)r\Delta t}} \cdot \prod_{i=n}^{k-1} (1 + h_{i}^{**} \cdot (e^{y_{i+1}} - 1)),
$$

and $h_{i}^{**}$ is defined in Lemma 4.4.

**Remark** There are particular cases depending on the structure of the transition matrix $P$, when we have the convergence rate $O(\Delta t)$ as above even for the power utility function. One of those cases is when the transition matrix has identical columns. We also have another case in [20] Appendix D, Theorem D.3 (ii) when the convergence rate is of a higher rate of $O(\Delta t)$.

**Remark** Let $u(x) = \frac{1}{2} x^{2}$. Define

$$
\eta_{N} := 1, \\
\eta_{n-1} := e^{-\gamma h_{n-1}^{**} \cdot (y_{n} - 1)}, \\
h_{n-1}^{**} := A_{n-1, 1} \cdot E_{n-1}[(e^{y_{n}} - 1)\eta_{n}], \\
U_{n}^{**} := u(x_{n-1} \cdot e^{r(N-n+1)\Delta t}) \cdot \prod_{i=n}^{N-1} (1 + h_{i}^{**} \cdot (e^{y_{i+1}} - 1)),
$$

where $n = N, \ldots, 1$. Then

$$
\|U_{0}(x_{0}) - U_{0}^{**}(x_{0})\| = o(1).
$$

**Remark** From the definition (20) of $\eta_{n}$, we can see that $\eta_{n}$ is the expected utility under strategy $h^{**}$, given $F_{n}$, and given $x_{n} = 1$. That is

$$
\eta_{n} = E[u(X_{n}^{**})|F_{n}, x_{n} = 1].
$$

**V. Simulations**

Generally, as a result it is not easy to compute the approximate optimal strategy $\{h_{n}^{**}\}_{n=0}^{N-1}$ in (18) in the case of power utility function. However we can get an estimation using a simplified strategy:

$$
h_{n}^{**} := -\frac{1}{1 - \alpha} A_{n-1, 1} \cdot E_{n-1}[(e^{y_{n}} - 1)], n = 0, 1, \ldots, N-1.
$$

From the definition of $V^{*}$, we see that the expected utility $E[u(x_{N}^{**})]$ associated with $h^{**}$ is a lower bound for $V^{*}$.

There are cases, however, where (18) is relatively easy to evaluate. For example, if the transition matrix has identical columns, then the conditional probability $P_{n}(Y_{n} = e^{k}|F_{n}), n \geq 1$, would be a constant regardless of $n, k$. Thus $\eta_{n}$ is a constant, and it can be excluded from the expression for $h_{n}^{**}$ (18). Therefore the strategy $h^{**}$ is the same as $h^{**}$.

We apply the results of Section IV to obtain an applicable representation of the strategy for the case of $m = 1$,

$$
h_{n-1}^{**} = \sum_{k=1}^{d} P_{n-1}(Y_{n-1} = e^{k}|F_{n-1})(\sigma(e^{k} - r)) + O(\Delta t),
$$

where $n = 1, 2, \ldots, N$. The simulations in this section deal with (21). We use (9) to calculate $P_{n}(Y_{n} = e^{k}|F_{n})$ recursively.

Comparing the strategies (21) with Merton strategies, we can see that in our case the constant drift or the constant volatility in the expression for Merton strategies are replaced by linear combinations of the drifts or of the volatilities corresponding to different states of the Markov chain. The weights of the linear combinations are probabilities that the Markov chain is in those states. We divide the time interval $[0, T]$ into $N$ parts, and assume the transition of the Markov chain occurs only at those points of time. Hence we have the Merton model on each interval. The consequence is that we might obtain a solution directly like (21) in the case of the logarithmic utility function. However, it is not true for the power or the exponential utility functions.

To illustrate this point, let us assume that the transition matrix does not have identical columns. Then in (18) we can not choose $\eta_{n} = 1$ as in the case of logarithmic utility function, nor can we simplify the expression by cancelling $\eta_{n}$ as in the case, when all the columns of the transition
matrix are identical. A representation as simple as (21) cannot be obtained. We have to employ the Monte Carlo method to calculate the portfolio (18). However, we may use (21) to get a lower bound for $V^*$.

In our simulations, $W_0$ stands for the initial wealth. The default value of $W_0$ is 1. $P$ denotes the transition matrix. The interest rate $r$ is equal to $0.06$. The time horizon $T$ is 1, and it is divided in $N = 1000$ parts, i.e. $\Delta t = 10^{-3}$.

We compare our optimal strategy with the Merton strategy. Since the Markov chain has several states, we use the Merton strategy replacing the drift and the volatility in it with those obtained from taking average of the drift and volatility respectively over different states of the Markov chain. The resulting Merton’s strategy is

$$h_n = \frac{\bar{\mu} - \bar{\sigma}}{(1 - \alpha)\bar{\sigma}^2}, n = 1, \ldots, N,$$

where $\bar{\mu} = \sum_i \mu(e_i)/d, \bar{\sigma} = \sum_i \sigma(e_i)/d$. One may think of using $\sum_i \sigma'(e_i)^2/d$ instead of $\bar{\sigma}^2$ in the formula. However, our simulation shows that it does not provide a better result.

We also compare our optimal strategy with the buy-and-hold strategy which is denoted as “b/h”. The buy-and-hold strategy means to buy the stock using all cash available at the beginning, then hold the stock until the end. We generate the wealth process 1000 times and calculate the average of the utilities from the terminal wealth.

Table I lists the result for a Markov chain which has a transition matrix with identical columns. For the power utility function, we can obtain that our optimal strategy is a constant 0.1133 while the Merton strategy is also a constant 0.3636 different from ours as seen from (21). Table I &II show that our optimal strategy on average gives better utilities with smaller standard deviations for both the logarithm and the power law utility functions. The last lines of Table I&II show the number of simulations in which our optimal strategy generates a better utility than the Merton strategy or the "b/h" strategy. Note that in the case of the power utility function, even though our optimal strategy only generates 487 better than the Merton strategy among 1000 simulations, the average (-0.2748) is still significantly higher than the one (-0.2974) generated by the Merton strategy, and the standard deviation (0.0380) is also significantly less than 0.1348.

In Table III&IV, we use the same parameters except the transition matrix is replaced by a matrix with non-identical columns. In this case for the power utility function, the Merton strategy is still a constant ($=0.3636$). However, our optimal strategies $h_*$ varies from 0.1294 to 0.3324 with a mean 0.1450 while it is a constant 0.1133 in the previous case. The result are similar to those of Table I & II though.

The average utility may vary slightly if more wealth processes are generated in the simulation. However, we always find our optimal strategy generates on average the best utilities. The results in Table V&VI for 5000 and 10000 simulations show that although the utilities vary slightly, our optimal strategy still has the best performance among three strategies.

For one more example, we choose the same parameters as in the example 1 of Sass and Haussmann [18]. The results for this example are listed in Table VII&VIII. Table IX&X are copies of Table 2 of Sass and Haussmann [18]. One can see that our optimal strategy generates the average utilities(0.3969 for the logarithm, -0.1128 for the power) very close to theirs(0.399 for the logarithm, -0.121 for the power). It is not surprising because our model can be viewed as an extension of theirs in an approximate sense. Therefore similar results are expected when the same parameters are employed.

Finally, we provide standard deviations in Table I&II and Table III&IV. One can see that the standard deviation associated with the optimal strategy is always the smallest.

### Table I

<table>
<thead>
<tr>
<th>$u(x)$</th>
<th>$\log(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>strategy</td>
<td>opt</td>
</tr>
<tr>
<td>av. $u(x)$</td>
<td>0.0772</td>
</tr>
<tr>
<td>med $u(x)$</td>
<td>0.0781</td>
</tr>
<tr>
<td>std $u(x)$</td>
<td>0.1871</td>
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<tr>
<td>opt better</td>
<td>574</td>
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</table>

### Table II

<table>
<thead>
<tr>
<th>$u(x)$</th>
<th>$-x^{-3}/3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>strategy</td>
<td>opt</td>
</tr>
<tr>
<td>av. $u(x)$</td>
<td>-0.2748</td>
</tr>
<tr>
<td>med $u(x)$</td>
<td>-0.2719</td>
</tr>
<tr>
<td>std $u(x)$</td>
<td>0.0380</td>
</tr>
<tr>
<td>opt better</td>
<td>487</td>
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</table>

### Table III

<table>
<thead>
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<th>$\log(x)$</th>
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</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>opt</td>
</tr>
<tr>
<td>av. $u(x)$</td>
<td>0.0946</td>
</tr>
<tr>
<td>med $u(x)$</td>
<td>0.0930</td>
</tr>
<tr>
<td>std $u(x)$</td>
<td>0.2703</td>
</tr>
<tr>
<td>opt better</td>
<td>565</td>
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### Table IV

<table>
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</thead>
<tbody>
<tr>
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<td>opt</td>
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<tr>
<td>av. $u(x)$</td>
<td>-0.2715</td>
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<tr>
<td>med $u(x)$</td>
<td>-0.2664</td>
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<tr>
<td>std $u(x)$</td>
<td>0.0557</td>
</tr>
<tr>
<td>opt better</td>
<td>470</td>
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</table>
TABLE V

<table>
<thead>
<tr>
<th>u(x)</th>
<th>log(x)</th>
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</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>opt</td>
</tr>
<tr>
<td>5000 sim.</td>
<td>0.0923</td>
</tr>
<tr>
<td>10000 sim.</td>
<td>0.0966</td>
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</table>

VI. Acknowledgements

The authors are grateful to the Associate Editor and two anonymous referees for their constructive suggestions, especially the comments on the structure of the paper and the model setting. The authors are also indebted to J. Zhang and J. Cvitanić for constructive conversations.

REFERENCES


TABLE VII

$$\mu = [0.8, -0.4]', \sigma = [0.2, 0.2]', Q = [-30, 24; 30, -24], P = e^{Q/2\Delta t}, \Delta t = 1/250, 500 simulations.$$ 

<table>
<thead>
<tr>
<th>u(x)</th>
<th>log(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>opt</td>
</tr>
<tr>
<td>av. u(x)</td>
<td>0.3969</td>
</tr>
<tr>
<td>med u(x)</td>
<td>0.2965</td>
</tr>
<tr>
<td>opt better</td>
<td>319</td>
</tr>
</tbody>
</table>

TABLE VIII

$$\mu = [0.8, -0.4]', \sigma = [0.2, 0.2]', Q = [-30, 24; 30, -24], P = e^{Q/2\Delta t}, \Delta t = 1/250, 500 simulations.$$ 

<table>
<thead>
<tr>
<th>u(x)</th>
<th>−x^{-3}/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>opt</td>
</tr>
<tr>
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</tr>
<tr>
<td>10000 sim.</td>
<td>−0.2709</td>
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TABLE IX

SASS and HAUSSMANN’S Table 2, known Parameters

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</thead>
<tbody>
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<td>Strategy</td>
<td>opt</td>
</tr>
<tr>
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<td>0.399</td>
</tr>
<tr>
<td>med u(x)</td>
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</tr>
<tr>
<td>opt better</td>
<td>296</td>
</tr>
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</table>

TABLE X

SASS and HAUSSMANN’S Table 2, known Parameters

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</thead>
<tbody>
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<td>Strategy</td>
<td>opt</td>
</tr>
<tr>
<td>av. u(x)</td>
<td>−0.121</td>
</tr>
<tr>
<td>med u(x)</td>
<td>−0.091</td>
</tr>
<tr>
<td>opt better</td>
<td>359</td>
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</tbody>
</table>