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Simulation-aided Reachability and Local Gain Analysis for Nonlinear Dynamical Systems

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Abstract— **We analyze reachability properties and local input/output gains of systems with polynomial vector fields. Upper bounds for the reachable set and nonlinear system gains are characterized by polynomial Lyapunov (storage) functions satisfying certain bilinear constraints. A methodology utilizing information from simulations to generate Lyapunov function candidates satisfying necessary conditions for bilinear constraints is proposed. The suitability of Lyapunov function candidates are assessed solving linear sum-of-squares optimization problems. Qualified candidates are used to compute upper bounds for the reachable set and nonlinear system gains and to initialize further coordinate-wise affine optimization. We illustrate the method on several examples from the literature.**

I. Introduction

We consider the problem of computing upper bounds for the reachable set and local input-to-output (IO) gains of nonlinear dynamical systems with polynomial vector fields around asymptotically stable equilibrium points. Similar problems were studied in [1], [2], [3], [4], [5]. This paper is an extension of the work reported in [2] and proposes a methodology that incorporates prior information from simulations in reachability and local gain analysis of nonlinear systems.

Following [1], [2], we characterize upper bounds on the reachable sets and the local IO gains due to bounded $\mathcal{L}_2$ (and possibly $\mathcal{L}_\infty$) disturbances by Lyapunov (storage) functions which satisfy certain "local" dissipation inequalities [6]. Using sum-of-squares (SOS) relaxations for polynomial non-negativity [7], it is possible to search for polynomial Lyapunov functions for systems with polynomial (and/or rational) dynamics using semidefinite programming [3], [8], [1], [2]. However, the SOS relaxations for the problems in the papers lead to bilinear matrix inequality (BMI) constraints. Motivated by the difficulties associated with bilinear programming, we here propose a methodology that incorporates information from system simulations in formal proof construction (computing certifying Lyapunov functions) and extend the applicability of reachability and local IO gains analysis for nonlinear systems. Although the information from simulations is inconclusive, i.e., cannot be used to prove certain properties (such as upper bounds for reachable sets and local IO gains or lower bounds for regions-of-attraction) of the system, it provides insight into the system behavior. Following a similar technique developed for computing invariant subsets of the (robust) regions-of-attraction in [9], the method proposed here relaxes the bilinear problem (using a specific system theoretic interpretation of the corresponding optimization, specifically SOS optimization, problem) to a linear problem, and the true feasible set is a subset of the linear problem's feasible set. If large amount of simulation data is used, samples from the linear problem's feasible set often are suitable Lyapunov functions, i.e., provide suboptimal solutions for the actual problem, and certify certain upper bounds. They can also be used as initial seeds for further optimization (e.g. coordinate-wise linear search).

We demonstrate the development of the method for computing upper bounds for the reachable set due to bounded $\mathcal{L}_2$ disturbances (or local $\mathcal{L}_2 \to \mathcal{L}_\infty$ gain) in detail and only comment on how to modify the method for computing upper bounds for local $\mathcal{L}_2 \to \mathcal{L}_2$ gain. The rest of the paper is organized as follows: Characterization of upper bounds for the reachable set with Lyapunov functions and a method for computing lower bounds are explained in section II. Section III is devoted to the simulation-based relaxation for the bilinear problem. The method is demonstrated on a few examples in section IV. Extensions of the simulation-aided technique to local $\mathcal{L}_2 \to \mathcal{L}_2$ gain analysis is discussed in section V, which is followed by concluding remarks.

Notation: $\mathcal{R}^n$ denotes the n-dimensional Euclidean space. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ ($Q \succ 0$) means that $x^TQx \geq 0$ (≤ 0) for all $x \in \mathcal{R}^n$. $\mathbb{R}[x]$ represents the set of polynomials in x of certain finite degree with real coefficients. The subset $\Sigma[x] := \{\pi \in \mathbb{R}[x] : \pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_m^2,\ \pi_1, \cdots, \pi_m \in \mathbb{R}[x]\}$ of $\mathbb{R}[x]$ is the set of SOS polynomials. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of π. We do not specify the degree of polynomials in $\mathbb{R}[x]$ and $\Sigma[x]$, unless it causes confusion, and it is to be understood in the context. $\triangleleft$

II. Reachable Set due to $\mathcal{L}_2$ Disturbances

Consider the nonlinear dynamical system

$$\dot{x}(t) = f(x(t), w(t)) \quad (1)$$

where $x(t) \in \mathcal{R}^n$, $w(t) \in \mathcal{R}^{n_w}$, and f is a n-vector with elements in $\mathbb{R}[(x, w)]$ such that $f(0, 0) = 0$. Let $\phi(t; \xi, w)$ denote the solution to (1) at time t with the initial condition $x(0) = \xi$ driven by the input/disturbance w. The set Ω_γ of points reachable from the origin under (1), provided that the

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disturbance satisfies $\|w\|_2^2 := \int_0^T w(t)^T w(t)\,dt \leq \gamma$, $T \geq 0$ is defined

$$\Omega_\gamma := \left\{\phi(T;0,w) \in \mathcal{R}^n \ : \ T \geq 0,\ \|w\|_2^2 \leq \gamma\right\}.$$

Next, we review characterizations of the upper and lower bounds of the reachable set.

A. Upper bound of the reachable set

Following a Lyapunov-like argument in [10, §6.1.1], Lemma 1 provides a characterization of sets containing Ω_γ. For $\gamma > 0$ and a function $V : \mathcal{R}^n \to \mathcal{R}$, define

$$\Omega_{\gamma,V} := \{\xi \in \mathcal{R}^n \ : \ V(\xi) \leq \gamma\}.$$

Lemma 1: [1], [2] If a continuously differentiable function V satisfies

$$V(x) > 0 \text{ for all } x \in \mathcal{R}^n\backslash\{0\} \text{ with } V(0) = 0, \text{ and} \quad (2)$$

$$\frac{\partial V}{\partial x} f(x,w) \leq w^T w \text{ for all } x \in \Omega_{\gamma,V}, w \in \mathcal{R}^{n_w}, \quad (3)$$

then $\Omega_{\gamma,V}$ contains Ω_γ. ◁

In order to compute a tight upper bound for the reachable set by choice of V, we introduce a fixed positive definite convex function p and a variable sized region $\mathcal{P}_\beta := \{\xi \in \mathcal{R}^n \ : \ p(\xi) \leq \beta\}$ and minimize β imposing the constraint $\Omega_{\gamma,V} \subseteq \mathcal{P}_\beta$. This can be written as

$$\min_{\beta>0, V\in\mathcal{V}} \beta \text{ subject to } (2),\ (3),\ \text{and } \Omega_{\gamma,V} \subseteq \mathcal{P}_\beta. \quad (4)$$

Here $\mathcal{V}$ denotes the set of candidate Lyapunov functions over which the maximum is computed.

In order to make (4) amenable to numerical optimization (specifically SOS optimization), we restrict V and p to be all polynomials of some fixed degree and use the well-known sufficient condition for polynomial non-negativity: for a polynomial $\pi \in \mathbb{R}[\xi]$ with $\xi \in \mathcal{R}^n$, if $\pi \in \Sigma[\xi]$, then π is positive semidefinite [7]. Using simple generalizations of the S-procedure (Lemma 3), we obtain sufficient conditions for the set containment constraint in (3). Specifically, let l_1 be a positive definite polynomial (typically $\epsilon x^T x$ for some small real number ϵ). Then, the constraint

$$V - l_1 \in \Sigma[x] \quad (5)$$

and $V(0) = 0$ are sufficient conditions for the constraints in (2). By Lemma 3, if $s_1 \in \Sigma[x]$ and $s_2 \in \Sigma[(x,w)]$, then

$$(\beta - p) - (\gamma - V)s_1 \in \Sigma[x] \ \text{ and} \quad (6)$$

$$-\left[(\gamma - V)s_2 + \frac{\partial V}{\partial x} f(x,w) + w^T w\right] \in \Sigma[(x,w)] \quad (7)$$

imply $\Omega_{\gamma,V} \subseteq \mathcal{P}_\beta$ and the constraint in (3), respectively. Using these sufficient conditions, an upper bound on optimal value of β in (4) can be defined as an optimization:

Proposition 1: Let β_B^* be defined as

$$\beta_B^*(\mathcal{V}_{poly}, \mathcal{S}) := \min_{V,\beta,s_1,s_2} \beta \text{ subject to } (5) - (7), \quad (8)$$

$V(0) = 0$, $V \in \mathcal{V}_{poly}$, $s_1 \in \mathcal{S}_1$, $s_2 \in \mathcal{S}_2$, and $\beta > 0$. Here, $\mathcal{V}_{poly} \subset \mathcal{V}$ and $\mathcal{S}$'s are prescribed finite-dimensional subspaces of $\mathbb{R}[x]$ and $\Sigma[x]$, respectively. Then, $\beta_B^*(\mathcal{V}_{poly}, \mathcal{S})$ is an upper bound for the optimal value of β in (4). ◁

Although β_B^* depends on $\mathcal{V}_{poly}$ and $\mathcal{S}$, it will not always be explicitly notated. The optimization problem in (8) is bilinear because of the product terms Vs_1 in (6) and Vs_2 in (7). However, the problem has more structure than a general BMI problem. If V is fixed, the problem becomes affine in s_1 and s_2 and vice versa. In section III, we will construct a convex outer bound on the set of feasible V, sample from this outer bound set to obtain candidate V's, and then solve (8) for S, holding V fixed. Construction of this outer bound on the set of feasible V relies on data provided by simulation trajectories of (1) driven by admissible disturbance signals. Next, we propose a method adapted from [11] for computing lower bounds of the reachable set. The input signals generated by this procedure will be used both in assessing the suboptimality of the upper bounds and in forming the simulation-based relaxation for the bilinear SOS problem (8).

B. Lower bound

For any positive T, it follows that

$$\max_{\substack{w\in\mathcal{L}_2[0,T] \\ \|w\|_2^2\leq\gamma}} p(x(T)) \leq \max_{\substack{w\in\mathcal{L}_2[0,\infty) \\ \|w\|_2^2\leq\gamma}} p(x(t)) \leq \beta_B^*. \quad (9)$$

The conditions for stationarity of the finite horizon maximum in (9) are the existence of signals (x, λ) and w which satisfy $\dot{x} = f(x,w)$, $\|w\|_2^2 = \gamma$, $\lambda(T) = \frac{\partial p(x(T))}{\partial x}^T$, $\dot{\lambda}(t) = -\frac{\partial f(x(t),w(t))}{\partial x}^T \lambda(t)$, and $w(t) = \mu \frac{\partial f(x(t),w(t))}{\partial w}^T \lambda(t)$ for $t \in [0,T]$, where μ is chosen such that $\|w\|_2^2 = \gamma$. Tierno et al. [11] proposed a power-like method to solve a similar maximization, and this method was adapted for computing lower bounds of the reachable set in [2]:

1) Pick $T > 0$ and w with $\|w\|_2^2 = \gamma$.
2) Compute solution $\phi(T;0,w)$ of (1).
3) Set $\lambda(T) = \frac{\partial p(x(T))}{\partial x}^T$.
4) Compute solution of $\dot{\lambda}(t) = -\frac{\partial f(x(t),w(t))}{\partial x}^T \lambda(t)$, $t \in [0,T]$.
5) Update $w(t) = \mu \frac{\partial f(x(t),w(t))}{\partial w}^T \lambda(t)$.
6) Repeat steps (2) – (5) until w converges.

In practice, step (2) of each iteration gives a valid lower bound on maximum (over $\|w\|_2^2 = \gamma$) of $p(x(T))$, independent of whether the iteration converges.

III. Simulation based relaxation for the bilinear SOS problem

The usefulness of using simulation data in local stability analysis was demonstrated in [9] where a methodology that incorporates simulation data in formal proof construction for estimating the region-of-attraction of locally stable equilibrium points of autonomous dynamical systems with polynomial vector fields. In this section, we propose a similar technique for reachability analysis for nonlinear dynamical systems with polynomial vector fields. To this end, for given $\beta > 0$ and disturbance level $\gamma > 0$, let's ask the question whether the reachable set Ω_γ is contained in the set $\mathcal{P}_\beta$. Certainly, just one disturbance signal w (with $\|w\|_2^2 \leq \gamma$) that leads to a system trajectory on which p takes on a value larger than β certifies that $\Omega_\gamma \not\subseteq \mathcal{P}_\beta$. Conversely, a large collection of input signals w with $\|w\|_2^2 \leq \gamma$ that do not drive the system out of $\mathcal{P}_\beta$ hints to the likelihood that indeed $\Omega_\gamma \subseteq \mathcal{P}_\beta$. In this latter case, let $\mathbf{W}$ be a finite collection of signals

$$\mathbf{W} := \left\{ (w,x) \;:\; \begin{array}{c} w \in \mathcal{L}_2[0,\infty), \\ \|w\|_2^2 \leq \gamma, \\ p(\phi(t;0,w)) \leq \beta,\ \forall t \geq 0 \end{array} \right\}.$$

With β and γ fixed, the set of Lyapunov functions which certify that $\Omega_\gamma \subseteq \mathcal{P}_\beta$, using conditions (5)-(7), is simply $\{V \in \mathbb{R}[x] : (5)-(7) \text{ hold for some } s_i \in \Sigma[x]\}$. Of course, this set could be empty, but it must be contained in the convex set $\{V \in \mathbb{R}[x] \;:\; (10) \text{ holds}\}$, where

$$l_1(x(t)) \leq V(x(t)), \tag{10a}$$

$$V(x(t)) \leq \gamma, \tag{10b}$$

$$\frac{\partial V(x(t))}{\partial x} f(x(t), w(t)) \leq w(t)^T w(t) \tag{10c}$$

for all $(w,x) \in \mathbf{W}$ and $t \geq 0$. Informally, these conditions simply say that, on the trajectories starting at the origin and driven by disturbance signals with $\|w\|_2^2 \leq \gamma$, any V which verifies that $\Omega_\gamma \subseteq \mathcal{P}_\beta$ using the conditions (5)-(7) must take on values between 0 and γ and its rate of change along the corresponding system trajectory cannot exceed $w(t)^T w(t)$.

A. Affine relaxation

Let $\mathcal{V}$ be linearly parameterized as $\mathcal{V} := \{V \in \mathbb{R}[x] \;:\; V(x) = \varphi(x)^T\alpha\}$, where $\alpha \in \mathcal{R}^{n_b}$ and φ is n_b-dimensional vector of polynomials in x. Given $\varphi(x)$, constraints in (10) can be viewed as constraints on $\alpha \in \mathcal{R}^{n_b}$ yielding the convex set $\{\alpha \in \mathcal{R}^{n_b} \;:\; (10) \text{ holds for } V = \varphi(x)^T\alpha\}$. For each $(w,x) \in \mathbf{W}$, let $\mathcal{T}_w$ be a finite subset of the interval $[0,\infty)$. A polytopic outer bound for this set described by finitely many constraints is $\mathcal{Y}_{sim} := \{\alpha \in \mathcal{R}^{n_b} \;:\; (11) \text{ holds}\}$, where

$$l_1(x(t)) \leq \varphi(x(t))^T\alpha, \tag{11a}$$

$$\varphi(x(t))^T\alpha \leq \gamma, \tag{11b}$$

$$\frac{\partial \varphi(x(t))^T\alpha}{\partial x} f(x(t), w(t)) \leq w(t)^T w(t) \tag{11c}$$

for all $(w,x) \in \mathbf{W}$ and $t \in \mathcal{T}_w$.

The constraint that $\frac{\partial V(x)}{\partial x} f(x,w) \leq w^T w$ be satisfied on γ sublevel set of V and for all w implies that $\frac{\partial V(x)}{\partial x} f(x,w) \leq w^T w$ holds on a neighborhood of the origin in the (x,w)-space. While it is possible to approximately impose this constraint on a sample in a small enough neighborhood of the origin, in some case (e.g. exponentially stable linearization) it is easy to analytically express as a constraint on the low order terms of the polynomial Lyapunov function. To this end, let $\lambda > 1$ and $P^T = P \succ 0$ be such that $x^T P x$ is the quadratic part of V and define $\text{Lin}(P)$ as

$$\text{Lin}(P) := \begin{bmatrix} \frac{\partial f(0,0)}{\partial x}^T P + P\frac{\partial f(0,0)}{\partial x} & P\frac{\partial f(0,0)}{\partial u} \\ \frac{\partial f(0,0)}{\partial u}^T P & -\lambda I \end{bmatrix}.$$

Then, if (3) holds, it must be

$$\text{Lin}(P) \preceq 0, \tag{12}$$

in other words, (12) is a necessary condition for (3) (consequently for (7)). Now, let $\mathcal{Y}_{lin} := \{\alpha \in \mathcal{R}^{n_b} \;:\; P = P^T \succ 0 \text{ and } \text{Lin}(P) \preceq 0\}$. It is well-known that $\mathcal{Y}_{lin}$ is convex [10]. Furthermore, define $\mathcal{Y}_{SOS} := \{\alpha \in \mathcal{R}^{n_b} \;:\; (5) \text{ holds}\}$. By [7], $\mathcal{Y}_{SOS}$ is convex. Since $\mathcal{Y}_{sim}$, $\mathcal{Y}_{lin}$ and $\mathcal{Y}_{SOS}$ are convex, $\mathcal{Y} := \mathcal{Y}_{sim} \cap \mathcal{Y}_{lin} \cap \mathcal{Y}_{SOS}$ is a convex set in $\mathcal{R}^{n_b}$. Equations (11) and (12) constitute a set of necessary conditions for (5)-(7); thus, we have $\mathcal{Y} \supseteq \mathcal{B} := \{\alpha \in \mathcal{R}^{n_b} \;:\; \exists s_1 \in \Sigma[x], s_2 \in \Sigma[(x,w)] \text{ such that } (5)-(7) \text{ hold}\}$. Since (6) and (7) are not jointly convex in V and the multipliers, $\mathcal{B}$ may not be a convex set and even may not be connected.

A point in $\mathcal{Y}$ can be computed solving an affine (feasibility) SDP with the constraints (5), (11) and (12). An arbitrary point in $\mathcal{Y}$ may or may not be in $\mathcal{B}$. However, if we generate a collection $\mathcal{A} := \{\alpha^{(k)}\}_{k=0}^{N_V-1}$ of N_V points distributed approximately uniformly in $\mathcal{Y}$, it may be that some of the points are in $\mathcal{B}$. To this end, we use the so-called "Hit-and-Run" ($H\&R$) random point generation algorithm as described in [12].

B. Algorithms

Since an appropriate (feasible and not too conservative) value of γ is not known a priori, an iterative strategy to simulate and collect trajectories is necessary. This process when coupled with the $H\&R$ algorithm constitutes the Lyapunov function candidate generation.

<u>Simulation and Lyapunov function generation ($SimLFG$) algorithm</u>: Given positive definite convex $p \in \mathbb{R}[x]$, a vector of polynomials $\varphi(x)$, positive constants β, γ, N_{SIM} (integer), N_V (integer), $\gamma_{shrink} \in (0,1)$, and empty set $\mathbf{W}$.

1) Generate N_{SIM} input signals w with $\|w\|_2^2 \leq \gamma$ and integrate (1) from the origin.
2) If all trajectories stay in $\mathcal{P}_\beta$, add the input signals and corresponding trajectories to $\mathbf{W}$ and go to step (3). Otherwise, set γ to $\gamma_{shrink}\gamma$ and go to step (1).
3) Find a feasible point for (5), (11) and (12). If (5), (11) and (12) are infeasible, set $\gamma = \gamma_{shrink}\gamma$, and go to step (1). Otherwise, go to step (4).
4) Generate N_V Lyapunov function candidates using $H\&R$ algorithm, and return γ and Lyapunov function candidates. ◁

Step (1) of *SimLFG* algorithm requires to generate input signals w with $\|w\|_2^2 \leq \gamma$. In the current implementation of this algorithm, we use randomly generated piecewise constant input signals and signals generated by the method from section II-B using the given shape factor p and additional randomly generated shape factors.

The suitability of a Lyapunov function candidate is assessed by solving the optimization problem

<u>Affine Problem</u>: Given $V \in \mathbb{R}[x]$ (from $SimLFG$ algorithm), $p \in \mathbb{R}[x]$, and β, define

$$\begin{array}{c} \gamma_L^* := \max\limits_{\gamma, s_1, s_2} \gamma \text{ subject to} \\ s_1 \in \Sigma[x], s_2 \in \Sigma[(x,w)],\ \gamma > 0, \\ (\beta - p) - (\gamma - V)s_1 \in \Sigma[x], \\ -\left[(\gamma - V)s_2 + \frac{\partial V}{\partial x} f(x,w) + w^T w\right] \in \Sigma[(x,w)]. \end{array} \tag{13}$$

Although γ_L^* depends on the allowable degree of s_1 and s_2, this is not explicitly notated. Note that for given V and fixed γ the problem in (13) is affine in the multipliers and can be solved as a linear SDP via a line search on the parameter γ.

Assuming *Affine Problem* is feasible, it is true that $\Omega_\gamma \subseteq \Omega_{\gamma,V} \subseteq \mathcal{P}_\beta$. The solution to *Affine Problem* provides a feasible point for the problem in (8). This feasible point can be further improved by solving the problem in (8) using iterative coordinate-wise affine optimization schemes, one of which is given next.

Coordinate-wise optimization (CWOpt) algorithm: Given $V \in \mathbb{R}[x]$, positive definite $l_1 \in \mathbb{R}[x]$, positive definite convex $p \in \mathbb{R}[x]$, a constant $\varepsilon_{iter} > 0$ (stopping tolerance), and maximum number of iterations N_{iter}, set $k = 0$

1) Solve *Affine Problem*.
2) Given s_1 and s_2 from step (1), solve (8) for V and γ, and set $\gamma_L^* = \gamma_B^*$.
3) If $k = N_{iter}$ or the increase in γ_L^* between successive applications of step (2) is less than ε_{iter}, return V and γ_L^*. Otherwise, set k to $k+1$ and go to step (1). $\triangleleft$

Next, we review the following upper bound refinement procedure from [2] that is applied once a Lyapunov function V and corresponding γ satisfying the conditions in (5)-(7) are computed.

Upper bound refinement from [2]: Let V and γ satisfy (5)-(7), $m \geq 0$ be an integer, define $\epsilon := \gamma/m$, and partition the set $\Omega_{\gamma,V}$ into m subregions $\Omega_{\gamma,V,k} := \{x \in \mathcal{R}^n \,:\, (k-1)\epsilon \leq V(x) \leq k\epsilon\}$ for $k = 1, \dots, m$. If

$$\frac{\partial V}{\partial x} f(x,w) \leq h_k w^T w \text{ for all } x \in \Omega_{\gamma,V,k},\ w \in \mathcal{R}^{n_w}, \quad (14)$$

holds for some $h_k > 0$, then for any $k \leq m$, the system in (1) with piecewise continuous w starting from the origin satisfies

$$\int_0^T w^T w\, dt < \epsilon \left(\frac{1}{h_1} + \cdots + \frac{1}{h_k} \right) \Rightarrow V(\phi(T;0,w)) \leq k\epsilon.$$

Note that (14) already holds with $h_k = 1$. Therefore, it may be possible to make $\epsilon(1/h_1 + \dots, 1/h_k)$ (in particular $k = m$) greater than $\gamma k/m$ (in particular γ) by minimizing h_k such that (14) holds. S-procedure and SOS programming based sufficient conditions for (14) are proposed in [2].

IV. Examples

In the following examples, $l_1(x) = 10^{-6} x^T x$, $\partial(V) = 2$, $\partial(s_1) = 0$, $\partial(s_2) = 2$ (s_2 does not have constant and linear terms), $N_{SIM} = 100$ in *Example 1* and 600 in *Examples 2* and *3*, $N_V = 1$ (in *SimLFG* algorithm), $\varepsilon_{iter} = 0.01$, $N_{iter} = 20$, and $m = 10$ (in the refinement procedure). We applied *SimLFG* and *CWOpt* algorithms for different but fixed values of β and computed corresponding values of γ. Then, using the refinement procedure, new values γ_{refine} of $\|w\|_2^2$ are computed such that $\Omega_{\gamma,V}$ contains the reachable set for disturbance signals $\|w\|_2^2 \leq \gamma_{refine}$. We also solved the problem of maximizing γ subject to (5)-(7) for fixed values of β using PENBMI. When there runs return a feasible solution, the optimal value of γ is usually equal or close to the value shown (as dots on solid curves) in following figures. But, PENBMI runs often terminate with numerical problems or infeasible results (although a solution is known to exist using the procedure proposed in this paper). For these values of β, we put circles around corresponding dots on the solid curves in the following figures.

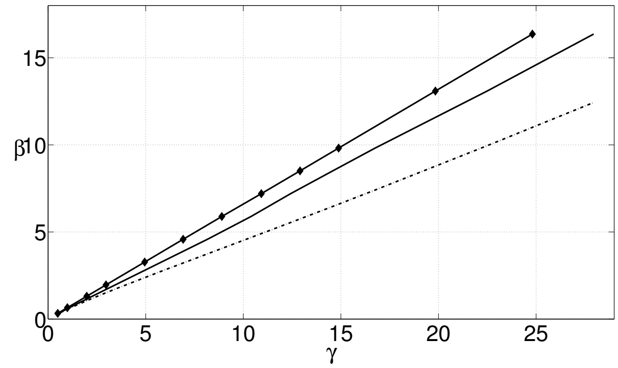

Fig. 1. Bounds on reachable sets due to disturbance w with $\|w\|_{\mathcal{L}_2}^2 \leq \gamma$ for *Example* 1 without delay (solid curve with markers: before refinement, solid curve: after refinement, and dash-dot curve: the lower bound.

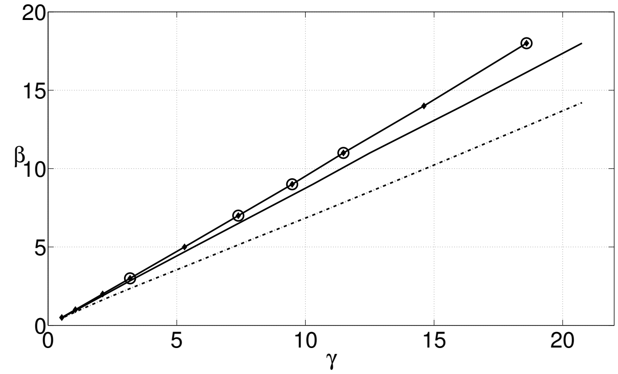

Fig. 2. Bounds on reachable sets due to disturbance w with $\|w\|_{\mathcal{L}_2}^2 \leq \gamma$ for *Example* 1 with delay (solid curve with markers: before refinement, solid curve: after refinement, dash-dot curve: the lower bound, and circles: failed PENBMI runs).

a) Example 1 : Consider the nonlinear system from [1], [2]

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_1 x_2^2, & \dot{x}_2 &= u - x_1^2 x_2 + w \\ y &= x_2, & u &= -y \end{aligned}$$

with $x_1(t), x_2(t), w(t) \in \mathcal{R}$ and $p(x) = x_1^2 + x_2^2$. Figure 1 shows the upper bounds for β versus γ before and after applying the refinement procedure and the lower bounds.

Next, we introduced $t_d = 0.4$ units of time delay at the input and modeled the delay using a (balanced) first order Pade approximation $\dot{x}_3(t) = -2/t_d x_3(t) + 2/\sqrt{t_d} y(t)$ and replaced the input by $u(t) = 2/\sqrt{t_d} x_3(t) + y(t)$, where $x_3(t) \in \mathcal{R}$, x_3 is the state of delay dynamics, and used $p(x) = x_1^2 + x_2^2 + 0.01 x_3^2$. Figure 2 shows the upper and lower bounds for β versus γ before and after applying the refinement procedure.

b) Example 2: (Pendubot dynamics) The pendubot is an underactuated two-link pendulum with torque action only

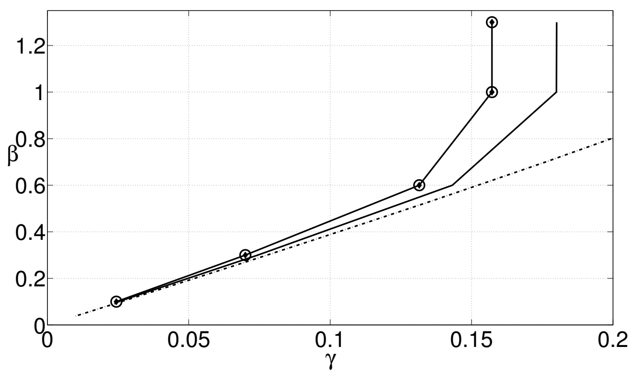

Fig. 3. Bounds on reachable sets due to disturbance w with $\|w\|^2_{\mathcal{L}_2} \leq \gamma$ for *Example* 2 (solid curve with markers: before refinement, solid curve: after refinement, dash-dot curve: the lower bound, and circles: failed PENBMI runs).

on the first link. We designed an LQR controller to balance the two-link pendulum about its upright position. Third order polynomial approximation of the closed-loop dynamics with a disturbance signal inserted to the input signal is $\dot{x}_1 = x_2$, $\dot{x}_2 = 45w + 782x_1 + 135x_2 + 689x_3 + 90x_4$, $\dot{x}_3 = x_4$ and $\dot{x}_4 = 279x_1x_3^2 + 273x_3^3 - 85w - 1425x_1 - 257x_2 - 1249x_3 - 171x_4$. Here, x_1 and x_3 are angular positions of the first link and the second link (relative to the first link) and w is the scalar disturbance. Figure 3 shows the upper and lower bounds for β versus γ before and after applying the refinement procedure.

c) Example 3 : (Controlled short period aircraft dynamics) Consider the plant dynamics

$$\begin{aligned}\dot{x}_{AC} = &\begin{bmatrix} -3 & -1.35 & -0.56 \\ 0.91 & -0.64 & -0.02 \\ 1 & 0 & 0 \end{bmatrix} x_{AC} \\ &+ \begin{bmatrix} 0.08x_1x_2 + 0.44x_2^2 + 0.01x_2x_3 + 0.22x_2^3 \\ -0.05x_2^2 + 0.11x_2x_3 - 0.05x_3^2 \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} 1.35 - 0.04x_2 \\ 0.4 \\ 0 \end{bmatrix} (u + w)\end{aligned} \tag{15}$$

$y = [x_1,\ x_3]^T$, where $x_{AC} = [x_1,\ x_2, x_3]^T$ is the plant state, and x_1, x_2 and, x_3 are the pitch rate, the angle of attack, and the pitch angle, respectively. The input u is the elevator deflection and determined by

$$\dot{x}_C = \begin{bmatrix} -0.60 & 0.09 \\ 0 & 0 \end{bmatrix} x_C + \begin{bmatrix} -0.06 & -0.02 \\ -0.75 & -0.28 \end{bmatrix} y \tag{16}$$

$u = x_4 + 2.2x_5$, where $x_C = [x_4\ \ x_5]^T$ is the controller state, and $w(t) \in \mathcal{R}$ is the input disturbance. We applied the algorithms as described above with $p(x) = x^T x$ and showed the upper and lower bounds before and after the refinement in Figure 4

V. Extensions to Local $\mathcal{L}_2 \to \mathcal{L}_2$ Gain Analysis

We now discuss a straightforward extension of simulation-aided analysis for computing upper bounds for $\mathcal{L}_2 \to \mathcal{L}_2$ IO

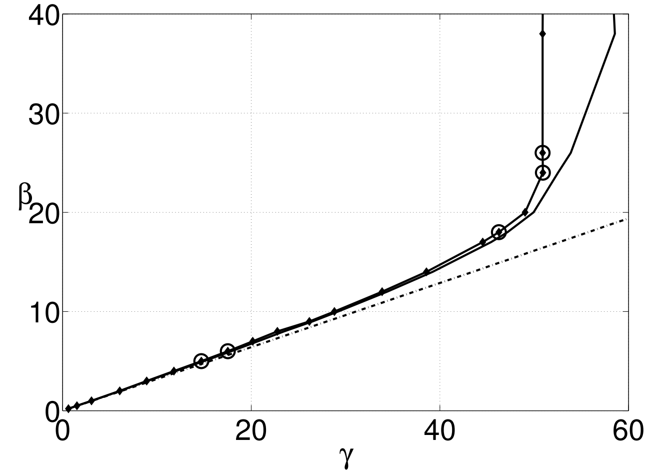

Fig. 4. Bounds on reachable sets due to disturbance w with $\|w\|^2_{\mathcal{L}_2} \leq \gamma$ for *Example* 3 without delay (solid curve with markers: before refinement, solid curve: after refinement, dash-dot curve: the lower bound, circles: failed PENBMI runs).

gains of nonlinear systems. Consider the dynamical system governed by

$$\dot{x} = f(x, w) \quad \text{and} \quad z = h(x), \tag{17}$$

where x, w, and f are as before and h is an n_z-vector of polynomials in x such that $h(0) = 0$. We use the following lemma characterizing an upper bound for the w to z induced $\mathcal{L}_2 \to \mathcal{L}_2$ gain of the system in (17) is proven in [2].

Lemma 2: If there exists a real scalar $\kappa > 0$ and a continuously differentiable function V such that

$$V(0) = 0 \text{ and } V(x) \geq 0, \tag{18}$$

$$\frac{\partial V}{\partial x} f(x, w) \leq w^T w - \frac{z^T z}{\kappa} \quad \forall x \in \Omega_{\gamma,V}, \text{ and } w \in \mathcal{R}^{n_w} \tag{19}$$

then the system in (17) starting from the origin and $\|w\|_2^2 \leq \gamma$, we have $\|z\|_2^2 \leq \kappa\gamma$. ◁

By Lemma 3, existence of $s_3 \in \Sigma[(x, w)]$ such that

$$-\left[(\gamma - V)s_3 + \frac{\partial V}{\partial x} f(x, w) - w^T w + \kappa^{-1} z^T z\right] \in \Sigma[(x, w)] \tag{20}$$

implies (19). The constraint in (20) is bilinear with the similar structure to the constraint in (7) that it becomes affine in V with fixed s_3 and vice versa. Therefore, if we generate qualified Lyapunov function candidates, we can assess these functions solving affine SDPs or further optimize by using coordinate-wise affine schemes (similar to *CWOpt* for reachability analysis). To this end, let $\kappa, \gamma > 0$ be given and define $\mathbf{G}$ be a finite collection of signals

$$\mathbf{G} := \left\{ (w, x, z) \ : \ \begin{array}{c} w \in \mathcal{L}_2[0, \infty),\ \|w\|_2^2 \leq \gamma, \\ \|h(\phi(t; 0, w))\|_2^2 \leq \gamma\kappa \end{array} \right\}.$$

Then, the conditions in (21) for $(w, x, z) \in \mathbf{G}$ and $t \geq 0$.

$$\begin{array}{c} 0 \leq V(x(t))\gamma \\ \frac{\partial V(x(t))}{\partial x} f(x(t), w(t)) \leq w(t)^T w(t) - \kappa^{-1} z(t)^T z(t) \end{array} \tag{21}$$

are necessary conditions for (20). These necessary conditions along with that on the linearized dynamics can be used to generate Lyapunov function candidates in a manner parallel to the *SimLFG* algorithm for reachability analysis.

VI. CONCLUSIONS

We analyzed reachability properties and local input/output gains of systems with polynomial vector fields. Upper bounds for the reachable set and nonlinear system gains are characterized with Lyapunov (storage) functions satisfying certain conditions. Finite dimensional polynomial parameterizations for Lyapunov functions were used. A methodology utilizing information from simulations to generate Lyapunov function candidates satisfying necessary conditions for bilinear constraints was proposed. The suitability of Lyapunov function candidates were assessed solving linear sum-of-squares optimization problems. Qualified candidates were used to compute upper bounds for the reachable set and nonlinear system gains and to initialize further coordinate-wise affine optimization. We illustrated the method on several examples from the literature.

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VIII. BACKGROUND

The following lemma is a generalization of the well known $\mathcal{S}$-procedure [10], and is a special case of the Positivstellensatz theorem [13, Theorem 4.2.2].

Lemma 3 (Generalized $\mathcal{S}$-procedure): Given $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. If there exist $\{s_k\}_{i=1}^m \in \Sigma_n$ such that $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, then

$$\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \ : \ p_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \,|\, p_0(x) \geq 0\}.$$