Static Output Stabilisation of Singular LPV Systems: \textit{LMI} Formulation

M. Chadli, M. Darouach and J. Daafouz

Abstract—This paper addresses the design of static output feedback controller for singular discrete-time LPV systems. Sufficient synthesis conditions are derived in Linear Matrix Inequalities (\textit{LMI}) formulation. To introduce more of relaxation, polyquadratic Lyapunov functions is proposed instead of quadratic method, the number of \textit{LMI} conditions is reduced and extra degrees of freedom are included. Academic example is proposed to illustrate the effectiveness of the derived results. \textbf{Keyword:} Singular systems, LPV systems, static output control, Lyapunov method, \textit{LMI}.

I. INTRODUCTION

These last years, many efforts have been carried out for singular systems [1]. Many works dealing with the control of impulsive and switching singular systems and singular LPV systems have been studied recently for the theoretical and practical point of view (see for example [2], [3], [4], [6], [10], [9]). For the switching systems the control techniques based on switching between different controllers have been applied extensively in recent years, due to their advantages in achieving stability [15], [7], [8], [10], [13] whereas gain-scheduling controllers techniques are used for LPV systems [12], [14]. For singular LPV systems, stability and stabilisation are studied using gain-scheduling controllers techniques [19] and recently using multiple state feedback controllers for the class of polytopic LPV [20]. However, if there exist a lot of works on singular linear systems (see for example [16], [17] and references therein), to our knowledge there are few studies on singular LPV systems and their corresponding control problems.

In this paper we consider the stabilisation of the class of polytopic singular LPV systems using static output feedback controller. While a single control is often used, this paper proposes to design multiple static output feedback controller obtained by interpolation of linear static output feedback control law. In fact, such controller introduces more of relaxation since a single/common gain fails to solve many control problems.

This paper studies the design of static output controllers for singular LPV systems in polytopic form. \textit{LMI} formulation and polyquadratic Lyapunov approach are used to design static output controller obtained by interpolation of multiple linear static output gain. For more of relaxation, a transformation technique of the system is proposed to reduce the number of \textit{LMI} constraints and extra degree of freedom is introduced and exploited.

The paper is organized as follows. In section 2, the considered class of a discrete-time singular LPV systems is described. In section 3, a static static output controller is introduced and the stabilisation of the considered class of singular LPV systems is studied. The main result is proposed in section 3-B where relaxations are introduced to design multiple controller gains in \textit{LMI} formulation. Example is given in section 4.

Notation. The following notations are used. $R^n$ and $R^{n \times m}$ denote, respectively, the $n$ dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript $^{-T}$ denotes matrix transposition, the notation $X \geq Y$ (respectively, $X \succ Y$) where $X$ and $Y$ are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite) and the symbol $(\ast)$ denotes the transpose elements in the symmetric positions. $I_n$ is the identity matrices with compatible dimensions and $I_N = \{1, 2, \cdots, N\}$.

II. PROBLEM STATEMENT

The considered singular LPV system is as follows

\[ E \dot{x}(t + 1) = A(\rho(t))x(t) + B(\rho(t))u(t) \]
\[ y(t) = Cx(t) \]

(1)

Two important classes of LPV systems can be distinguished; the affine LPV where the state space matrices depend affinely on $\rho(t)$ and the polytopic LPV where the parameter $\rho(t)$ varies in polytope of vertices $\rho_i$ such that $\rho(t) \in Co\{\rho_1, \rho_2, \cdots, \rho_r\}$ [12]. In the sequel only the second class is used in the following form

\[ E \dot{x}(t + 1) = \sum_{i=1}^{N} \xi_i(\rho(t))(A_i x(t) + B_i u(t)) \]
\[ y(t) = C x(t) \]

(2)

where

\[ \xi_i(\rho(t)) \geq 0, \sum_{i=1}^{N} \xi_i(\rho(t)) = 1 \]

(3)

with $x(t) \in R^n$ is the state vector, $y(t) \in R^p$ is the output vector, $u(t) \in R^m$ is the input vector, $A_i \in R^{n \times n}$, $B_i \in R^{n \times m}$ and $C \in R^{p \times n}$. The matrix $E$ may be singular with $0 \leq \text{rank}(E) = n_E < n$. 

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Before studying the static output feedback stabilisation, let us now recall some basic stability conditions for the unforced singular system (2). Consider the Lyapunov dependent parameter function of the form [20]:

\[ V(x(t), \rho(t)) = x(t)^T \mathcal{P}(\rho(t))x(t) \]  

(4)

with

\[ \mathcal{P}(\rho(t)) = \sum_{i=1}^{N} \xi_i(\rho(t)) E^T P_i E, \quad E^T P_i E \geq 0, \quad i \in I_N \]  

(5)

The difference of (4) along the solution of the unforced system of (2) is given by

\[ \Delta V = V(x(t+1), \rho(t+1)) - V(x(t), \rho(t)) = x(t+1)^T \mathcal{P}(\rho(t+1))x(t+1) - x(t)^T \mathcal{P}(\rho(t))x(t) \]  

(6)

Thus, the unforced singular system of (2) is stable if there exist nonsingular symmetric matrices \( P_i \) such that the following hold for all \( (i, j) \in I^2_N \):

\[ E^T P_i E \geq 0 \]  

(7)

\[ A_i^T P_j A_i - E^T P_i E < 0 \]  

(8)

To design control laws, introduction of extra degrees of freedom is largely used in literature (see for example [18], [15] and references therein). These techniques are very helpful for the development of an \( \mathcal{LMI} \)-based conditions for controllers design and particular for static output feedback controller. Thus the unforced singular system of (2) is stable if there exist nonsingular symmetric matrices \( P_i \), matrices \( F_i \) and \( G_i \) such that the following \( \mathcal{LMI} \) hold for all \( (i, j) \in I^2_N \):

\[ EP_i E^T \geq 0 \]

\[ \begin{pmatrix} -EP_i E^T + A_i F_i + (A_i F_i)^T -F_i^T + A_i G_j & -G^T + A_i G_j \\ P_j - (G_j + G_i^T) & \end{pmatrix} < 0 \]  

(9)

(10)

The stability conditions (10) and (8) are equivalents. However, this last formulation is very helpful for the control problem design.

In this paper our objective is to synthesis static output controller for discrete-time singular systems. Sufficient conditions for controller design will be derived based on the Lyapunov theory and \( \mathcal{LMI} \) formulation.

III. STABILISATION OF DISCRETE-TIME LPV SINGULAR SYSTEMS

A. Static output feedback controller design

Consider the following control law obtained by interpolation of linear static output controller:

\[ u(t) = \sum_{i=1}^{N} \xi_i(\rho(t)) K_i y(t) \]  

(11)

where \( K_i \in \mathbb{R}^{m \times p} \). The closed loop system is

\[ E x(t+1) = \sum_{i=1}^{N} \sum_{j=1}^{N} \xi_i(\rho(t)) \xi_j(\rho(t)) A_{ij} x(t) \]  

(12)

with

\[ A_{ij} = A_i + B_i K_j C \]  

(13)

In the sequel, we assume

Assumption 1: The matrix \( C \) is full row rank.

To derive stability conditions of (12) it is possible to substitute \( A_i \) by \( A_i + B_i K_j C \) in conditions (10). However the obtained conditions are Bilinear Matrix Inequalities in \( K_i, F_i \) and \( G_i \). \( \mathcal{LMI} \) conditions can be obtained by using the transformation used in [21] and choosing \( F_i = G_i = G \).

**Theorem 1:** The singular system (12) is stable if there exist nonsingular matrices \( G, P_i, M \) and \( N_j \) such that the following \( \mathcal{LMI} \) hold for all \( (i, j, k) \in I^3_N \):

\[ EP_i E^T \geq 0 \]

\[ \begin{pmatrix} -EP_i E^T + \Phi_{ij} + \Phi_{ij}^T & -G^T + \Phi_{ij} \\ \Phi_{ij} & P_k - (G + G^T) \end{pmatrix} < 0 \]  

(14)

(15)

\[ CG = MC \]  

(16)

with

\[ \Phi_{ij} = A_i G + B_i N_j C \]  

(17)

The controller gains are defined by:

\[ K_i = N_i M^{-1} \]  

(18)

Proof: Multiplying (15) by \( \sum_{i=1}^{N} \sum_{j=1}^{N} \xi_i(\rho(t)) \xi_j(\rho(t)) \) and according to (18) and (16), we obtain

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \xi_i(\rho(t)) \xi_j(\rho(t)) \]

\[ \begin{pmatrix} -EP_i E^T + A_i G + (A_i G)^T & -G^T + A_i G \\ \Phi_{ij} & P_k - (G + G^T) \end{pmatrix} < 0 \]  

(19)

Which is only stability conditions (10) by substituting \( A_i \) by \( A_i + B_i K_j C \) and \( F_i = G_i = G \).

In the following subsection, we introduce multiple matrices \( G_i \) instead of single matrix \( G \) which leads to more relaxation.
B. Main result

To give more relaxation, we propose to derive LMI conditions, to introduce different matrices $G_i$, and to reduce the number of LMI constraints. For this, we propose to write the model (2) and the controller (11) as follows

$$Ex(t + 1) = \sum_{i=1}^{N} \xi_i(\rho(t))A_i x(t) \quad (20)$$

Where

$$x(t) = (x(t)^T, u(t)^T)^T, \quad A_i = \begin{pmatrix} A_i & B_i \\ K_i C & -I \end{pmatrix} \quad (21)$$

$$E = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \quad (22)$$

Thus, to design static output controller it suffices to substitute $A_i$ by $A_i$ in conditions (10). However the obtained conditions are BMI in $K_i, F_i$ and $G_i$ even if we choose $F_i = G_i$.

Without lost of generality, we will further assume that $C$ has a specific structure $C = (I, 0)$ where $I \in R^{p \times p}$ is the identity matrix and $0 \in R^{p \times (n-p)}$ is the null matrix. Note that this is not a restricting condition. Indeed, if $C$ is a full rank matrix (assumption 1) then there always exist a transformation matrix that transforms $C$ into the above form.

In the following we propose LMI conditions obtained by considering the output matrix structure $C = (I, 0)$ and also specific structure of matrices $F_i = G_i$.

**Theorem 2**: The singular system (20) is stable if there exist nonsingular symmetric matrices $P_{1i}, P_{3i}$ and matrices $P_{2i}, G_{2i}, G_{3i}$ and $G_{1i}$ of the form

$$G_{1i} = \begin{pmatrix} G_{11i} & 0 \\ G_{12i} & G_{13i} \end{pmatrix} \quad (23)$$

such that the following LMI hold for all $(i, j) \in P_{N}$.

$$EP_i E^T \geq 0 \quad (24)$$

$$\begin{pmatrix} \Sigma_i & B_i G_{3i} + N_i^T - G_{3i}^T \\ N_i^T - G_{3i}^T & (A_i G_{1i} + B_i G_{2i})^T + B_i^T G_{3i} \end{pmatrix} \begin{pmatrix} -G_{2i}^T \\ +B_i^T \end{pmatrix} \begin{pmatrix} \Sigma_i & B_i G_{3i} + N_i^T - G_{3i}^T \\ N_i^T - G_{3i}^T & (A_i G_{1i} + B_i G_{2i})^T + B_i^T G_{3i} \end{pmatrix} \begin{pmatrix} -G_{2i}^T \\ +B_i^T \end{pmatrix} = 0 \quad (25)$$

with

$$\Sigma_i = -EP_i E^T + A_i G_{1i} + B_i G_{2i} + (A_i G_{1i} + B_i G_{2i})^T \quad (26)$$

$$N_i = (N_i, 0) \quad (27)$$

The controller gains are defined by:

$$K_i = N_i G_{1i}^{-1} \quad (28)$$

**Proof**: Substituting $A_i$ by $A_i$ in conditions (10) with $F_i = G_i$ we get

$$EP_i E^T \geq 0 \quad (29)$$

$$\begin{pmatrix} -EP_i E^T + A_i G_i + (A_i G_i)^T & -G_i^T + A_i G_i \\ P_2 - (G_i + G_i^T) \end{pmatrix} < 0 \quad (30)$$

The obtained conditions (30) are BMI in $K_i$ and $G_i$. To derive LMI conditions we propose to chose matrix $G_i$ with the following structure

$$G_i = \begin{pmatrix} G_{1i} & 0 \\ G_{2i} & G_{3i} \end{pmatrix} \quad (31)$$

and $G_{1i}$ has the structure (23). Thus

$$A_i G_i = \begin{pmatrix} A_i G_{1i} + B_i G_{2i} \\ K_i C G_{1i} - G_{2i} \\ -G_{3i} \end{pmatrix} \quad (32)$$

Tacking account the structure of the matrices $G_{1i}$ (23) and $C$ and condition (28), we get

$$A_i G_i = \begin{pmatrix} A_i G_{1i} + B_i G_{2i} \\ B_i G_{3i} \end{pmatrix} \quad (33)$$

Writing the symmetric matrices $P_i$ as follows

$$P_i = \begin{pmatrix} P_{1i} & P_{2i} \\ P_{2i}^T & P_{3i} \end{pmatrix} \quad (34)$$

and according to (33) we obtain (25) from (30). Conditions (24) are obtained directly from (29).

The result of theorem 2 leads to more relaxation compared with the result of the theorem 1 by reducing the number of constraint to $(N^2 + N)$ LMI instead of $(N^3 + N)$ and by introducing different matrices $G_i$.

**IV. Numerical Example**

Consider a numerical example with the following data:

$$A_1 = \begin{pmatrix} 2.018 & 1.326 & -0.08691 & -0.1644 \\ 2.277 & 2.018 & -0.1359 & -0.2908 \\ 0.4259 & 0.1666 & 0.5714 & 0.1732 \\ 0.6659 & 0.4259 & -0.4822 & -0.152 \end{pmatrix} \quad (35)$$

$$A_2 = \begin{pmatrix} 1 & 1 & -0.06988 & -0.1351 \\ 0 & 1 & -0.07577 & -0.1835 \\ 0 & 0 & 0.5807 & 0.1894 \\ 0 & 0 & -0.4546 & -0.1013 \end{pmatrix} \quad (36)$$
$B_1 = \begin{pmatrix} 0.6047 \\ 1.644 \\ -1.709 \\ -1.732 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0.5422 \\ 1.351 \\ -1.747 \\ -1.894 \end{pmatrix}$ (37)

$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$

The singular matrix, $E$, are given by:

$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (38)

Solving $\mathcal{LMI}$ (24)-(25), we get feasible problem:

$G_{11} = \begin{pmatrix} 0.0058 & -0.0042 & 0 & 0 \\ -0.0043 & 0.0056 & 0 & 0 \\ -0.0027 & -0.0018 & 0.0786 & 0.0020 \\ 0.0012 & -0.0026 & 0.0490 & 0.1963 \end{pmatrix}$ (40)

$G_{12} = \begin{pmatrix} 0.0071 & -0.0019 & 0 & 0 \\ -0.0029 & 0.0053 & 0 & 0 \\ 0.0027 & -0.0028 & 0.0790 & 0.0030 \\ 0.0188 & -0.0046 & 0.0470 & 0.1924 \end{pmatrix}$ (41)

which give the controller gains

$K_1 = \begin{pmatrix} -2.3661 \\ -2.4425 \end{pmatrix}$ (42)

$K_2 = \begin{pmatrix} -0.3311 \\ -0.8934 \end{pmatrix}$ (43)

We note that the synthesis conditions of theorem 1 fail to prove the stabilisation of the given example.

V. CONCLUSION

This paper deals with the output stabilisation of class of singular LPV systems. Thus, static output stabilisation controller is studied for polytopic singular LPV systems. Using polyquadratic Lyapunov functions, sufficient conditions to design multiple static output controllers are developed in $\mathcal{LMI}$ terms. For more of relaxation, a technique to reduce the number of $\mathcal{LMI}$ constraints is proposed and different extra degrees of freedom are introduced. Numerical example is given to illustrate the benefit of derived results.

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