Robust Constraint Satisfaction for Continuous-Time Nonlinear Systems

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Abstract—A method to design feedback control laws that guarantee state-constraint satisfaction for a class of uncertain nonlinear systems is proposed. A recursive procedure is used to construct robustly controlled invariant sets for input affine nonlinear systems with bounded disturbances or parametric uncertainties. Constraint satisfaction is achieved by a modification of the control input on the boundaries of the constructed sets. The results are illustrated on a design example.

I. INTRODUCTION

Every physical control system must deal with some limits or constraints in the operation space. Such limits can arise from physical constraints imposed on the system by its surroundings. They can also arise from safety constraints. In process control, for example, it is often economically desirable to operate close to limits of the feasible region. Unfortunately, such operating practices are not sustainable due to the potential damage and safety problems associated with the violations of the operation bounds. To avoid such problems, it is imperative that one incorporates all process constraints, and especially safety constraints, as an essential performance requirement in the design of a control system.

While the problem of saturated actuators has already gained much interest in recent years [5], structural results for state-constrained problems remain rare. A good insight in the problem is given in [12] and the references therein where state-constrained linear systems are considered. In the study of state-constrained nonlinear systems, different approaches have been proposed over the last few years. Basically two types of approaches have been considered. The first category consists of controller design methods in which state constraints are taken into account explicitly during the controller design procedure. Such approaches result in control schemes that guarantee closed-loop stability, constraint satisfaction and other specifications. In [10], an approach based on backstepping is proposed. A second category of strategies seeks to modify the command signal on-line to prevent constraint violations. A very successful example of such methods is the well-known Nonlinear Model Predictive Control [3], where optimization methods are used to determine a control signal that avoids constraint violation. Other approaches include so-called override schemes [7], [14] and reference governors [1], [6]. Of particular interest in this study is the invariance control approach proposed in [16]. In this method, an invariance control law is designed a priori and a subset of the state space that can be rendered invariant by this control law is defined. Using this method, it is possible to apply a nominal controller inside the invariant set, while switching to the invariance controller on the boundaries guarantees constraint satisfaction. The approach presented in the following is based on the ideas of invariance control. The advantage of this approach is that the controller design method is greatly simplified by considering the stabilization and the constraint satisfaction problem separately. The key point of the proposed approach is the construction of a robustly controlled invariant set. An interesting feature is that finally only a first order constraint has to be considered to guarantee the invariance of the resulting set. This leads directly to conditions on the control input that can be integrated in the controller design. The proposed design procedure is flexible and can handle bounded uncertainties in the system.

The remainder of the paper is organized as follows. In section II, the relevant class of systems is identified and a general problem formulation is given. The main result of this work is stated in section III where a method to construct robustly controlled invariant sets is presented. The controller synthesis is explained in section IV. In section V, the proposed design procedure is illustrated on a simulation example. A summary and an outlook on future work are given in section VI.

II. PROBLEM FORMULATION

Throughout this paper we consider the following class of uncertain input affine control systems

$$\dot{x} = f(x) + g(x)u + q(x, \theta(t))$$

(1)

where $x \in \mathbb{R}^n$ are the state variables and $u \in \mathbb{R}$ is the input variable. The vector fields $f(x)$ and $g(x)$ are assumed to be sufficiently smooth on a domain $\mathcal{D} \subset \mathbb{R}^n$. The uncertainty vector $\theta(t) = [\theta_1, \ldots, \theta_p] \in \Omega \subset \mathbb{R}^p$ belongs to a known compact set $\Omega \subset \mathbb{R}^p$. Furthermore the uncertainty is assumed to be linearly parametrized, i.e.,

$$q(x, \theta(t)) = \sum_{i=1}^{p} \rho_i \theta_i(t)$$

State constraints of the system are given by a set of output functions

$$y_i = h_i(x) \leq 0, \quad i = 1, \ldots, m$$

(2)

where $h_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ are sufficiently smooth functions. Each output is assumed to have a well defined relative degree $\rho_i$, that is

$$L_{\rho_i}h_i(x) = \cdots = L_{\rho_i}^{\rho_i-2}h_i(x) = 0, \quad L_{\rho_i}^{\rho_i-1}h_i(x) \neq 0$$

for all $x \in \mathcal{D}$, where $L_{\rho_i}h_i(x)$ denotes the standard Lie (directional) derivative of $h_i(x)$ along the vector field $g(x)$. The disturbances are assumed to satisfy the triangularity condition [9, Theorem 3.1.1]. Therefore the following assumption is made.
Assumption 1. For each output function $h_i(x)$, there exists a change of coordinates

$$ T(x) = \begin{bmatrix} \Phi_1(x) \\ \vdots \\ \Phi_{n-\rho}(x) \\ h_i(x) \\ \vdots \\ L_f^{\rho-1}h_i(x) \end{bmatrix} =: \begin{bmatrix} \xi \\ z \end{bmatrix}, \quad (3) $$

where $\Phi_1$ to $\Phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on the domain $D \subset \mathbb{R}^n$, that transforms the system into the normal form

$$ \dot{\xi} = \Phi(\xi, z, \theta(t)) \quad (4) $$

$$ \dot{z}_j = z_{j+1} + \sum_{l=1}^{p_j} \phi_{ji}(z_1, \ldots, z_j)\theta_l(t), \quad 1 \leq j \leq \rho_i - 1 \quad (5) $$

$$ y_i = z_1. $$

Note that no further assumption on the $\xi$-dynamics needs to be made at this point. It will become obvious in the following that for the purpose of constraint satisfaction no other requirement, i.e. no minimum-phase assumption is needed since only the $z$-dynamics will be considered for the construction of robustly controlled invariant sets. Thus, a transformation $T(x)$ can be found in a straightforward manner for many control problems. The most restrictive assumption concerns the existence of the $b(z, \xi)$ term, that is the well defined relative degree. This term has to exist and be bounded in specific regions of the state space yet to be determined.

A feedback controller for the system (1) can be designed using conventional design techniques while ignoring the constraints. Well known robust controller design techniques can be found in [4] or [9]. In the following, this robust feedback controller is termed nominal controller and is denoted by $u_{nom}$. It is assumed that the nominal controller stabilizes the system (1) with respect to an equilibrium $x_d$ or a time-varying trajectory $x_d(t)$. Furthermore, it is assumed that a suitable Lyapunov function $V(e), e = x - x_d$, for the closed-loop system is known.

### III. Design Procedure

In this section, we state the main contribution of this study. A design method, based on the idea of backstepping, is proposed for the construction of robustly invariant sets that are constraint admissible. First, a well known result is reviewed for the sake of clarity.

**Definition** [[2], Def. 2.3] The set $\mathcal{S} \subset \mathbb{R}^n$ is said to be robustly controlled invariant for the system

$$ \dot{x} = f(x(t), u(t), \theta(t)), \quad \theta(t) \in \Omega, $$

if there exists a feedback control law $u(t) = k(x(t))$ which assures the existence and uniqueness of the solution on $\mathbb{R}^+$ and is s.t. for all $x(0) \in \mathcal{S}$ the solution $x(t) \in \mathcal{S}$ for $t > 0$.

Clearly not every subset of the state space is robustly controlled invariant. In the following, a procedure is proposed to design suitable sets and to derive corresponding conditions on the control inputs for systems of the form (1),(2). The basic idea is to restrict the set of safe initial conditions step by step by constructing new constraints, whereas satisfaction of each constraint guarantees satisfaction of the previous constraint. The procedure is repeated until one obtains a constraint whose time derivative depends explicitly on the control input. Starting inside the constructed set, the control input can be used to guarantee constraint satisfaction for the last and therefore for all previous constraints. The construction algorithm is first introduced for a single constraint. However, it is possible to repeat the algorithm for each constraint $h_i(x)$ and under some additional conditions on the constraint relation it is possible to guarantee satisfaction for the whole set of constraints (2). This topic will be discussed later on. Under the previous assumptions, the main result can be stated as follows.

**Theorem 1:** Consider the uncertain nonlinear system (1) subject to one of the constraints (2). Suppose furthermore that assumption 1 holds, then a constraint admissible safe set $\mathcal{S}$ can be constructed and there exists a control law $u_{inv}$ that guarantees robust invariance of this set.

**Remark** The proof of this result is constructive and introduces an algorithm for the construction of robustly controlled invariant sets. For simplicity, $\rho_i$ will be denoted by $p$ and the $\xi$-dynamics is neglected in the following. This does not impose a problem as clarified later.

**Proof.** By assumption 1, there exists a change of coordinates that transforms the system into the normal form

$$ \dot{z}_j = z_{j+1} + \sum_{i=1}^{p} \phi_{ji}(z_1, \ldots, z_j)\theta_i(t), \quad 1 \leq j \leq p - 1 $$

$$ \dot{z}_p = a(z) + b(z)u + \sum_{i=1}^{p} \phi_p(z_1, \ldots, z_p)\theta_i(t), $$

where the constraint is now imposed on the coordinate $z_1$, i.e. $y = z_1 \leq 0$. To construct a set of safe initial conditions, one starts with the first subsystem

$$ \dot{z}_1 = z_2 + \sum_{i=1}^{p} \phi_{1i}(z_1)\theta_i(t). \quad (6) $$

Since the uncertainties belong to a known and compact set $\Omega$ one can find a bound such that $|\theta_i(t)| \leq \bar{\theta}_i$. To guarantee safety in the system, one has to consider the worst case
disturbance and find an upper bound for the right hand side of (6). This can be done in different ways, but it is advantageous to find a smoothly differentiable bounding function. In general, it is possible to find some smooth functions $\Xi$ and $\Psi$ that guarantee

$$
\sum_{i=1}^{p} \phi_{l}(z_{1})\theta_{i}(t) \leq \Xi_{l}(z_{1}) + \Psi_{l}(\bar{\theta})
$$

(7)

for $t > 0$ in the subset of interest in the state-space, i.e. where $z_{1} \leq 0$. Now, the right hand side of (6) can be overestimated and rewritten as

$$
\dot{z}_{1} = z_{2} + \sum_{i=1}^{p} \phi_{l}(z_{1})\theta_{i}(t) \leq z_{2} + \Xi_{l}(z_{1}) + \Psi_{l}(\bar{\theta})
$$

$$
= -k_{1}z_{1} + \left[ z_{2} + k_{1}z_{1} + \Xi_{l}(z_{1}) + \Psi_{l}(\bar{\theta}) \right],
$$

where $k_{1}$ is a positive constant. The new variable

$$
v_{1} := z_{2} + k_{1}z_{1} + \Xi_{l}(z_{1}) + \Psi_{l}(\bar{\theta})
$$

can be introduced and one can rewrite the upper system as

$$
\dot{z}_{1} \leq -k_{1}z_{1} + v_{1}.
$$

The comparison principle [8, Lemma 3.4] can be used to provide an upper bound for the trajectories of the differential inequality above by those of the comparison system $\dot{z}_{1} = -k_{1}\dot{z}_{1} + v_{1}$ (i.e. whenever it holds that $z_{1}(t_{0}) \leq z_{1}(t_{0})$ then $z_{1}(t) \leq \dot{z}_{1}(t)$, $\forall t > 0$). The main objective here is to guarantee that $z_{1}(t) \leq 0$ which implies that $z_{1}(0) \leq 0$. Considering now the comparison system, one can use the Lyapunov function $V_{1} = \frac{1}{2}z_{1}^{2}$ that satisfies $\dot{V} \leq -k_{1}z_{1}^{2} + \dot{z}_{1}v_{1}$ to show that if $\dot{z}_{1}(0) \leq 0$ and $v_{1}(t) \leq 0$ then it is guaranteed that $\dot{z}_{1}(t) \leq 0$ and therefore that $z_{1}(t) \leq 0$. The second requirement for constraint satisfaction is that $\sup_{l} v_{l}(t) \leq 0$ which yields a new constraint given by

$$
v_{l} := z_{l+1} + \Xi_{l+1}(z_{1}, \ldots, z_{l}) + \Psi_{l}(\bar{\theta}), \quad l = 1 \ldots n - 2.
$$

Taking their time derivatives, one obtains the following expressions

$$
\dot{v}_{l} = z_{l+2} + \Xi_{l+1}'(z_{1}, \ldots, z_{l+1}) + \sum_{i=1}^{p} \phi_{l+1,i}\theta_{i}(t),
$$

where all terms without uncertainty resulting from the derivatives of $k_{l}v_{l-1}$ and $\Xi_{i}$ are collected in the new expression $\Xi_{l+1}$. Using the inequality conditions, this derivative can be bounded by a smoothly differentiable function, i.e.

$$
\dot{v}_{l} \leq z_{l+2} + \Xi_{l+1}(z_{1}, \ldots, z_{l+1}) + \Psi_{l+1}(\bar{\theta})
$$

with $\Xi_{l+1}$ resulting from $\Xi_{l+1}$ and the bounding of $\sum_{i=1}^{p} \phi_{l+1,i}\theta_{i}(t)$. Again, applying the same reasoning as above, the next constraint can be defined as

$$
v_{l+1} = z_{l+2} + k_{l+1}v_{l} + \Xi_{l+1}(z_{1}, \ldots, z_{l+1}) + \Psi_{l+1}(\bar{\theta}) \leq 0
$$

The last constraint that has to be defined is the constraint $v_{p-1}$. As above, this constraint takes the form

$$
v_{p-1} := z_{p} + \Xi_{p-1}(z_{1}, \ldots, z_{p-1}) + \Psi_{p-1}(\bar{\theta}),
$$

but its derivative can be controlled directly by the input, i.e.

$$
\dot{v}_{p-1} \leq a(z) + b(z)u + \Xi_{p}(z_{1}, \ldots, z_{p}) + \Psi_{p}(\bar{\theta}).
$$

By the input-output linearity of the nominal system, it follows that $b(z) \neq 0$. Therefore the control input $u$ can be used to guarantee $\dot{v}_{p-1} \leq 0$ using an appropriate control law $u_{ins}(x)$. More precisely, any control law that guarantees

$$
a(z) + b(z)u + \Xi_{p}(z_{1}, \ldots, z_{n}) + \Psi_{p}(\bar{\theta}) \leq 0,
$$

(8)

guarantees constraint satisfaction. There are different ways to construct such a controller. One possible choice will be proposed later in this paper. The constraint admissible set of safe initial conditions is given by

$$
S = \{ x \in \mathbb{R}^{n} | z_{i} \leq 0, v_{i} \leq 0, i = 1, \ldots, p - 1 \}
$$

(9)

This completes the proof.

At this point, the results can be shortly summarized as follows. Starting with the given state constraint $y = h(x) \leq 0$, $\rho - 1$ new constraints $v_{1}(x), \ldots, v_{p-1}$ are designed recursively. The satisfaction of each constraint $v_{i} \leq 0$ guarantees satisfaction of the previous constraint $v_{i-1} \leq 0$ directly. Therefore, if the initial state is such that $z_{1}(x(0)) \leq 0, v_{i}(x(0)) \leq 0, i = 1 \ldots p - 1$, it is sufficient to verify satisfaction of the last constraint $v_{p-1}(x(t)) \leq 0, \forall t > 0$, which can be achieved with the control input. The function $v_{p-1}(x)$ is negative inside the safe set that should be rendered invariant, it is zero on its boundary and positive in the unsafe region. Due to the special importance of this last constraint the function $B(x) := v_{p-1}$ referred to as a barrier certificate [11], [15] for the constraint admissible set $S$. Control inputs that guarantee the invariance of the set $S$ are described by the condition (8). It remains to select a suitable control input in accordance to this condition.

The advantage of the backstepping-like procedure proposed here is that, like the original backstepping methods, it allows one to tailor the approach to address many problem dependent issues often encountered in the study of nonlinear control systems. Note that it is not necessary to transform each constraint’s dynamics to $\dot{v}_{i} \leq -k_{i}v_{i} + v_{i+1}$. Alternatively, one could easily assign another function with similar properties e.g. $\dot{v}_{i} \leq -\alpha(v_{i}) + v_{i+1}$, where $\alpha$ is a Lipschitz function in $(0, \infty)$. For most cases however, the proposed approach yields good results.

Remark When the system (1) is not subject to disturbances, all inequalities become equalities and the system takes the
form

\[\begin{align*}
\dot{z}_1 &= -k_1 z_1 + v_1 \\
\dot{v}_1 &= -k_2 v_1 + v_2 \\
&\vdots \\
\dot{v}_{\rho-1} &= a(z) + b(z)u.
\end{align*}\]

If furthermore the remaining \(\xi\)-dynamics are input-to-state stable, then the controller 
\(u = (-a(z) - v_{\rho-1})/b(z)\) stabilizes the system on the original constraint 
\(y = h(x) = 0\).

If the system (1) is subject to more than one constraint, the
proposed algorithm has to be repeated for each constraint
\(h_i(x)\) to construct the different sets \(S_i\). The overall safe
set is then given by \(S = \bigcap S_i\) and is characterized by
the barrier certificate \(B(x) = \max_i B_i(x)\). In this case it
might happen that two or more constraints are active at
the same time. It is necessary to guarantee that there is no
contradiction in the constraint satisfaction. In general, the
question is whether there exists at least one control input
\(u_{inv}\) that decreases simultaneously the barrier certificates
of all active constraints. This very general question of existence
is hard to answer and therefore a simplifying assumption on
the intersecting constraints is made here.

**Assumption 2** For constraint satisfaction of multiple
intersecting constraints, it is assumed that the \(L_y B_i(x)\)-terms have
the same sign at the intersection points. That is,

\[\text{sign} L_y B_i(x) = \text{sign} L_y B_j(x), \quad \text{if } B_{i,j}(x) = 0, \quad x \in S.\quad (10)\]

If this requirement is not met, it might still be possible to
find an inner approximation of one constraint or to introduce
an additional constraint such that the condition is satisfied.
The reasoning for this additional assumption is the following.
When the given condition is satisfied, both invariance control
laws have the same sign and therefore \(u_{inv} = u_{inv,1} + u_{inv,j}\)
renders \(B_i(x)\) as well as \(B_j(x)\) negative. However, even if
the given condition on the constraint relation seems to be
satisfied quite often, situations where the condition is not
met require clearly further investigation.

**IV. Controller Synthesis**

In this section, some basic concepts associated with the
controller design procedure and the controller implementation
are presented. The basic premise of the controller design
and implementation is to use the nominal controller inside
the safe set and to design a complementary controller that
renders the safe set robustly positively invariant. The nominal
controller \(u_{nom}\) is assumed to be known. The complementary
controller implementation seeks to minimize its effects on
the nominal controller. The strategy is to use the function \(B(x)\)
as a decision variable for the activation of the complementary
controller.

The first step is to design controllers, \(u_{inv,i}\) that guarantee
the invariance of the sets \(S_i\). A good choice, derived from
(8), could be the following

\[
u_{inv,i} = \begin{cases} 
  u_{nom, i} , & \text{if } u_{nom, i} \text{ sat. (8)} \\
  b(z)^{-1}(-a(z) - \Xi_0 - \Psi_{\rho}(\bar{\theta})), & \text{o.w.}
\end{cases}
\]

This control law \(u_{inv,i}\) guarantees that

\[
\dot{B}_i(x) = L_f B_i(x) + L_y B_i(x) u_{inv,i}(x) + L_p B_i(x) \leq 0,
\]

for all disturbances \(\theta \in \Omega\). Note that the function \(u_{inv,i}\) is
continuous since there exists a unique \(u\) that makes the condi-
tion (8) hold with an equality sign. To avoid discontinuities
in the control law, the switching procedure is implemented
using a sigmoidal switching surface as proposed in [15]. The
applied control input is defined by

\[
u = k(x) = \prod_{i=1}^{m} (1 - \sigma_i(x)) \cdot u_{nom} + \sum_{i=1}^{m} \sigma_i(x) u_{inv,i} \quad (11)
\]

with

\[
\sigma_i(x) = \begin{cases} 
  1, & -B_i(x) < 0 \\
  1 - 2 \left( \frac{B_i(x)}{\epsilon} \right)^3 - 3 \left( \frac{B_i(x)}{\epsilon} \right)^2, & 0 \leq -B_i(x) < \epsilon \\
  0, & -B_i(x) > \epsilon
\end{cases}
\]

for some small constant \(\epsilon > 0\). The control law \(u\) acts like a
feed-through term for the nominal controller inside the set
\(S\). If the systems’ states come close to a given constraint,
the proposed controller structure guarantees that the corre-
sponding invariance controller is activated, ensuring that
\(\dot{B} \leq 0\). Hence the switching controller affects the
nominal controller as little as possible while guaranteeing
constraint satisfaction for the system. As mentioned in
the previous section, the sum of two invariance control laws
 guarantees constraint satisfaction for both constraints. The
proposed structure is a suitable choice for a continuous
controller that guarantees constraint satisfaction.

A crucial problem in systems with switching controls is the
question of stability. One way to ensure complete integration
of the constraints into a Lyapunov based control approach is
to further impose the restriction that

\[
\text{sign} \{ L_y V(x) \} = \text{sign} \{ L_y B_i(x) \}, \quad (12)
\]

whenever \(-\epsilon \leq B(x) \leq 0\). In this case the control actions
of both controllers act in the same direction. More precisely,
the invariance controller decreases the Lyapunov function
and stability is guaranteed since \(V(x)\) can be seen as a
common Lyapunov function for the switching control law.
This case is very convenient, since stabilization of the
desired equilibrium is possible without violation of the
constraint. This property may not be satisfied everywhere
on the constraint or at all times, e.g. it may be violated
when a reference signal to be tracked lays outside of the set
\(S\). However, many situations may arise where the geometry
of the constraint \(vis-a-vis\) the system dynamics is such that
the sign-condition is violated (even in situations where the
reference point lies inside the set \(S\)). Such cases remain
highly problematic because they single out areas on the
boundary of the set \(S\) where constraint satisfaction and
minimization of the Lyapunov function are in contradiction.
Since this situation can lead to convergence of the closed-
loop system on the boundary of \(S\), it is of interest to change
the overall control objective to stabilization on or close to the
constraint. Note that, as mentioned in the previous section, this is always possible for minimum phase constraints and undisturbed systems. In this case stability can be concluded using simple dwell-time considerations. Unfortunately, the quest for Lyapunov functions of the switch controller remains an open problem at this and is beyond the scope of the current study. It will be considered further in future work.

V. SIMULATION EXAMPLE

In this section, the design procedure is illustrated on an Active Magnetic Bearing (AMB). An AMB is an apparatus that is used to bear a rotating mass between two electromagnets. It is advantageous for various reasons. It avoids friction losses due to the lack of contact between the rotor and the stator. It also provides active disturbance rejection. A simplified one-dimensional AMB system, shown in figure 1 is considered in the following. A nominal model for the system is proposed in [13]. In this study, a time-varying disturbance is added to the nominal system yielding the following uncertain system representation

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= cx_3 + x_3|x_3| + \theta x_1 \sin(bt) \\
\dot{x}_3 &= u,
\end{align*} \]  

(13)

where \( x_1 \) represents the position and \( x_2 \) the velocity of the mass. The third state \( x_3 \) is proportional to the magnetic flux. The system parameter \( \epsilon \geq 0 \) is in general smaller than one. In the following it is assumed that \( \epsilon = 0.1 \). The voltage \( V_1 = V \) and \( V_2 = -V \) is used as the control input \( u \). A stabilizing controller for the nominal system is proposed in [13], i.e. \( u_{nom} = -1.7538x_1 - 6.6957x_2 - 2.5883x_3 - 3.1582x_3|x_3| \). The additional uncertainty influences the acceleration, i.e. it can be seen as an external force. In the disturbance term, the constant \( \theta \) is known, but \( b \) is unknown. To avoid collisions with the magnet, the position \( x_1 \) has to be constrained such that \( |x_1| \leq a \). The nominal controller cannot stabilize the disturbed problem adequately. Nevertheless one needs to guarantee that the mass does not come too close to the magnets. This should be done via a robust invariant control.

The design starts with the upper constraint \( x_1 < a \). For the nominal system and with respect to the fictive output \( y = x_1 - a \) one can define a coordinate transformation \( z_1 := x_1 - a \), \( z_2 := x_2 \), \( z_3 := \epsilon x_3 + x_3|x_3| \). In the new coordinates the system can be written as

\[ \begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 + \theta(z_1 + a) \sin bt \\
\dot{z}_3 &= (\epsilon + 2|x_3|)u.
\end{align*} \]  

(14)

Note that the disturbance does not satisfy the matching condition but the triangularity condition is met. The backstepping procedure can now be applied to design a robustly invariant set and the corresponding controller. One starts with the \( z_1 \) subsystem and rewrites it as

\[ \dot{z}_1 = -k_1z_1 + [z_2 + k_1z_1] = -k_1z_1 + v_1. \]

The second constraint is now given by \( v_1 := z_2 + k_1z_1 \leq 0 \). The dynamics of the constraint is given by

\[ \dot{v}_1 = k_1z_2 + \theta(z_1 + a) \sin bt + z_3. \]

Therefore a third constraint can be defined by considering

\[ \begin{align*}
z_3 + k_2(z_2 + k_1z_1) + k_1z_1 + \theta(z_1 + a) \sin bt \\
\leq z_3 + (k_2 + k_1)z_2 + k_2k_1z_1 + \theta|z_1| + \theta|a| \\
\leq z_3 + (k_2 + k_1)z_2 + k_2k_1z_1 - \theta z_1 + \theta a,
\end{align*} \]

since \( z_1 \leq 0 \). Hence, the third relevant constraint is set as

\[ v_2 := z_3 + (k_2 + k_1)z_2 + (k_2k_1 - \theta)z_1 + \theta a \leq 0. \]  

(15)

This is the \( p \)-th constraint and, by construction, provides a barrier certificate for the constrained system. Its derivative depends directly on the input, i.e.

\[ \dot{v}_2 = (\epsilon + 2|x_3|)u + (k_2 + k_1)z_3 + (k_2k_1 - \theta)z_2 + (k_2 + k_1)\theta(z_1 + a) \sin bt. \]

Since \((k_2 + k_1)\theta(z_1 + a) \sin bt \leq (k_2 + k_1)\theta|z_1| + a| \), any control law that satisfies the inequality

\[ u \leq \frac{1}{(\epsilon + 2|x_3|)}(-k_2 + k_1)z_3 - (k_2k_1 - \theta)z_2 - (k_2 + k_1)\theta|z_1| + a \]

is an invariance controller for the designed safe set \( S = \{ x \in \mathbb{R}^n | z_1(x) \leq 0, v_1(x) \leq 0, v_2(x) \leq 0 \} \).

The same design procedure can be applied to design a invariance controller that guarantees safety for the lower constraint \( x_1 \geq -a \). Proceeding as above, the following new states are defined, i.e. \( \xi_1 = -x_1 - a \), \( \xi_2 = -x_2 \), \( \xi_3 = -\epsilon x_3 - x_3|x_3| \). In these new coordinates a barrier certificate can be defined as \( B_2 = (k_1 + k_2)\xi_2 + (k_1k_2 - \theta)\xi_1 + \xi_3 + \theta a \), and a corresponding invariance control law has to satisfy

\[ u \geq -\frac{1}{(\epsilon + 2|x_3|)}(-(k_1k_2 - \theta)\xi_2 + (k_1 + k_2)\xi_3 + (k_1 + k_2)\theta|\xi_1 - a|). \]

The control law is implemented as proposed in section IV. For both constraints an invariance control law \( u_{inv} \) and a switching surface \( \sigma \) are designed as proposed previously. The control input \( u \) is synthesized in the form (11).

Simulation results for the given problem with \( a = 1 \), \( b = 0.2 \)
and $\theta = 0.3$ are shown in figure 2. The nominal controller fails to stabilize the disturbed system in this case. It has to be redesigned. Nevertheless the invariance control guarantees that the mass does not hit the magnet, while this would clearly happen without the invariance control. In figure 3 the time trajectories of the three constraint variables $z_1, v_1, v_2$ are shown. All three constraints are kept below zero at all times, hence the overall constraint is satisfied. The constraint $v_2$ (solid line) would be the first to become positive, which is avoided by the invariance controller.

VI. Conclusion

An algorithm to construct control laws for uncertain nonlinear systems under state constraints is proposed. In a recursive design procedure, a set of safe initial conditions is constructed that is robustly controlled invariant. The design leaves some degree of freedom that can be used to adapt the procedure to specific requirements. A switching control is designed to guarantee positive invariance of the set and to satisfy a nominal control objective whenever possible. The proposed method allows one to consider constraints on single states as well as on combinations of states and can handle multiple constraints. The applicability of the proposed procedure is illustrated on a design example. It is shown that the ideas are easily applicable to constraints with a high relative degree.

Even if the current work focuses primarily on single input systems, a generalization to multi-input systems can be easily established at this point. An open problem that remains is with regards to the simultaneous satisfaction of state and input constraints. This question is connected to the problem of maximizing the set $S$. Furthermore it is of interest to focus on robust stabilization of the system on the constraint if the nominal control objective and constraint satisfaction contradict each other.

![Fig. 2. Position $x_1$ of the uncertain Active Magnetic Bearing System for the initial condition $x(0) = [0.5, 0, 0]$ with (solid) and without (dotted) the invariance control. The systems constraints are indicated by the solid horizontal lines.](image2.png)

![Fig. 3. Constraint variables $z_1$ (dotted), $v_1$ (dashed), and $v_2$ (solid) for the upper constraint of the uncertain Active Magnetic Bearing System for the initial condition $x(0) = [0.5, 0, 0]$ and the safety control law.](image3.png)

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