Virtual Closed Loop Identification: 
A generalized tool for identification in closed loop

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Abstract—In this paper we propose a virtual closed loop model parameterization to perform system identification. This parameterization is designed to achieve specific goals. We show that the method includes, as special cases, known methods for closed loop identification and also offers additional flexibility. We analyze the ramifications of the new tailor-made parameterization for systems operating in closed loop. The approach exploits a property of Box-Jenkins models in order to minimize the bias arising from feedback and noise model mismatch.

I. INTRODUCTION

Identification of systems operating in closed loop has received considerable attention in the System Identification literature [8], [14], [15], [4], [12]. There are safety and economic reasons to perform identification experiments in closed loop. Also, it is known that the optimal experiment is usually performed in closed loop [17], [11], [15], [5], [10]. Indeed, recent research has established that, for a general class of systems, and when there is a constraint on the output power, the optimal experiment is necessarily closed loop [3].

Unfortunately, the identification of systems operating under the presence of feedback presents several difficulties [15], [4]. For example, correlation between the input signal and the noise is problematic in the context of several identification techniques. In fact, it is well known that the Prediction Error Method (PEM) provides a non-consistent estimate in the presence of under-modeling of the noise transfer function [4].

Several attempts to overcome this difficulty have been made. In particular, indirect identification is a popular approach to mitigate this difficulty. Traditional indirect identification is a two step procedure where the identification of a plant object is first obtained and then the open loop system is unraveled from this preliminary estimate. Here, and in the sequel, we use the term “plant object” to refer to a transfer function that depends on the system. In traditional indirect identification, the plant object to be identified is usually the complementary sensitivity transfer function relating the reference signal to the output [14]. However, several difficulties are known to exist with this approach. For example, it is common that the estimate of the open loop process is not necessarily stabilized by the controller used in the identification experiment, even though it is known that the real system is stabilized by this controller. This difficulty can be overcome by using a particular parameterization of the system, the so called Dual-Youla parameterization [9], [13], [16]. Also, it is often true in practice that the controller can have certain non-linearities e.g. anti-windup schemes [7]. This difficulty renders the usual indirect identification approaches unusable on many problems. Recent research reported in [2], [1], [6] has proposed an alternative method which tackles the difficulty of non-linear or partially known controllers. The method is based on a “virtual controller” which approximates the true one. The method is, in general, non-consistent when PEM is utilized. However, the asymptotic bias is small in the frequency range where the virtual controller approximately matches the true one. This method leads to a new class of estimators for systems operating in closed loop.

This paper we generalize the virtual closed loop approach. We show that, by suitable choice of parameters, the method specializes to known closed loop identification schemes. We propose a parameterization of the process which is designed in order to achieve different goals. In particular, we focus on the minimization of the asymptotic bias due to feedback and noise model mismatch.

The remainder of the paper is organized as follows: In Section II we describe the scheme of interest in a general non-linear setting. In Section III we specialize to linear systems. In Section IV we show that the virtual closed loop method generalizes known schemes for closed loop identification. In Section V we show how the choice of parameters in the virtual closed loop affects the asymptotic bias in the identification of systems operating in closed loop. In Section VI we present a numerical example. Finally in Section VII we draw conclusions.

II. GENERAL SYSTEM OF INTEREST

We consider the following general non-linear system (see Figure 1):

\[ y_t = G_o(u_t, u_{t-1}, \ldots, w_t, w_{t-1}, \ldots) \]  

(1)

where \( u_t, y_t \) are the input and output signals and \( w_t \) is zero mean Gaussian white noise of variance \( \sigma_w^2 \). We assume that the input and output signals are bounded. In this section, we do not make assumptions on the experimental conditions. However, the system may have been operating in closed loop in order to ensure bounded input-output signals for and open loop unstable process.

In order to develop the identification procedure proposed...
in this paper, we first define the following signals:

\[ x_t = F_1 u_t + F_2 y_t \]  
\[ z_t = F_3 u_t + F_4 y_t \]

(2)  
(3)

where \( F_1, F_2, F_3 \) and \( F_4 \) are stable filters. We assume that \( F_1 \) is bi-proper. Notice that the signals \( x \) and \( z \) are bounded because they are generated by passing bounded signals through stable filters.

We assume that the filters \( F_i \) are written in terms of the following polynomials:

\[ F_i = N_i D_i^{-1} \]

(4)

where \( D_i \) roots are inside the stability boundary.

This is the reason for the term “Virtual Closed Loop” (VCL). Also, stability is not an issue for the system of Figure 2 since we already know that all signals are bounded. In the remainder of the paper we will explore the implications of this configuration in System Identification. We propose to identify a model relating the signals \( x_t \) and \( z_t \) shown in Figure 2.

Towards this goal, we conceive of a model of the same structure as that shown in Figure 2 but parameterized by a vector \( \theta \). This is shown in Figure 3. The basic idea of virtual closed loop identification is to estimate \( \theta \) by minimizing some function of the error between the measured signal \( z \) and the model output \( \hat{z} \). Note that stability is an issue and this model as we are treating \( x_t \) as an external signal.

We note that the signals \( x_t \) and \( z_t \) are readily obtained from knowledge of \( u_t \) and \( y_t \). We then have the following core result:

**Theorem 1:** The signal transformations (2) and (3) induce the “virtual closed loop” shown in Figure 2.

**Proof:** The result is obtained by re-arranging equations (2) and (3) and using some block algebra.

Notice that the feedback loop in Figure 2 has nothing to do with the existence of otherwise of a real closed loop system.

![Fig. 1. Signal Generation in the Virtual Closed Loop Method.](image1)

![Fig. 2. Generalized Virtual Closed Loop.](image2)

![Fig. 3. Predictor Model for the Generalized Virtual Closed Loop.](image3)

**III. Specialization to Linear Systems**

For the case of linear systems, the model (1) can be written as:

\[ y_t = G_o(q^{-1}) u_t + v_t \]  
\[ v_t = H_o(q^{-1}) w_t \]

(5)  
(6)

where \( q^{-1} \) is the back-shift operator, and the transfer functions \( G_o \) and \( H_o \) are defined as follows:

\[ G_o = B_o A_o^{-1} \]  
\[ H_o = P_o Q_o^{-1} \]

(7)  
(8)

where \( B_o, A_o, P_o, Q_o \) are polynomials and \( P_o \) and \( Q_o \) are monic and have roots inside the stability boundary.

Using equations (5), (2) and (3) we obtain the following set of equations describing the virtual closed loop system:

\[
\begin{bmatrix}
1 & -F_3 & -F_4 \\
0 & F_1 & F_2 \\
0 & -G_o & 1
\end{bmatrix}
\begin{bmatrix}
z_t \\
u_t \\
y_t
\end{bmatrix} =
\begin{bmatrix}
0 \\
x_t \\
v_t
\end{bmatrix}
\]

(9)

Solving for \( z, u \) and \( y \) we have the following:

\[ z_t = \frac{F_3 + F_4 G_o}{F_1 + F_2 G_o} x_t + \frac{F_1 F_4 - F_2 F_3}{F_1 + F_2 G_o} H_o w_t \]

(10)

\[ u_t = \frac{1}{F_1 + F_2 G_o} x_t - \frac{F_3}{F_1 + F_2 G_o} H_o w_t \]

(11)

\[ y_t = \frac{F_3 + F_4 G_o}{F_1 + F_2 G_o} x_t + \frac{F_1 F_4 - F_2 F_3}{F_1 + F_2 G_o} H_o w_t \]

(12)
The core idea of the approach proposed in the current paper is to identify the virtual closed loop system in (10). Accordingly, we define the following transfer functions

\[ R_o = \frac{F_3 + F_2G_o}{F_1 + F_2G_o}, \quad K_o = \frac{F_1F_4 - F_2F_3}{F_1 + F_2G_o}, \quad H_o \] (13)

We will use a Box-Jenkins type of model for the virtual closed loop system in which we treat \( F_3 + F_2G(\rho) \) as a Tailormade parameterization and we independently parameterize \( K \) in terms of a parameter \( \eta \). Hence, the parameters \( \rho \), and \( \eta \) are estimated by minimizing a criterion of the form:

\[ J = \sum_{t=1}^{N} \epsilon_t^2 \] (14)

where

\[ \epsilon_t = K(\eta)^{-1} [z_t - R(\rho)x_t] \]

\[ R(\rho) = \frac{F_3 + F_4G(\rho)}{F_1 + F_2G(\rho)} \] (15)

In the subsequent analysis we will analyze the impact of the following two issues:

1) \( x_t \) is not, in general, an exogenous signal but is potentially correlated with the noise \( u_t \).
2) The class of models used for \( K(\eta) \) may not include the true noise model \( K_o \) e.g. we might decide to use a fixed noise model \( K \neq K_o \).

IV. SPECIALIZATION TO DIRECT AND INDIRECT CLOSED LOOP IDENTIFICATION METHODS

Here, we show that the Virtual Closed Loop method generalizes known methods for closed loop identification. In particular, it is readily seen that:

- Direct identification (see e.g. [12]) is obtained by the choice \( F_1 = F_4 = 1, \; F_2 = F_3 = 0 \). This results in \( x_t = u_t, \; z_t = y_t, \; R_o = G_o, \; K_o = H_o \).

- Traditional Indirect identification ([14]) is obtained by the choice \( F_1 = C_o^{-1}, \; F_2 = F_4 = 1, \; F_3 = 0 \) where \( C_o \) is the (assumed known and linear) true controller. This results in \( x_t = r_t, \; z_t = y_t, \; R_o = \frac{G_oC_o}{1 + G_oC_o}, \; K_o = \frac{1}{1 + G_oC_o}, \; H_o \).

- The Dual Youla method ([9], [13], [16]) results from the choice \( F_1 = D_c, \; F_2 = N_c, \; F_3 = -N_x, \; F_4 = D_x \) where \( M = N_cN_x + D_cD_x \), is stable, minimum phase, and bi-proper (same number of poles and zeros) and where \( N_cD_c^{-1} \) is a co-prime representation of the (assumed known and linear) true controller \( C_o \) and where \( G_x = N_xD_x^{-1} \) is a co-prime representation of an a-priori given estimate for \( G_o \).

- The “whitening procedure” (see e.g. [12]) is obtained by the choice \( F_1 = F_4 = F \) and \( F_2 = F_3 = 0 \). In this case we have \( x_t = u_t, \; z_t = y_t, \; R_o = G_o, \; K_o = FH_o \).

Note that if \( F \approx H_o^{-1} \), then we might consider using a fixed filter \( K = 1 \) in the estimates.

V. ANALYSIS OF VIRTUAL CLOSED LOOP IDENTIFICATION

Here we revert to the general scheme given in (2), (3). We will hypothesize that the true system operates in closed loop with either a non-linear controller or a linear controller which are only partially known.

**Remark 1**: As a preliminary observation, we see that, when \( x_t \) is considered as an exogenous signal, then the virtual closed loop of Figure 3 will be stable if and only if the polynomial \( N_1D_2A + N_2D_1B \) has its roots inside the stability boundary where \( G = B/A \). Thus, if the virtual controller \( \hat{C} = F_2F_1^{-1} \) is known to stabilize the true system when \( x_t \) is exogenous, then it suffices to search for estimated models such that the tailor-made parameterization of \( \hat{R} \) is stable.

The key tool that we will utilize to analyze the estimates provided by the virtual closed loop schemes is the following:

**Lemma 1**: Consider the parameter estimation scheme described in (14) to (16) where \( z_t \) is related to \( x_t \) as in (10).

- For general, possibly non-linear, feedback of the form

\[ x_t = \Gamma(x_{t-1}, x_{t-2}, \cdots, z_t, z_{t-1}, \cdots, r_t, r_{t-1}, \cdots) \] (17)

where \( r_t \) is a given exogenous reference signal, then the asymptotic bias in the resulting estimate of \( R_o \) is

\[ B_R = R - R_o = [K_o - K] \begin{bmatrix} \Phi_{w,x} \\ \Phi_x \end{bmatrix} \] (18)

where \([\Phi]_+ \) represents the causal part of \( \Phi \), and where \( \Phi_{w,x} \) and \( \Phi_x \) respectively denote the cross spectrum between \( x_t \) and \( u_t \) and the spectrum of \( x_t \).

- For the case of linear feedback, where the feedback takes the form

\[ x_t = \gamma_o(q^{-1})(r_t - z_t) \] (19)

then the asymptotic bias can be evaluated explicitly as

\[ B_R = [K - K_o] \begin{bmatrix} (\gamma_oK_oS_o)\sigma^2_w \\ |S_o|^2\Phi_x + |\gamma_oK_oS_o|^2\sigma_o^2 \end{bmatrix} \] (20)

where \( \ast \) denotes complex conjugate and \( S_o \) represents the sensitivity function given by

\[ S_o = \frac{1}{1 + R_o\gamma_o} \] (21)

**Proof**: Essentially as in [4], [12] with a change of notation. \( \Box \)

**Remark 2**: If we apply Lemma 1 to direct identification, then we see that the estimates will be biased when \( K \) differs from \( K_o = H_o \). This is a well known problem with direct identification when the noise model is ill-defined (e.g. time varying).

We can apply Lemma 1 to the Virtual Closed Loop scheme. To do this, we need to evaluate \( \Phi_{w,x} \) and \( \Phi_x \) in terms of other signals. To do this we will assume either that

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The true system operates under non-linear feedback of the form
\[ u_t = K\{u_{t-1}, u_{t-2}, \ldots, y_t, y_{t-1}, \ldots \} \]  
(22)
or linear feedback of the form
\[ u_t = C_o(q^{-1})(r_t - y_t) \]  
(23)

Note that the controllers described above are not the controllers given (17) and (19). Of course, there is a relationship between the feedback laws found by solving the various system of equations. Indeed, this underlies the methodology used to prove the following result:

**Theorem 2:** Consider the virtual closed loop estimates described in (14) to (16). Also, assume that

- \( G_o \) lies in the model class \( G(\rho) \) for some \( \rho = \rho_o \).
- \( H \) is the implicit equivalent noise model induced by the relationship

\[
H := \frac{F_1 + F_2 G_o}{M} K \quad (24)
\]

Then, the asymptotic bias in the estimate of \( G_o \) induced by solving \( R = \frac{F_3 + F_2 G_o}{F_1 + F_2} \) for \( G \) is

\[
G_o - G \approx (H_o - H)(F_1 + F_2 G_o) \left[ \frac{1}{F_1 + F_2 G_o} \frac{\beta}{\alpha} \right] \quad (26)
\]

where

\[
\beta := \Phi_{wu} + (\tilde{C} H_o \tilde{S}_o)^* \sigma_{w}^2 \quad (27)
\]

\[
\alpha := \Phi_{uu} + |\tilde{C} H_o \tilde{S}_o|^2 \sigma_{w}^2 + 2Re \{(\tilde{C} H_o \tilde{S}_o)^* \Phi_{wu}\} \quad (28)
\]

\[
\tilde{C} := \frac{F_2}{F_1}, \quad \tilde{S}_o := \frac{1}{1 + G_o \tilde{C}} \quad (29)
\]

Moreover, when the true controller is linear and with transfer function is \( C_o \), we have that

\[
\beta = [(\tilde{C} - C_o) H_o S_o \tilde{S}_o] \sigma_{w}^2, \quad S_o = \frac{1}{1 + G_o \tilde{C}} \quad (30)
\]

**Proof:** See the Appendix.

Theorem 2 provides a basis for choosing suitable values for \( F_1, F_2, F_3, F_4 \). In particular, we see from (26) and (30) that the asymptotic bias is small under either of the following two conditions

- \( H_o - H \) is small
- \( C - C_o \) is small

Note that this holds on a frequency by frequency basis so it suffices for \( \tilde{C} \) to be near the true controller when \( H_o - H \) is large or for \( H_o - H \) to be small when \( \tilde{C} \) is a poor representation of the true controller.

Hence, it makes sense to choose \( F_1, F_2 \) such that \( \tilde{C} = F_2 F_1^{-1} \) is close to the true controller. For example, if the true controller is a linear controller with anti-windup protection, then \( \tilde{C} \) could be chosen as the linear controller without anti-windup.

From (10), it may be tempting to think that a good to choice for \( F_3, F_4 \) would be such that \( F_1 F_4 = F_2 F_3 \) since this removes all noise from (10). However, in this case, \( R_o = F_4 F_3^{-1} \) i.e. we learn nothing about \( G_o \). Thus, it is necessary to design the filters \( F_i \) such that \( M \) is different from zero in the frequency range of interest.

An alternative choice of \( F_3, F_4 \) would be to use a-priori estimates \( G_x, H_o \) for \( G_o, H_o \) to render \( K_o \approx 1 \). In this case, we might try using a fixed value for \( K \) (namely 1) in (15). The virtual closed loop scheme then reduces to an output error method linking the measured variable \( z_t \) to the model output \( \hat{z}_t \). Of course, based on Theorem 2, bias may result if \( F_1 F_2 G_o \) is significantly different form \( H_o \) in frequency ranges where \( \tilde{C} \) is a poor approximation to the true controller.

**VI. A Numerical Example**

Consider the following system:

\[
G_o = \frac{b_0 q^{-1}}{1 - a_1 q^{-1}} \quad (31)
\]

\[
H_o = \frac{1 + c_1 q^{-1} + c_2 q^{-2} + c_3 q^{-4} + c_4 q^{-4}}{1 + d_1 q^{-1} + d_2 q^{-2} + d_3 q^{-4} + d_4 q^{-4}} \quad (32)
\]

with \( a_1 = 0.6, b_1 = 0.4, c_1 = 1.851, c_2 = -1.976, c_3 = -0.7605, c_4 = 0, d_1 = -1.2, d_2 = 0.3309, d_3 = -0.6484, d_4 = 0.605 \). The true control law is given by

\[
u_t = C_o(q^{-1})(r_t - y_t) \quad (33)
\]

\[
C_o(q^{-1}) = \frac{0.5q^{-1}}{1 - 0.5q^{-1}} \quad (34)
\]

where \( r_t \) zero mean Gaussian noise of variance \( \sigma^2 = 10 \) passing trough the filter \( 1 - 0.5q^{-1} \). We use \( N = 10000 \) data points. The ratio between the variance of the output noise \( \sigma_x^2 \) and the variance of the output is \( \sigma_o^2 / \sigma_y^2 \approx 0.4 \) for all the experiments.

For the Virtual Closed Loop identification method we choose \( F_1 = 1, F_2 = 1, F_3 = 0, F_4 = 1 \). This implies that \( \tilde{C} = F_2/F_1 = 1 \). We use an output error model for the virtual closed loop, i.e. \( K(q^{-1}) = 1 \).
Figure 4 shows the Bode diagrams for $G_o(q^{-1})$ and $R_o(q^{-1})$. The Bode diagram for $1 - H_o(q^{-1})$ and for $1 - K_o(q^{-1})$ are shown in Figure 5. This figure shows that $H_o$ and $R_o$ are different from 1 in the range of frequencies where the magnitude of $G_o$ and $R_o$ is significant. This implies a difficulty for using an output error model to identify $G_o$ and $R_o$.

We identify the system by using direct identification with the following model for the transfer function $H_o(q^{-1})$:

$$H(q^{-1}) = \frac{1 + c_1 q^{-1} + \cdots + c_n q^{-n}}{1 + d_1 q^{-1} + \cdots + d_n q^{-n}}$$

(35)

for different values of $n$.

Figure 6 shows the parameter estimates for 300 Monte-Carlo Experiments. We see that, even though we use an output error model for the virtual closed loop, the bias in the estimated model is small. This is due to the fact that the model for the controller is correct in the frequency region of interest. This, actually shows an advantage of using the VCL method. On the other hand, the bias of the models obtained by using direct identification for a Box-Jenkins model is only reduced when the noise model order $(n)$ is increased. Moreover, we see in Figure 6 that the parameters estimated with direct identification are unbiased only when there is no under-modelling $(n = 4)$.

It is important to note that direct identification is also covered by the VCL method ($F_2 = F_3 = 0$, $F_1 = F_4 = 1$). However, the VCL method provides additional flexibility which is useful to reduce the bias in the estimates.

VII. Conclusions

In this paper we have generalized the virtual closed loop (VCL) approach to System Identification. We have focused on systems operating in closed loop and we have analyzed the asymptotic bias due to feedback and noise model mismatching. We have shown that the new parameterization generalizes known methods for closed loop identification and also offers additional flexibility. A numerical example has confirmed the claimed merits of the approach.

REFERENCES


APPENDIX

A. Proof of Theorem 2:

Using the model in equation (10) and equation (18), we have that:

$$R_o - R = (K_o - K) [\kappa]_+$$

(36)

where

$$\kappa = \frac{1}{F_1 + F_2 G_o} \beta$$

(37)

$$\beta := \Phi_{wu} + (\bar{C}_H \bar{S}_o)^\ast \sigma_w^2$$

(38)

$$\alpha := \Phi_u + |\bar{C}_H \bar{S}_o|^2 \sigma_w^2 + 2 \text{Re} \{ (\bar{C}_H \bar{S}_o) \ast \Phi_{wu} \}$$

(39)
We have that the difference between \( R_o \) and \( R \) is given by:
\[
R_o - R = (H_o - H) \frac{M}{F_1 + F_2 G_o} [\kappa]_+ \tag{40}
\]
On the other hand, the difference between \( R_o \) and its estimate \( R \) is also given by:
\[
R_o - R = \frac{F_3 + F_2 G_o - F_3 + F_4 G}{F_1 + F_2 G_o} \frac{F_3 + F_4 G}{F_1 + F_2 G} \tag{41}
\]
\[
= \frac{(F_1 F_4 - F_2 F_3)(G_o - G)}{(F_1 + F_2 G_o)(F_1 + F_2 G)} \tag{42}
\]
Solving for \( G \) we have:
\[
G = \frac{MG_o - F_1(F_1 + F_2 G_o)(R_o - R)}{M + F_2(F_1 + F_2 G_o)(R_o - R)} \tag{43}
\]
and then calculating the difference between \( G_o \) and \( G \) we have that:
\[
G_o - G = \frac{(F_1 + G_o F_2)^2(R_o - R)}{M + F_2(F_1 + F_2 G_o)(R_o - R)} \tag{44}
\]
Then, using (40) we have that:
\[
G_o - G = \frac{(F_1 + G_o F_2)(H_o - H) [\kappa]_+}{1 + F_2(H_o - H) [\kappa]_+} \tag{45}
\]
We have that the asymptotic bias on the estimate of \( G_o \) is given by:
\[
G_o - G = (F_1 + F_2 G_o) \frac{(H_o - H) [\kappa]_+}{1 + F_2(H_o - H) [\kappa]_+} \tag{46}
\]
Using a Taylor expansion of first order we have that the asymptotic bias on the estimate of \( G_o \) can be approximated as follows:
\[
G_o - G \approx (H_o - H)(F_1 + F_2 G_o) \left[ \frac{1}{F_1 + F_2 G_o} \frac{\beta}{\alpha} \right]_+ \tag{47}
\]
which finishes the proof.