On Design of Reduced-Order $H_\infty$ Filters for Discrete-Time Systems from Incomplete Measurements

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Abstract—In networked control systems, when data are transmitted over wireless channels, measurements of the transmitted data are often incomplete due to transmission errors and/or packet losses. The problem of designing reduced-order $H_\infty$ filters for discrete-time systems from incomplete measurements is investigated in this paper. A Bernoulli distributed white sequence is adopted as a model for the normal operating condition of packet delivery and transmission failure. The reduced-order filter to be designed from incomplete measurements is required to ensure the mean-square stability of the filtering error system and to guarantee a prescribed $H_\infty$ filtering performance level. It is shown that such a desired reduced-order $H_\infty$ filter can be constructed under a sufficient condition expressed in terms of two linear matrix inequalities (LMIs) subject to a rank constraint.

I. INTRODUCTION

The problem of $H_\infty$ filtering for dynamic systems is concerned with designing an estimator which guarantees that the $L_2$-induced gain from the noise signals to the estimation error is less than a prescribed level. During the past decade, design of $H_\infty$ filters has been an active area of research (see, e.g., [4], [6], [22], [25]).

It is noticed that in the existing literature most approaches to $H_\infty$ filter design assume that the measurements of the system output are complete without any loss. How to use lossy measurements in estimating signals presents a practically important issue that requires much attention. The lossy measurements maybe arise in information transmissions across limited bandwidth wireless channels [2], [14]. In memoryless communication channel, the lossy measurement is commonly modeled as a stochastic Bernoulli process while in fading communication channel it is modeled as a finite-state Markov chain [3], [9]. Estimation with lossy measurement for dynamic systems has been a hot research topic recently [2], [5], [14]. Some signal estimation results for dynamic systems with lossy measurements have been reported in the literature [2], [14], [17], [21]. A jump estimation technique was presented to cope with lossy information in [2]. The authors in [14] investigated the Kalman filtering problem with intermittent observations. A variance-constrained filtering approach was proposed for systems with lossy measurements in [17].

On the other hand, in many real-world problems, there is need to utilize lower order filters, which has inspired the research on reduced-order $H_\infty$ filtering. Based on the projection lemma, the authors in [19] proposed an LMI (linear matrix inequality) with rank constraint approach to the reduced-order $H_\infty$ filter design for a class of stochastic systems. The reduced-order $H_\infty$ filtering problem for linear systems with Markovian jump parameters was studied in [16]. More recently, an LMIs with equality constraint approach to the fixed-order $H_\infty$ filtering problem of uncertain systems with Markovian jump parameters was presented in [18].
The purpose of this paper is to investigate the reduced-order $H_\infty$ filtering problem for dynamic systems with lossy measurements. The works in [14], [19], [24] provide the impetus to carry out the present investigation. Specifically, we are interested in designing reduced-order filters by using lossy measurement such that the filtering error system is exponentially mean-square stable and a prescribed $H_\infty$ filtering performance level is achieved.

Notations: Throughout this paper, $Z^+$ denotes the set of positive integers; $\mathbb{R}^n$ denotes the n dimensional Euclidean space; $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$ real matrices. A real symmetric matrix $P > 0(\geq 0)$ denotes $P$ being a positive definite (or positive semi-definite) matrix, and $A > (\geq) B$ means $A - B > (\geq) 0$. $I$ denotes an identity matrix of appropriate dimension. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations. The superscript ‘$\tau$’ represents the transpose. * is used as an ellipsis for terms that are induced by symmetry. For a $x \in \mathbb{R}^n$,

$$||x||^2 := x^T x.$$ 

Any matrix whose columns form the basis of the right null space of $M$ is denoted by $\mathcal{N}(M)$ or $\mathcal{N}_M$. The notation $l_2[0, \infty)$ represents the space of square summable infinite vector sequences with the usual norm $||\cdot||_2$. A sequence

$$v = \{v_k\} \in l_2[0, \infty)$$

if

$$||v||_2 = \sqrt{\sum_{k=1}^{\infty} v_k^2 v_k} < \infty.$$

$Prob\{\cdot\}$ stands for the occurrence probability of an event; $E\{\cdot\}$ denotes the expectation operator with respect to some probability measure.

II. Problem Formulation

Consider the discrete-time dynamic system $\Sigma$:

$$x_{k+1} = Ax_k + A_\omega \omega_k$$

$$z_k = Lx_k + L_\omega \omega_k$$  \hspace{1cm} (1)

where $x_k \in \mathbb{R}^n$ is the state; $\omega_k \in \mathbb{R}^p$ is the deterministic disturbance signal in $l_2[0, \infty)$; $z_k \in \mathbb{R}^q$ is the signal to be estimated; and $A, A_\omega, L$ and $L_\omega$ are known constant matrices with compatible dimensions. The measurement is modeled by

$$y_k = Cx_k$$

$$y_{ck} = (1 - \theta_k) y_k + \theta_k y_{k-1}$$  \hspace{1cm} (4)

where $y_k \in \mathbb{R}^p$ is the output, $y_{ck} \in \mathbb{R}^p$ is the measured output, $C \in \mathbb{R}^{p \times n}$ is a known matrix, and the stochastic variable $\theta_k$ is a Bernoulli distributed white sequence taking value on 0 and 1 with

$$Prob\{\theta_k = 1\} = E\{\theta_k\} = \rho$$

$$Prob\{\theta_k = 0\} = E\{1 - \theta_k\} = 1 - \rho$$  \hspace{1cm} (5)

where $\rho \in [0, 1]$ and is a known constant.

Remark 1: The system measurement modeled in (3) and (4) was first introduced in [13] and has been used to characterize the effect of communication data loss in information transmissions across limited bandwidth communication channels over a wide area, such as navigating a vehicle based on the estimations from a sensor web of its current position and velocity [14]. The output $y_k$ produced at a time $k$ is sent to the observer through a communication channel. If no packet-loss occurs, the measurement output $y_{ck}$ takes value $y_k$; otherwise, the measurement output $y_{ck}$ takes value $y_{k-1}$. When the probability of event packet-loss occurring is assumed as $\rho$, the measurement output $y_{ck}$ in (4) thus takes the value $y_k$ with probability $1 - \rho$, and the value $y_{k-1}$ with probability $\rho$. 

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We consider the following filter for the estimation of $z_k$:

\[
\begin{align*}
\hat{x}_{k+1} & = A_f \hat{x}_k + B_f y_{ck} \\
\hat{z}_k & = C_f \hat{x}_k + D_f y_{ck}
\end{align*}
\]

(7)

where $\hat{x}_k \in \mathbb{R}^\hat{n}$, $0 < \hat{n} \leq n$, and $\hat{z}_k \in \mathbb{R}^q$. $A_f$, $B_f$, $C_f$ and $D_f$ are to be determined.

Remark 2: The filter in the form of (7) reduces to a full order one when $\hat{n} = n$. There are many results for the designs of the full order filters (see, e.g., [12], [25]).

Combining (1)-(4) and (7) together, the filtering error dynamics can be represented as $\tilde{\Sigma}$:

\[
\begin{align*}
\tilde{x}_{k+1} & = A(\theta_k)\tilde{x}_k + A_1(\theta_k)H\tilde{x}_{k-1} + A_\omega \omega_k \\
\tilde{z}_k & = L(\theta_k)\tilde{x}_k + L_1(\theta_k)H\tilde{x}_{k-1} + L_\omega \omega_k
\end{align*}
\]

(8)

where

\[
\begin{align*}
\tilde{x}_k & = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T \\
\tilde{z}_k & = z_k - \hat{z}_k \\
H & = \begin{bmatrix} I & 0 \end{bmatrix}
\end{align*}
\]

\[
A(\theta_k) = \begin{bmatrix} A & 0 \\
(1-\theta_k)B_fC & A_f \end{bmatrix}
\]

\[
A_1(\theta_k) = \begin{bmatrix} 0 \\
\theta_kB_fC \end{bmatrix}
\]

\[
A_\omega = \begin{bmatrix} A_\omega \\
0 \end{bmatrix}
\]

\[
L(\theta_k) = \begin{bmatrix} L - (1-\theta_k)D_fC, & -C_f \end{bmatrix}
\]

\[
L_1(\theta_k) = -\theta_kD_fC \\
L_\omega = L_\omega
\]

(10)

Let

\[
F = \begin{bmatrix} A_f & B_f \\
C_f & D_f \end{bmatrix}
\]

(11)

It can be checked via (10) that

\[
\begin{bmatrix} A(\rho) & A_1(\rho) & A_\omega \\
L(\rho) & L_1(\rho) & L_\omega \end{bmatrix}
\]

\[
= \begin{bmatrix} A & 0 & 0 \\
0 & 0 & 0 \\
L & 0 & 0 \end{bmatrix}
\]

\[
+ \begin{bmatrix} 0 & I & 0 \\
0 & 0 & -I \\
0 & \rho C & 0 \end{bmatrix} F \begin{bmatrix} 0 & I \\
(1-\rho)C & 0 \\
\rho C & 0 \end{bmatrix}
\]

(12)

where the $\rho$-dependent matrices are defined as in (10) with $\theta_k$ replaced by $\rho$.

Throughout the paper, we make the following assumptions for system (1)-(4).

Assumption 1: The matrix $A$ is Schur stable (i.e., all eigenvalues of $A$ are located within the unit circle in the complex plane).

Assumption 2: $x_{-1} = 0$.

Remark 3: Assumption 1 is a common assumption in dealing with the filtering problem. Assumption 2 implies from (3) that $y_{-1} = 0$, which gives the initial condition for the lossy measurement model (4).

It is noted that the filtering error dynamics (8) is a system with stochastic parameters since some of the parametric matrices in (10) are associated with the stochastic variable $\theta_k$. For the problem formulation, we adopt the notion of stochastic stability in the mean-square sense from [15].

Definition 1: The filtering error dynamics $\tilde{\Sigma}$ is said to be exponentially mean-square stable if with

\[
\omega_k \equiv 0,
\]

there exist constants $\alpha > 0$ and $\tau \in (0, 1)$ such that

\[
\mathbb{E}\{\|\tilde{x}_k\|^2\} \leq \alpha \tau^k \mathbb{E}\{\|\tilde{x}_0\|^2\},
\]

for all $\tilde{x}_0 \in \mathbb{R}^{n+\hat{n}}$, $k \in \mathbb{Z}^+$.

The $H_\infty$-type filtering problem addressed in this paper is to design a filter in the form of (7) such that the filtering error system $\tilde{\Sigma}$ is exponentially mean-square stable and under the zero initial condition, the filtering error $\tilde{z}_k$ satisfies

\[
\sum_{k=0}^{\infty} \mathbb{E}\{\|\tilde{z}_k\|^2\} \leq \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|\tilde{\omega}_k\|^2\}
\]

(13)

for a given scalar $\gamma$ and all nonzero $\omega_k$. In such a case, the filtering error system is called to be exponentially mean-square stable with $H_\infty$ filtering performance $\gamma$. 1668
To begin with, we establish a condition of mean-square stability and $H_\infty$ performance for the filtering error dynamics $\hat{\Sigma}$, which will be fundamental in the derivation of our $H_\infty$ filter design methodology.

**Lemma 1:** Consider the filtering error dynamics $\hat{\Sigma}$. Given a scalar $\gamma > 0$, the filtering error system $\hat{\Sigma}$ is exponentially mean-square stable and has a guaranteed $\gamma$ level of disturbance attenuation, if there exist matrices $P$ and $Q$ such that

$$
\begin{bmatrix}
-P & 0 & PA_0 & PA_1 \\
-I & L_0 & L_1 & \ell_\omega \\
0 & -Q & 0 & -\gamma^2I \\
0 & 0 & 0 & -\gamma^2I
\end{bmatrix} < 0
$$

where $*$ denotes the corresponding transposed block matrix due to symmetry.

**Theorem 1:** Consider an $H_\infty$ filter (7), of order $n$, with the $H_\infty$ filtering performance level $\gamma$, for system (1)-(2) with lossy measurements (3)-(4). Suppose that $0 < \hat{n} \leq n$. There exist a filter matrix $F$, and matrices $P$ and $Q$ satisfying (14), if and only if there exist matrices $X > 0$, $Y > 0$ and $Q > 0$ such that

$$
\begin{bmatrix}
-Y & YA \\
-\gamma^2I & 0 & 0 \\
-\gamma^2I & -Q & 0 \\
-\gamma^2I & -\gamma^2I & -\gamma^2I
\end{bmatrix} < 0
$$

(15)

$$
\begin{bmatrix}
-X & 0 & XA_0 \\
-\gamma^2I & L_0 & LN_2 \\
-\gamma^2I & -\gamma^2I & 0 \\
-\gamma^2I & -\gamma^2I & -N_\omega^TQN_1 + N_\omega^T(Q - X)N_2
\end{bmatrix} < 0
$$

(16)

$$
X - Y \geq 0
$$

(17)

where

$$
\begin{bmatrix}
N_1 \\
N_2
\end{bmatrix} := N \begin{bmatrix}
\rho C \\
(1 - \rho)C
\end{bmatrix}
$$

(18)

and

$$\text{rank}(X - Y) \leq \hat{n}
$$

(19)

In this case, if matrices $X$, $Y$, and $Q$ are solutions to linear matrix inequalities (15)-(17) with rank constraint (19), then there always exist matrices $X_{22} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ with $X_{22} > 0$ and $X_{12} \in \mathbb{R}^{n \times \hat{n}}$ satisfying

$$
X_{12}X_{22}^{-1}X_{12}^{T} = X - Y
$$

(20)

The parametric matrix $F$ defined in (11), of the filter in the form of (7) with order $\hat{n}$, can be obtained via solving the linear matrix inequality:

$$
\Psi + U^T F^T V + V^T FU < 0
$$

(21)
where

\[ \Psi = \begin{bmatrix}
-X & -X_{12} & 0 & X \Lambda & 0 \\
* & -X_{22} & 0 & X_{12} \Lambda & 0 \\
* & * & -I & L & 0 \\
* & * & * & -\gamma^{2} I & 0 \\
* & * & * & * & * \\
* & * & * & * & * \\
0 & X \Lambda & 0 & X_{12} \Lambda & 0 \\
0 & L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & X_{12}^{T} & 0 & 0 & 0 \\
* & Q & 0 \\
* & * & * & * & * \\
\end{bmatrix} \]

(22)

\[ U = \begin{bmatrix}
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho C & 0 & 0 & (1-\rho) C \\
0 & 0 & 0 & 0 & 0 & 0 & \rho C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q & 0 \\
X_{12}^{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

(23)

\[ V = \begin{bmatrix}
X_{12}^{T} & X_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

(24)

**Remark 4:** Theorem 1 provides not only a sufficient condition for the solvability of the reduced-order $H_{\infty}$ filtering problem for the discrete-time systems with lossy measurement, but also an equivalent condition to Lemma 1. This equivalence implies that the filter design result derived from Lemma 1 is more general. It should be pointed out that the rank-constrained linear matrix inequalities (15)-(19) are non-convex due to the rank constraint (19). Many techniques have been presented to solve rank-constrained linear matrix inequalities (see [7], [11] and the references therein).

IV. CONCLUSIONS

This paper has examined the problem of designing reduced-order $H_{\infty}$ filters for a class of discrete-time systems with lossy measurements. The main contribution has been the development of a reduced-order $H_{\infty}$ filter design approach by using the projection lemma. The solvability of the reduced-order $H_{\infty}$ filtering problem with lossy measurements has been linked to the feasibility of two linear matrix inequalities with a rank constraint, which significantly facilitates finding out the desired solution.

ACKNOWLEDGEMENTS

This work was supported in part by the National Natural Science Foundation of P. R. China under Grants 60574080 and 60434020, and in part by a Research Grant from the Australian Research Council and a Research Grant from the University of Western Sydney, Australia.

REFERENCES


