Abstract—Exponential stability analysis and $L_2$-gain analysis are developed for uncertain distributed parameter systems. Scalar heat processes and distributed mechanical oscillators, governed by semilinear partial differential equations of parabolic and, respectively, hyperbolic types, are chosen for treatment. Sufficient exponential stability conditions with a given decay rate are derived in the form of Linear Matrix Inequalities (LMIs) for an uncertain heat conduction equation and for an uncertain wave equation. These conditions are then utilized to synthesize $H_\infty$ static output-feedback boundary controllers of the systems in question.

Keywords: distributed parameter systems, stability, $H_\infty$ control, Lyapunov functional, LMI.

I. INTRODUCTION

Many important plants, such as flexible manipulators and heat transfer processes are governed by partial differential equations and are often described by models with a significant degree of uncertainties. The existing results [2], [4], [5], [7] on robust control of distributed parameter systems, operating under uncertainty conditions, extend the state space systems. Scalar heat processes and distributed mechanical oscillators are developed for uncertain distributed parameter systems. The paper is organized as follows. Exponential performance in spite of significant model uncertainties. The existing results [2], [4], [5], [7] are among such methods and its primary concern of the present paper.

In our recent papers [6], [11] we have introduced LMI approach to the stability analysis of linear heat and wave equations with the Dirichlet boundary conditions. In the present paper we extend the LMI approach to the Neumann boundary control stabilization and $H_\infty$ control of uncertain systems. The paper is organized as follows. Exponential stability analysis and $L_2$-gain analysis are developed side by side in Sections 2 and 3 for scalar heat processes and, respectively, for distributed mechanical oscillators, governed by semilinear partial differential equations of parabolic and of hyperbolic types. Sufficient exponential stability conditions with a given decay rate are derived in the form of LMIs for these systems. Capabilities of the LMI approach are then tested for designing $H_\infty$ static output-feedback boundary controllers of the systems in question.

A. Notation and Preliminaries

The notation used throughout is fairly standard. The superscript $^T$ stands for matrix transposition, $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space with the norm $\parallel \cdot \parallel$, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n \times n}$ means that $P$ is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $\ast$.

Functions, continuous in all arguments and, respectively, continuously differentiable in all arguments, are referred to as of class $C$ and of class $C^1$.

$L_2(a, b)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi \in [a, b]$ with the corresponding norm.

$L_2(0, \infty; L_2(0, 1))$ is the Hilbert space of square integrable functions $w(\cdot, t) \in L_2(0, \infty)$ with values $w(\xi, \cdot) \in L_2(a, b)$ and with the corresponding norm.

For later use, we recall the following.

Lemma 1: [12]. Let $z \in W^{1,2}([a, b], R)$ be a scalar function with $z(a) = 0$. Then

$$\int_a^b z^2(\xi)d\xi \leq \frac{(b - a)^2}{2} \int_a^b (z'(\xi))^2d\xi.$$  

II. BOUNDARY STABILIZATION OF A SEMILINEAR HEAT EQUATION

A. Exponential Stability

Consider the heat equation

$$z_t(\xi, t) = \frac{\partial}{\partial \xi} [a(\xi, t, z(\xi, t))z(\xi, t)] + r_0(\xi, t, z(\xi, t))z(\xi, t) + r_1(\xi, t, z(\xi, t))z(1, t), \ t \geq t_0, \ 0 \leq \xi \leq 1$$

where $t_0 \in R$ is an initial time instant, $a(\xi, t, z)$ and $r_i(\xi, t, z)$, $i = 0, 1$ are functions of class $C^1$, which may be unknown and which satisfy the inequalities

$$|r_i| \leq \beta_i, \ a \geq a_1 > 0$$

for all $(\xi, t, z) \in [0, 1] \times \mathbb{R}^2$ and for some constants $\beta_0 \geq 0, \beta_1 \geq 0, a_1 > 0$, known a priori. Hereinafter, the dependence on time $t$ and spatial variable $\xi$ is suppressed whenever possible and the functions $a$ and $r_i$ will be written without arguments.

Equation (2) describes the nonlinear propagation of heat in a one-dimensional rod. Due to the presence of the boundary-value term $z(1, t)$ in the state equation, the above model particularly captures significant features of thermal instability in solid propellant rockets [1].

Let the above equation be coupled to the mixed boundary condition

$$z(0, t) = 0, \ z(1, t) = -kz(1, t), \ t \geq t_0$$
with a parameter \( k \geq 0 \) and to the initial condition
\[
z(\xi, t_0) = \phi(\xi) \in L_2(0, 1). \tag{5}
\]
Since the nonlinear terms in (2) locally meet the Lipshitz condition, a unique strong solution of the boundary-value problem (2)–(5) turns out to locally exist [9]. Throughout, only strong solutions are under study.

It is well-known [1] that the linear system (2), (4) with \( k = 0 \) and with constant coefficients \( r_0 = 0, a = 1 \) and \( r_1 > 2 \) is unstable. We are looking for exponential stability conditions for uncertain nonlinear system (2), (4) with \( k \geq 0 \).

Consider the following Lyapunov-Krasovskii functional
\[
V(z(\cdot, t)) = \int^1_0 z^2(\xi, t)d\xi. \tag{6}
\]
We aim to find conditions guaranteeing that along the solutions \( z(\xi, t) \) of (2), (4) the inequality
\[
\frac{d}{dt}V(z(\cdot, t)) + 2\delta V(z(\cdot, t)) \leq 0 \tag{7}
\]
holds. Then by the comparison principle argument (Khalil, 1992), it would follow
\[
\int^1_0 z^2(\xi, t)d\xi = V(z(\cdot, t)) \leq V(z(\cdot, t))e^{-2\delta(t-t_0)} = e^{-2\delta(t-t_0)}\int^1_0 \phi^2(\xi)d\xi.
\]
Therefore, solutions of the boundary value problem (2), (4) would be globally continuable to the right and would satisfy the inequality
\[
\int^1_0 z^2(\xi, t)d\xi \leq e^{-2\delta(t-t_0)}\int^1_0 \phi^2(\xi)d\xi, \quad \forall t \geq t_0. \tag{8}
\]
The heat process (2), (4) would thus be exponentially stable in \( L_2(0, 1) \) with the decay rate \( \delta \).

Differenitavating \( V \) along (2), integrating by parts and taking into account (4), we find that
\[
\frac{d}{dt}V + 2\delta V = 2\int^1_0 z(\xi, t)z_t(\xi, t)d\xi + 2\delta \int^1_0 z^2(\xi, t)d\xi
= 2\int^1_0 z(\xi, t)((\partial_\xi + r_0)z(\xi, t) + r_1 z(1, t))d\xi + 2\int^1_0 z^2(\xi, t)d\xi
+ 2\delta \int^1_0 z^2(\xi, t)d\xi = -2ka_1 z^2(1, t) - 2\int^1_0 az^2(\xi, t)d\xi + 2\int^1_0 (\delta + r_0) z^2(\xi, t)d\xi + 2r_1 \int^1_0 z(\xi, t)z(1, t)d\xi
\leq -2ka_1 z^2(1, t) - 2\int^1_0 az^2(\xi, t)d\xi + 2\int^1_0 (\delta + r_0) z^2(\xi, t)d\xi + 2r_1 \int^1_0 z(\xi, t)z(1, t)d\xi. \tag{9}
\]
Applying inequality (1), we have
\[
-2a_1 \int^1_0 z^2(\xi, t)d\xi \leq -4a_1 \int^1_0 z^2(\xi, t)d\xi.
\]
We thus derive that
\[
\frac{d}{dt}V + 2\delta V \leq \int^1_0 [z(\xi, t)z(1, t)]\Psi[z(\xi, t)z(1, t)]d\xi \leq 0
\]
provided that the following LMI
\[
\Psi \geq \left[ \begin{array}{cc} -4a_1 + 2(\delta + \beta_0) & r_1 \\ r_1 & -2ka_1 \end{array} \right] \leq 0 \tag{10}
\]
is feasible. Since LMI (10) is affine in \( r_1 \) and \( r_1 \in [-\beta_1, \beta_1] \), the latter LMI is feasible if the following LMI
\[
\left[ \begin{array}{cc} -4a_1 + 2(\delta + \beta_0) & \beta_1 \\ \beta_1 & -2ka_1 \end{array} \right] \leq 0
\]
is feasible. We note that the condition \( \beta_0 < 2a_1 \) is necessary for the feasibility of (11). For \( \beta_1 = 0 \) the system (2)–(5) is exponentially stable for all \( k \geq 0 \) and the resulting \( \delta \) is given by
\[
\delta = 2a_1 - \beta_0. \tag{12}
\]
For \( \beta_1 > 0 \) (2)–(5) is exponentially stable with the decay rate \( 0 < \delta < 2a_1 - \beta_0 \) for large enough \( k > 0 \) that can be found from the inequality
\[
-4a_1 + 2(\delta + \beta_0) + \frac{\beta_0^2}{2ka_1} \leq 0. \tag{13}
\]
Summarizing, the following result is concluded.

Theorem 1: Consider the boundary-value problem (2)–(5) with the assumptions above and with \( \beta_0 < 2a_1 \). Given \( \delta \in (0, 2a_1 - \beta_0) \), let there exist \( k \) such that LMI (11) is feasible. Then a unique solution of (2)–(5) is globally continuable to the right and it satisfies (8).

\( B. H_\infty \) Boundary Control

Let us, along with the homogeneous heat process (2), consider its perturbed version
\[
z(\xi, t) = \frac{\partial}{\partial x}[az(\xi, t)] + r_0 z(\xi, t) + r_1 z(1, t) + bw(\xi, t), \quad t \geq t_0, \quad 0 \leq \xi \leq 1 \tag{14}
\]
where \( w(\xi, t) \in L_2(0, \infty; L_2(0, 1)) \) is an external disturbance; \( b = b(\xi, t, z) \) is a function of class \( C^1 \), which is assumed to be uniformly bounded, i.e., \( |b(\xi, t, z)| \leq b_1 \) for all \( (\xi, t, z) \in [0, 1] \times \mathbb{R}^2 \) and some \( b_1 > 0 \).

While internally stabilizing the heat process, the influence of the admissible external disturbance \( w(\xi, t) \in L_2(0, \infty; L_2(0, 1)) \) on the controlled output
\[
z(\xi, t) = [\alpha(\xi, t, z(\xi, t))]z(\xi, t) + d(t, z(1, t))u(t)], \quad t \geq t_0. \tag{15}
\]
Hereinafter, \( u(t) \) is the control input, \( d \) and \( \alpha \) are continuous functions, which are uniformly bounded
\[
|\alpha(\xi, t, z)| \leq \alpha_1, \quad |d(t, z)| \leq d_1, \tag{17}
\]
for all \( (\xi, t, z) \in [0, 1] \times \mathbb{R}^2 \), where \( \alpha_1 \geq 0 \) and \( d_1 \geq 0 \) are some constants. Collocated sensing \( y(t) = z(1, t) \) at the boundary \( \xi = 1 \) is the only available information on the process.

The following \( H_\infty \) control problem is thus under study. Given \( \gamma > 0 \), it is required to find a linear static output feedback
\[
u(t) = -k z(1, t), \tag{18}
\]
that exponentially stabilizes the unperturbed process (4), (14) and leads to a negative performance index
\[
J = \int^\infty_{t_0} \int^1_0 \bar{z}^T(\xi, t)\bar{z}(\xi, t) - \gamma^2 \omega^2(\xi, t)d\xi dt < 0. \tag{19}
\]
for all \(0 \neq w(\xi, t) \in L_2(0, \infty; L_2(0, 1))\) and for all solutions of (14), (16), being initialized with zero data \(z(\xi, t_0) = 0\) and being globally continuable to the right.

In order to solve the problem we carry out conditions that guarantee the following:

\[
W(t) = p \frac{d}{dt} V + \int_0^1 \left[ z^T(\xi, t) z(\xi, t) - \gamma^2 w^2(\xi, t) \right] d\xi < 0,
\]

(20)

where \(p > 0\), \(V\) is given by (6) and the temporal derivative is computed along the trajectories of closed-loop system (4), (14). Then integrating (9) in \(t\) from \(t_0\) to \(\infty\) and taking into account that \(V \geq 0\) and \(V(0) = 0\) would yield (19).

It is worth noticing that

\[
\int_0^1 z^T(\xi, t) z(\xi, t) d\xi = \int_0^1 \alpha^2 z^2(\xi, t) d\xi + d^2 k^2 z^2(1, t).
\]

Then similar to the previous section, we obtain

\[
W = \int_0^1 [2p z(\xi, t) z_\xi(\xi, t) + z^T(\xi, t) z(\xi, t) - \gamma^2 w^2(\xi, t)] d\xi
\]

\[
= \int_0^1 [2p z(\xi, t) \frac{\partial}{\partial \xi} [a z(\xi, t)] + r_1 z(\xi, t) + r_1 z(1, t)
\]

\[
+ b w(\xi, t) + z^T(\xi, t) z(\xi, t) - \gamma^2 w^2(\xi, t)] d\xi
\]

\[
\leq (-2k_1 p + d^2 k^2) z^2(1, t) - 2 p a_1 z^2(\xi, t) d\xi
\]

\[
+ 2 \beta_0 p \int_0^1 z^2(\xi, t) d\xi + \int_0^1 [2p z(\xi, t) |r_1 z(1, t)]
\]

\[
+ b w(\xi, t) + \alpha_1^2 z^2(\xi, t) - \gamma^2 w^2(\xi, t)] d\xi.
\]

Furthermore, applying inequality (1) and setting \(\zeta^T = [z(\xi, t) z(1, t)] w(\xi, t)\), we find that

\[
W \leq \int_0^1 \zeta^T \Psi_\gamma \zeta d\xi < 0
\]

if

\[
\Psi_\gamma \triangleq \left[ \begin{array}{cc}
-4a_1 p + 2 \beta_0 p + \alpha_1^2 & r_1 p \\
\ast & -2 k a p + d^2 k^2 \\
\ast & \ast \\
\end{array} \right] < 0
\]

(21)

is feasible. By Schur complements, the latter inequality holds if

\[
\left[ \begin{array}{ccc}
-4a_1 p + 2 \beta_0 p + \alpha_1^2 & r_1 p & b p \\
\ast & -2 k a p + d^2 k^2 & 0 \\
\ast & \ast & \ast
\end{array} \right] < 0.
\]

(22)

Multiplying (23) by \(diag\{p^{-1}, p^{-1}, 1, 1\}\) from the right and from the left, we denote \(q = p^{-1}\) and \(g = p^{-1} k\). By Schur complements formula we arrive at

\[
\left[ \begin{array}{ccc}
-4a_1 q + 2 \beta_0 q & r_1 & b \\
\ast & -2 a_1 q & 0 \\
\ast & \ast & \ast
\end{array} \right] < 0.
\]

(23)

LMI (24) is affine in \(r_1\) and \(b\) and it is therefore feasible for all \(r_1 \in [-\beta_1, \beta_1]\), \(b \in [-b_1, b_1]\) if it is feasible for \(r_1 = \pm \beta_1\) and \(b = \pm b_1\), thereby yielding 4 LMIs. It is easy to see that these 4 LMIs are equivalent to the following LMI

\[
\left[ \begin{array}{cc}
-4a_1 q + 2 \beta_0 q & b_1 \\
\ast & -2 a_1 q & 0 \\
\ast & \ast & \ast
\end{array} \right] < 0.
\]

Thus, we proved the following.

**Theorem 2:** Consider the perturbed input-output system (14)-(16) with the assumptions above and with \(\beta_0 < 2 a_1\). Given \(\gamma > 0\), let there exist \(q > 0\) and \(g\) such that the LMI (25) is satisfied. Then the static output feedback (18) with \(k = q^{-1} g\) internally exponentially stabilizes the boundary-value problem (14), (16) and attenuates the admissible perturbations \(w(\xi, t) \in L_2(0, \infty; L_2(0, 1))\) in the sense of (19).

**C. Example**

Consider (14)-(17) with

\[
a_1 = 1, \quad b_1 = 1, \quad \beta_0 = 1, \quad \beta_1 = 3, \quad d_1 = 0.1, \quad \alpha_1 = 1.
\]

In this example \(\beta_0 < 2 a_1\) and \(\beta_1 > 0\). Therefore, by Theorem 1 the static output feedback (18) with large enough \(k > 0\) internally exponentially stabilizes the system which appears to be unstable for \(k = 0\) (since \(\beta_1 > 2 a_1\) cf. [1]). By using LMI toolbox of Matlab to verify the feasibility of LMI (25), we find that the static output feedback (18) with \(k = 10.1744\) internally exponentially stabilizes the system and leads to the disturbance attenuation level \(\gamma = 3\). Substituting the resulting \(k\) into (13), we find that this gain exponentially stabilizes the system with \(\delta = 0.7789\).

A lower \(L_2\)-gain \(\gamma = 1.1\) is achieved by a higher gain \(k = 106.01\). The decay rate by the latter gain is found to be \(\delta = 0.9788\).

**III. BOUNDARY STABILIZATION OF A SEMILINEAR WAVE EQUATION**

**A. Exponential Stability**

Consider the wave equation

\[
z_{tt}(\xi, t) = \frac{\partial}{\partial \xi} [a z(\xi, t)] + r_1 z(\xi, t) + r_1 z(1, t)
\]

\[
+ r_2 z(\xi, t),
\]

\[
t \geq t_0, \quad 0 \leq \xi \leq 1
\]

(26)

where \(a = a(\xi)\) and \(r_i = r_i(\xi, t, z, z_t)\), \(i = 0,1,2\) are functions of class \(C^1\). Equation (26) describes nonlinear oscillations of a string. As in the heat equation (2), the functions \(a\) and \(r_i\), \(i = 0,1,2\) are admitted to be unknown subject to inequalities (3) that hold for all \((\xi, t, z, z_t) \in [0,1] \times R^3\) with a priori known constants \(\beta_i \geq 0, i = 0,1,2\) and \(\alpha_i > 0\) whereas due to technical reasons, \(a(\xi)\) does not depend of \(t, z, z_t\) anymore.

To facilitate exposition, we have ignored restoring stiffness of the string, implicitly assuming that the corresponding terms (such as \(r(\xi, t, z_t) z(\xi, t)\) and \(r(\xi, t, z,z_t) z(1, t)\)) are negligible. Since the above simplified model captures all the essential features of the general treatment, the extension to a wave model with a nontrivial stiffness is indeed possible. For the sake of generality, we included the boundary-value...
term $r_1 z_t(1, t)$, similar to that of the parabolic equation (2). However, in contrast to the parabolic case, the stabilization problem still persists even without this term, because the wave equation (26) subject to the boundary condition (27), given below, may present instabilities under $r_1 \equiv 0$.

Let equation (26) be coupled to the mixed boundary condition
\[
\begin{align*}
    z(0, t) &= 0, \\
    z_\xi(1, t) &= -k z_t(1, t), \quad t \geq 0,
\end{align*}
\]  
with a parameter $k > 0$ and to the initial condition
\[
    z(\xi, t_0) = \phi(\xi) \in L_2(0, 1), \\
    z_t(\xi, t_0) = \phi_t(\xi) \in L_2(0, 1).
\]  
(28)

As the nonlinear terms in (26) locally meet the Lipshitz condition, a unique strong solution of the boundary-value problem (26)–(28) turns out to locally exist [9]. As in the heat equation case, only solutions of the boundary-value problem (26)–(28) are under study.

It should be pointed out that the linear system (26), (27), specified with $k = 0$, $a = 1$, and $r_1 = 0$, $i = 0, 1, 2$, generates oscillating solutions and it is therefore asymptotically unstable. As in the case of heat equation, we are looking for exponential stability for uncertain nonlinear system (26), (27) with $k > 0$.

On solutions of (26)–(28), consider the Lyapunov-Krasovskii functional
\[
    V(z(\xi, t), z_t(\xi, t)) = p \int_0^1 a z^2(\xi, t) d\xi + p \int_0^1 z^2(\xi, t) d\xi
    + 2\chi \int_0^1 z_\xi(\xi, t) z_t(\xi, t) d\xi
\]  
proposed in [10] with some constants $p > 0$ and $\chi$ such that
\[
    V(z(\xi, t), z_t(\xi, t)) \geq \varepsilon \int_0^1 [z^2(\xi, t) + z^2_t(\xi, t)] d\xi
\]  
for some $\varepsilon > 0$. The latter inequality holds if
\[
    \begin{bmatrix}
        a_1 p & \chi \\
        \chi & p
    \end{bmatrix} > 0, \quad \forall \xi \in [0, 1],
\]  
i.e., if
\[
    \begin{bmatrix}
        a_1 p & \chi \\
        \chi & p
    \end{bmatrix} > 0.
\]  
(31)

Our aim is to find conditions that would guarantee that along (26) the inequality $\frac{d}{dt} V + 2\delta V \leq 0$ holds. Then, by the comparison principle argument (Khalil, 1992), it would follow that
\[
    \varepsilon \int_0^1 [z^2(\xi, t) + z_t^2(\xi, t)] d\xi = V(z(\xi, t), z_t(\xi, t))
    \leq V(z(\xi, t_0), z_t(\xi, t_0)) e^{-2\delta(t-t_0)}
    \leq M e^{-2\delta(t-t_0)} \int_0^1 [\phi_\xi(\xi) + \phi_\xi(\xi)] d\xi
\]  
for some $M > 0$. Therefore, the solutions of the boundary value problem (26), (27) would globally be continuuable to the right and would satisfy the inequality
\[
    \int_0^1 [z^2(\xi, t) + z_t^2(\xi, t)] d\xi \leq \frac{M}{\varepsilon} e^{-2\delta(t-t_0)} \int_0^1 [\phi^2(\xi) + \phi_t^2(\xi)] d\xi,
\]  
for all $t \geq t_0$. The boundary value problem (26), (27) would thus be exponentially stable with the decay rate $\delta$.

For later use, we derive that
\[
    \frac{d}{dt} \left(2 \int_0^1 \xi_z(\xi, t) z(\xi, t) d\xi \right) = 2 \int_0^1 \xi_z(\xi, t) z_t(\xi, t) d\xi + 2 \int_0^1 \xi z_t(\xi, t) z(\xi, t) d\xi
    = 2 \int_0^1 \xi z_t(\xi, t) z(\xi, t) d\xi + 2 \int_0^1 \xi z_t(\xi, t) d\xi
    + 2 \int_0^1 \xi [r_1 z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t)] z(\xi, t) d\xi
    = 2 \int_0^1 \xi z_t(\xi, t) z(\xi, t) d\xi + 2 \int_0^1 \xi z_t(\xi, t) d\xi
\]  
and by taking into account that $a \geq a_1$, we conclude that
\[
    \frac{d}{dt} V + 2\delta V \leq 2 \int_0^1 \xi_z(\xi, t) z(\xi, t) d\xi < 0.
\]

After integrating by parts, we obtain
\[
    2 \int_0^1 \xi_z z_t d\xi = -2 \int_0^1 \xi_z z_z d\xi - 2 \int_0^1 \xi z_t^2 d\xi + 2z_t^2(1, t).
\]

Therefore,
\[
    2 \int_0^1 \xi_z z_t d\xi = - \int_0^1 \xi z_t^2 d\xi + z_t^2(1, t),
\]
which yields
\[
    \frac{d}{dt} \left(2 \int_0^1 \xi z_t(\xi, t) z(\xi, t) d\xi \right) = - \int_0^1 \xi z_t^2(\xi, t) d\xi + \chi z_t^2(1, t)
    + 2 \int_0^1 \xi z_t(\xi, t) d\xi
\]  
and to the initial condition
\[
    r_1 z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t) z(\xi, t) d\xi
    = - \int_0^1 \xi z_t^2(\xi, t) d\xi + \chi z_t^2(1, t)
\]  
Thus, differentiating $V$ along (26), we obtain
\[
    \frac{d}{dt} V + 2\delta V \leq -2a_1 k p z_t^2(1, t)
\]  
+ 2p \int_0^1 \xi z_t(\xi, t) d\xi + \frac{2}{\delta} \chi z_t^2(1, t)
\]  
+ \chi [\int_0^1 z_t^2(\xi, t) + z_\xi^2(\xi, t) + 2z_t^2(1, t) + a_1 \int_0^1 z_t^2(\xi, t) d\xi]
\]  
+ 2 \int_0^1 \xi [r_0 z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t)] z_\xi(\xi, t) + \chi z_t^2(1, t)
\]  
and to the initial condition
\[
    \begin{bmatrix}
        a_1 p & \chi \\
        \chi & p
    \end{bmatrix} > 0.
\]

Now integrating by parts and taking into account (26) and (27) yield
\[
    \frac{d}{dt} V + 2\delta V \leq -2a_1 k p z_t^2(1, t)
\]  
+ 2p \int_0^1 \xi z_t(\xi, t) d\xi + \chi z_t^2(1, t)
\]  
+ \chi [\int_0^1 z_t^2(\xi, t) + z_\xi^2(\xi, t) + 2z_t^2(1, t) + a_1 \int_0^1 z_t^2(\xi, t) d\xi]
\]  
+ \chi [\int_0^1 \xi z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t)] z_\xi(\xi, t) + \chi z_t^2(1, t)
\]
(34)

Since
\[
    2 \int_0^1 \xi [r_0 z_t(\xi, t) + r_1 z_t(1, t) + r_2 z_\xi(\xi, t)] z(\xi, t) d\xi
\]  
for some $s_0 > 0$ and $s_1 > 0$, by setting $\int (\xi, t) = \int (\xi, t) z(\xi, t) z(\xi, t) d\xi$ and by taking into account that $a \geq a_1$, we conclude that
\[
    \frac{d}{dt} V + 2\delta V \leq \int_0^1 \xi T(\xi, t) \Psi(\xi, t) d\xi < 0,
\]
(34)
if
\[ \Psi = \begin{bmatrix} \psi_1 + \frac{\beta_2^2}{s_1} & 0 & p_{r1} \\ \ast & \psi_2 & 2\chi\delta \xi + p_{r2} \\ \ast & \ast & \psi_3 + \frac{\beta_2^2}{s_0} \end{bmatrix} < 0 \tag{35} \]
where
\[ \psi_1 = -2a_1\kappa p + (1 + a_1k^2)\chi, \]
\[ \psi_2 = -a_1\chi + 2\delta_1p + s_0 + s_1 + 2\chi\xi \beta_2, \]
\[ \psi_3 = -\chi + 2\rho_1\delta_2 + 2\delta_1p. \tag{36} \]

By Schur complements (35) holds if
\[ \begin{bmatrix} \psi_1 & 0 & p_{r1} \\ \ast & \psi_2 & 2\chi\delta \xi + p_{r2} \\ \ast & \ast & \psi_3 \end{bmatrix} < 0. \tag{37} \]

It is worth noticing that given \( k, \) (37) is LMI which is affine in \( \xi \in [0,1], r_i \in [-\beta_i, \beta_i], i = 1,2. \) Therefore, LMI (37) is feasible if the following LMIs in the eight vertices are feasible:
\[ \begin{bmatrix} \psi_1 & \beta_1 & 0 \\ \ast & \psi_2 & 0 \\ \ast & \ast & s_0 \end{bmatrix} < 0, \]
\[ \psi_2^{(j)} = -a_1\chi + 2\delta_1p + s_0 + s_1 + 2\chi\xi \beta_j, \]
\[ i = 1, 2; j = 1, 2; \]
\[ r_m^{(1)} = \beta_m, r_m^{(2)} = -\beta_m, m = 1, 2; \]
\[ \xi(1) = 0, \xi(2) = 1. \tag{38} \]

Since LMIs (38) for \( r_m^{(i)} = \pm \beta_i \) are equivalent, it is sufficient to check the feasibility of the four LMIs (38), where \( r_1^{(1)} = \beta_1 \) and \( j = 1, 2, l = 1, 2. \) We note that for the stability analysis \( p \) can be chosen to be 1. Summarizing, the following result is obtained.

**Theorem 3:** Given \( k > 0 \) and \( \delta > 0, \) let the LMIs (31) and (38) with notations (36) and \( p = 1 \) hold for some \( \chi, s_0 \) and \( s_1. \) Then the boundary-value problem (26), (27) is exponentially stable with the decay rate \( \delta. \)

**B. \( H_\infty \) Boundary Control**

In addition to the wave equation (26), let us now consider its perturbed version
\[ z_{ul}(\xi, t) = \frac{2}{\mu}[a_2z_{l}] + r_0 z_{l}(\xi, t) + r_1 z_{l}(1, t) + r_2 z_{l}(\xi, t) + bw(\xi, t), \]
\[ t \geq 0, \ 0 \leq \xi \leq 1 \tag{39} \]
where \( w(\xi, t) \in L_2(0, \infty; L_2(0, 1)) \) is an external disturbance; \( b = b(\xi, t, z) \) is a function of class \( C^1, \) which is assumed to be uniformly bounded, i.e., \( \|b(\xi, t, z)\| \leq b_1 \) for all \( \xi, t, z \in [0,1] \times R^2 \) and some \( b_1 > 0. \) While internally stabilizing the wave process, the influence of the admissible external disturbance on the controlled output
\[ \bar{z}(\xi, t) = (az_{l} + a_2z_{l}) \xi + d(t, z_{l}(1, t))u(t), \tag{40} \]
is to be attenuated through the actuation at the right-hand end
\[ z(0, t) = 0, \ z_{l}(1, t) = u(t), \ t \geq t_0. \tag{41} \]

Hereinafter, \( u(t) \) is the control input, \( d \) and \( \alpha = (\xi, t, z), \ \alpha = \tilde{\alpha}(\xi, t, z) \) are continuous functions, which are uniformly bounded
\[ |\alpha(\xi, t, z, z_1)| \leq \alpha_0, \ |\tilde{\alpha}(\xi, t, z, z_1)| \leq \alpha_1, \ |d(t, z)| \leq d_1, \]
for all \( (\xi, t, z, z_1) \in [0,1] \times R^3, \) where \( \alpha_i \geq 0, i = 0, 1 \) and \( d_1 \geq 0 \) are some constants. Collocated sensing \( y(t) = z_{l}(1, t) \) at the boundary \( \xi = 1 \) is the only available information on the process.

The \( H_\infty \) control problem of interest is stated as follows. Given \( \gamma > 0, \) it is required to find a linear static output feedback
\[ u(t) = -kz_{l}(1, t), \tag{42} \]
that exponentially stabilizes the unperturbed system (26), (27) and leads to a negative performance index
\[ J = \int_0^\infty \int_0^1 [\bar{z}^T(\xi, t)\bar{z}(\xi, t) - \gamma^2w^2(\xi, t)]d\xi dt < 0 \tag{43} \]
for all \( 0 \neq w(\xi, t) \in L_2(0, \infty; L_2(0, 1)) \) and for all solutions of (39), (41), being initialized with zero data \( z(\xi, t_0) = z_{l}(1, t_0) = 0 \) and being globally continuable to the right.

For solving the stated problem, let us find conditions that guarantee the following:
\[ W(t) \leq \frac{d}{dt}V + \int_0^1 [\bar{z}^T(\xi, t)\bar{z}(\xi, t) - \gamma^2w^2(\xi, t)]d\xi dt < 0, \tag{44} \]
where \( V \) is given by (29), and the temporal derivative is computed along the closed-loop system (39), (41). We have
\[ \int_0^1 \bar{z}^T(\xi, t)\bar{z}(\xi, t)d\xi \leq \int_0^1 \left[ a_0^2z_{l}^2(\xi, t) + a_1^2z_{l}^2(\xi, t) \right]d\xi \leq \int_0^1 \left[ \frac{a_0^2}{b_0}z_{l}^2(\xi, t) + \frac{a_1^2}{b_1}z_{l}^2(\xi, t) \right]d\xi \]
\[ + d_1^2k_1^2z_{l}^2(1, t) \right]d\xi, \]
where inequality (1) has been used. In analogy to (34), we have
\[ \frac{d}{dt}V \leq -2a_1\kappa p + (1 + a_1k^2)\chi, \]
\[ +2p\int_0^1 z_{l}(\xi, t)|r_0 z_{l}(\xi, t) + r_1 z_{l}(1, t) + r_2 z_{l}(\xi, t) + bw|d\xi \]
\[ +\chi \left[ -\int_0^1 (z_{l}^2 + a_2z_{l}^2) d\xi + (1 + a_1k^2)z_{l}^2(1, t) \right] +2\int_0^1 \chi|z_{l}(\xi, t) + r_1 z_{l}(1, t) + r_2 z_{l}(\xi, t) + bw|z_{l}d\xi \]
\[ \leq \int_0^1 \left[ \frac{d_2}{b_0}z_{l}^2(\xi, t) + \frac{a_1^2}{b_1}z_{l}^2(\xi, t) + \frac{d_1^2}{b_2}w^2 \right]d\xi \]
\[ + (s_0 + s_1 + s_2 + 2\chi(\beta_2)z_{l}^2(\xi, t))d\xi, \]
for some \( s_0 > 0 \) and \( s_1 > 0. \) By taking into account that \( a \geq a_1, \) we conclude that
\[ W = \frac{d}{dt}V + \int_0^t [\bar{z}^T \bar{z} - \gamma^2w^2]d\xi \leq \bar{z}^T \Psi \bar{z}, \]
where
\[ \zeta^T = [z_t(1, t) \ z_\xi(\xi, t) \ z_t(\xi, t) \ w(\xi, t)], \]
\[ \Psi = \begin{bmatrix}
\psi_1 + \alpha_1^2 k^2 & \beta_0 & \beta_1 & \alpha_0 \\
\psi_{2\gamma} & \rho & \beta_0 & 0 \\
* & * & \gamma & 0 \\
* & * & * & s_0 \\
* & * & * & * & s_2
\end{bmatrix}, \]
and
\[ \psi_1 = -2a_1 k p + (1 + a_1 k^2) \chi, \]
\[ \psi_{2\gamma} = -a_1 \chi + \sum_{i=0}^2 s_i + \frac{1}{2} \alpha_0^2 + 2 \chi \beta_2, \]
\[ \psi_{3\gamma} = -\chi + 2 \beta \beta_0 + \alpha_1^2. \]

Therefore \( W < 0 \) if \( \Psi < 0 \), i.e. by Schur complements, if
\[ \begin{bmatrix}
\psi_1 + \alpha_1^2 k^2 & \beta_0 & \beta_1 & \alpha_0 \\
\psi_{2\gamma} & \rho & \beta_0 & 0 \\
* & * & \gamma & 0 \\
* & * & * & s_0 \\
* & * & * & * & s_2
\end{bmatrix} < 0. \]

LMI (46) is affine in \( r_i \in [-\beta_i, \beta_i], i = 1, 2 \) and \( b \in [-\beta_1, \beta_1] \). Therefore, it is feasible if it holds in the vertices \( r_i = \pm \beta_i \) and \( b = \pm b_1 \). It is easy to see that the eight LMIs in the vertices are equivalent to the following one
\[ \begin{bmatrix}
\psi_1 + \alpha_1^2 k^2 & 0 & \beta_0 & \beta_1 & 0 \\
\psi_{2\gamma} & \rho & 0 & 0 & 0 \\
* & * & \gamma & 0 & 0 \\
* & * & * & -s_0 & 0 \\
* & * & * & * & -s_2
\end{bmatrix} < 0. \]

We note that if (47) is feasible, then the LMIs (38) for exponential stability hold with small enough \( \delta > 0 \). We thus proved the following.

Theorem 4: Consider the perturbed input-output system (39)–(41) with the assumptions above. Given \( \gamma > 0 \) and \( k > 0 \), let there exist \( p > 0, \chi, s_0, s_1 \) and \( s_2 \) such that the LMIs (31) and (47) are satisfied with the notations given by (45). Then the static output feedback (42) internally exponentially stabilizes the boundary value problem (39), (41) and attenuates the external disturbances \( w(\xi, t) \in L_2(0, \infty; L_2(0, 1)) \) in the sense of (43).

C. Example

Consider (39)–(41) with
\[ \alpha_1 = 2, \quad \beta_0 = \beta_1 = 0.3, \quad \beta_2 = 0.4, \]
\[ \alpha_0 = \alpha_1 = 1, \quad d_1 = 0.1. \]

As mentioned above, the open-loop system is unstable. By using LMI toolbox of Matlab to verify LMIs (31) and (47), we find that the static output feedback (42) with \( k = 1 \) internally exponentially stabilizes the system and attenuates the external disturbances with \( \gamma = 8.9 \). By verifying (38) in the four vertices, we find that the resulting closed-loop system is internally exponentially stable with the decay rate \( \delta = 0.06 \).

IV. Conclusions

In the present paper an LMI approach is extended to \( H_\infty \) boundary control of uncertain semilinear heat equations and wave equations. The uncertainties are admitted to be time- space- and state-dependent with \( a \) priori known upper/lower bounds. Sufficient conditions for static output feedback stabilization are given in terms of LMIs. Numerical examples illustrate the efficiency of the method.

The proposed method seems to be extendible to dynamic output feedback \( H_\infty \) control and to other classes of distributed parameter systems. LMIs are thus expected to provide effective tools for robust control of distributed parameter systems.

REFERENCES