Controllability properties of discrete-spectrum Schrödinger equations

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Abstract—We state an approximate controllability result for the bilinear Schrödinger equation in the case in which the uncontrolled Hamiltonian has discrete non-resonant spectrum. This result applies both to bounded or unbounded domains and to the case in which the control potential is bounded or unbounded. In addition we get some controllability properties for the density matrix. Finally we show, by means of specific examples, how these results can be applied.

I. INTRODUCTION

In this paper we study the controllability of the bilinear Schrödinger equation. Its importance is due to applications to modern technologies such as Nuclear Magnetic Resonance, laser spectroscopy, and quantum information science (see for instance [1], [2], [3], [4]).

Many controllability results are available when the state space is finite dimensional, e.g., for spin systems or for molecular dynamics when one neglects interactions with highly excited levels (see for instance [5], [6]). When the state space is infinite-dimensional the controllability problem appears to be much more intricate.

We consider the controllability problem for the following bilinear system representing the Schrödinger equation driven by one external field

$$\frac{d\psi}{dt}(t) = (H_0 + u(t)H_1)\psi(t).$$  

(1)

Here the wave function $\psi$ evolves in an infinite-dimensional Hilbert space, $H_0$ is a self-adjoint operator called drift Hamiltonian (i.e. the Hamiltonian responsible for the evolution when the external field is not active), $u(t)$ is a scalar control function, and $H_1$ is a self-adjoint operator describing the interrelation between the system and the external field.

The reference case is the one in which the Hilbert space is $L^2(\Omega)$ where $\Omega$ is either $\mathbb{R}^d$ or a bounded domain of $\mathbb{R}^d$, and equation (1) reads

$$\frac{\partial \psi}{\partial t}(t, x) = (-\Delta + V(x) + u(t)W(x))\psi(t, x),$$  

(2)

where $\Delta$ is the Laplacian (with Dirichlet boundary condition in the case in which $\Omega$ is bounded) and $V$ and $W$ are suitably regular functions defined on $\Omega$. However the setting of the paper covers more general cases (for instance $\Omega$ can be a Riemannian manifold and $\Delta$ the corresponding Laplace-Beltrami operator).

Besides the fact that one cannot expect exact controllability on the whole Hilbert sphere (see [7], [8]) and some negative result (in particular [9], [10]) only few approximate controllability results are available and concern mainly special situations. It should be mentioned, however, that several results on efficient steering of the Schrödinger equation without any controllability assumptions are available, e.g., [11], [12], [13]. For optimal control results for finite dimensional quantum systems see, for instance, [14], [15], [16], [17].

In [18], [19] Beauchard and Coron study the controllability of a quantum particle in a 1D potential well with $W(x) = x$. Their results are highly nontrivial and are based on Coron’s return method (see [20]) and Nash–Moser’s theorem. In particular, they prove that the system is exactly controllable in the unit sphere of the Sobolev space $H^7$ (implying in particular approximate controllability in $L^2$). One of the most interesting corollaries of this result is exact controllability between eigenstates.

A different result is given in [21], where adiabatic methods are used to prove approximate controllability for systems having conical eigenvalue crossings in the space of controls.

Another controllability result has been proved by Mirrahimi in [22] using Strichartz estimates and concerns approximate controllability for a certain class of systems such that $\Omega = \mathbb{R}^d$ and whose drift Hamiltonian has mixed spectrum (discrete and continuous).

The aim of the present paper is to state a general approximate controllability result for a large class of systems for which the drift Hamiltonian $H_0$ has discrete spectrum satisfying a non-resonance condition, while $H_1$ couples each pair of distinct eigenstates of $H_0$. The proof of this controllability result relies on finite-dimensional techniques applied to the Galerkin approximations and it is presented in [23].

A feature of our method is that the infinite-dimensional system inherits, in a suitable sense, controllability results for the group of unitary transformations from those of the Galerkin approximations. This permits to extract controllability properties for the density matrix. Let us stress that, as it happens in finite dimension, controllability properties for the density matrix cannot in general be deduced from those of the wave function (see for instance [24]).

The paper is organized as follows. In Section II we present the general functional analysis setting and we state our main result (Theorem 2.4) for the control system (1). We also show how this result applies to the Schrödinger equation (2) when...
\( \Omega \) is both bounded or unbounded. In Section III we give a sketch of the proof of the main result. In Section IV we extend Theorem 2.4 to the controlled evolution of the density matrix (Theorem 4.2). Finally in Section V we show how Theorem 2.4 and Theorem 4.2 can be applied to specific cases, namely the harmonic oscillator and the 3D potential well, for suitable controlled potentials. In particular, we show how to get controllability results even in cases in which \( V \) does not satisfy the required non-resonance hypothesis, using perturbation arguments.

II. MATHEMATICAL FRAMEWORK AND STATEMENT OF THE MAIN RESULT

Hereafter \( N \) denotes the set of strictly positive integers. Definition 2.1 below provides the abstract mathematical framework that will be used to formulate the controllability results later applied to the Schrödinger equation (2).

Definition 2.1: Let \( \mathcal{H} \) be a complex Hilbert space and \( U \) be a subset of \( \mathbb{R} \). Let \( A, B \) be two, possibly unbounded, operators on \( \mathcal{H} \) with values in \( \mathcal{H} \) and denote by \( D(A) \) and \( D(B) \) their domains. The control system \((A, B, U)\) is the formal controlled equation

\[
\frac{d\psi}{dt}(t) = A\psi(t) + u(t)B\psi(t), \quad u(t) \in U. \tag{3}
\]

We say that \((A, B, U)\) is a skew-adjoint discrete-spectrum control system if the following conditions are satisfied: (H1) \( A \) and \( B \) are skew-adjoint, (H2) there exists an orthonormal basis \( \{\phi_n\}_{n \in \mathbb{N}} \) of \( \mathcal{H} \) made of eigenvectors of \( A \), (H3) \( \phi_n \in D(B) \) for every \( n \in \mathbb{N} \).

In order to give a meaning to the evolution equation (3), at least when \( u \) is constant, we should ensure that the sum \( A + uB \) is well defined. The standard notion of sum of operators seen as quadratic forms (see [25]) is not always applicable under the sole hypotheses (H1), (H2), (H3). An adapted definition of \( A + uB \) can nevertheless be given as follows: hypothesis (H3) guarantees that the sum \( A + uB \) is well defined on \( V = \text{span}\{\phi_n \mid n \in \mathbb{N}\} \). Any skew-Hermitian operator \( C : V \to \mathcal{H} \) admits a unique skew-adjoint extension \( \mathcal{E}(C) \). We identify \( A + uB \) with \( \mathcal{E}(A|_V + uB|_V) \).

Let us notice that when \( A + uB \) is well defined as sum of quadratic forms and is skew-adjoint then the two definitions of sum coincide. This happens in particular for the Schrödinger equation (2) in most physically significant situations.

A crucial consequence of what precedes is that for every \( u \in U \) the skew-adjoint operator \( A + uB \) generates a group of unitary transformations \( e^{t(A+uB)} : \mathcal{H} \to \mathcal{H} \). In particular, the unit sphere \( S \) of \( \mathcal{H} \) satisfies \( e^{t(A+uB)}(S) = S \) for every \( u \in U \) and every \( t \geq 0 \).

Due to the dependence of the domain \( D(A + uB) \) on \( u \), the solutions of (3) cannot in general be defined in classical (strong, mild or weak) sense. Let us mention that, in some relevant cases in which the spectrum of \( A \) has a nontrivial continuous component the solution can be defined as in [26], [27] by means of Strichartz estimates.

We will say that the solution of (3) with initial condition \( \psi_0 \in \mathcal{H} \) and corresponding to the piecewise constant control \( u : [0, T] \to U \) is the curve \( t \mapsto \psi(t) \) defined by

\[
\psi(t) = e^{t(A+u(t)B)}\psi_0, \quad t \in [0, T],
\]

where \( \sum_{j=1}^{m-1} t_j \leq t < \sum_{j=1}^{m} t_j \) and \( u(t) = u_j \) if \( \sum_{j=1}^{m-1} t_j \leq t < \sum_{j=1}^{m} t_j \). Notice that such a \( \psi(t) \) satisfies, for every \( n \in \mathbb{N} \) and almost every \( t \in [0, T] \), the differential equation

\[
\frac{d}{dt}\langle \psi(t), \phi_n \rangle = -\langle \psi(t), (A + u(t)B)\phi_n \rangle. \tag{5}
\]

Remark 2.2: The notion of solution introduced above makes sense in very degenerate situations and can be enhanced when \( B \) is bounded. Indeed, well-known results assert that in this case if \( u \in L^1([0, T], U) \) then there exists a unique weak (and mild) solution \( \psi \in C([0, T], \mathcal{H}) \) which coincides with the curve (4) when \( u \) is piecewise constant. Moreover, if \( \psi_0 \in D(A) \) and \( u \in C^1([0, T], U) \) then \( \psi \) is differentiable and it is a strong solution of (3). (See [7] and references therein.)

Definition 2.3: Let \((A, B, U)\) be a skew-adjoint discrete-spectrum control system. We say that \((A, B, U)\) is approximately controllable if for every \( \psi_0, \psi_1 \in S \) and every \( \varepsilon > 0 \) there exist \( k \in \mathbb{N}, t_1, \ldots, t_k > 0 \) and \( u_1, \ldots, u_k \in U \) such that

\[
\|\psi_1 - e^{t_k(A+u_kB)} \cdots e^{t_1(A+u_1B)}(\psi_0)\| < \varepsilon.
\]

Let, for every \( n \in \mathbb{N} \), \( i\lambda_n \) denote the eigenvalue of \( A \) corresponding to \( \phi_n \) (\( \lambda_n \in \mathbb{R} \)). Our general result is the following.

Theorem 2.4: Let \( \delta > 0 \) and \((A, B, (0, \delta))\) be a skew-adjoint discrete-spectrum control system. If the elements of the sequence \( (\lambda_{n+1} - \lambda_n)_{n \in \mathbb{N}} \) are \( \mathbb{Q} \)-linearly independent and if \((B\phi_n, \phi_{n+1}) \neq 0 \) for every \( n \in \mathbb{N} \), then \((A, B, (0, \delta))\) is approximately controllable.

Recall that the elements of the sequence \( (\lambda_{n+1} - \lambda_n)_{n \in \mathbb{N}} \) are said to be \( \mathbb{Q} \)-linearly independent if for every \( N \in \mathbb{N} \) and \((q_1, \ldots, q_N) \in \mathbb{Q}^N \setminus \{0\} \) one has \( \sum_{n=1}^{N} q_n(\lambda_{n+1} - \lambda_n) = 0 \).

Under suitable assumptions on the domain \( \Omega \) and on the potentials \( V, W \) it turns out that the Schrödinger equation (2) falls into the previous abstract setting (see for instance [28, Theorem 1.2.2] and [29, Theorems XIII.69 and XIII.70]). In particular we have the following result.

Corollary 2.5: Let \( \Omega \) be an open subset of \( \mathbb{R}^d \), \( V, W \) be two real-valued functions defined on \( \Omega \), and \( U \) be a subset of \( \mathbb{R} \). Assume either that (i) \( \Omega \) is bounded, \( V, W \) belong to \( L^\infty(\Omega, \mathbb{R}) \) or that (ii) \( \Omega = \mathbb{R}^d \), \( V, W \) belong to \( L^1_{loc}(\mathbb{R}^d, \mathbb{R}) \), the growth of \( W \) at infinity is at most exponential and, for every \( u \in U \), \( \lim_{|x| \to +\infty}(V(x) + uW(x)) = +\infty \) and \( \inf_{x \in \mathbb{R}^d}(V(x) + uW(x)) > -\infty \).

Denote by \( (\lambda_k)_{k \in \mathbb{N}} \) the sequence of eigenvalues of \( -A + V + W \) and by \( (\phi_k)_{k \in \mathbb{N}} \) an orthonormal basis of \( L^2(\Omega, C) \) of corresponding real-valued eigenfunctions. Assume, in addition to (i) or (ii), that \( U \) contains the interval \((0, \delta)\) for some \( \delta > 0 \), that the elements of \( (\lambda_{k+1} - \lambda_k)_{k \in \mathbb{N}} \) are \( \mathbb{Q} \)-linearly independent, and that \( \int_{\Omega} W(x)\phi_k\phi_k dx \neq 0 \) for every \( k \in \mathbb{N} \). Then the controlled Schrödinger equation (2) associated with \( \Omega, V, W \) and \( U \) is approximately controllable.
III. SCHEME OF THE PROOF

In this section we briefly outline the main steps leading to the proof of Theorem 2.4 (details can be found in [23]).

First step. First remark that, if \( u \neq 0 \), \( e^{t(A+B)} = e^{tu((1/2)A+B)} \). Theorem 2.4 is therefore equivalent to the following property: if the terms \((\lambda_{j+1} - \lambda_j)_{j \in \mathbb{N}}\) are \(Q\)-linearly independent and if \( \langle B\phi_j, \phi_{j+1} \rangle \neq 0 \) for every \( n \in \mathbb{N} \), then for every \( \delta, \varepsilon > 0 \) and every \( \psi_0, \psi_1 \in S \) there exist \( k \in \mathbb{N} \), \( t_1, \ldots, t_k > 0 \) and \( u_1, \ldots, u_k > \delta \) such that

\[
\|\psi_1 - e^{t_k(u_kA+B)} \circ \cdots \circ e^{t_1(u_1A+B)}(\psi_0)\| < \varepsilon. \tag{6}
\]

In other words, the system for which the roles of \( A \) and \( B \) as drift and controlled field are inverted, namely,

\[
\frac{d\psi}{dt}(t) = u(t)A\psi(t) + B\psi(t), \quad u(t) \in U, \tag{7}
\]

is approximately controllable provided that the control set \( U \) contains a half-line. The notion of solution of (7) corresponding to a piecewise constant control function is defined as in (4).

Second step. Let, for every \( j, k \in \mathbb{N} \), \( b_{jk} = \langle B\phi_j, \phi_k \rangle \) and \( a_{jk} = \langle A\phi_j, \phi_k \rangle = i\lambda_j b_{jk} \). Define, for every \( n \in \mathbb{N} \), the two complex-valued \( n \times n \) matrices \( A(n) = (a_{jk})_{1 \leq j, k \leq n} \) and \( B(n) = (b_{jk})_{1 \leq j, k \leq n} \). These two matrices give rise to the Galerkin approximations of (7). To these finite-dimensional systems we can apply the following controllability result.

Proposition 3.1: Let \( \mathcal{A} = (a_{jk})_{n \times n} \), \( \mathcal{B} = (b_{jk})_{n \times n} \) be two skew-symmetric matrices and assume that \( \mathcal{A} \) is diagonal and \( \mathcal{B} \) is connected (i.e. for every pair of indices \( j, k \in \{1, \ldots, n\} \) there exists a finite sequence \( r_1, \ldots, r_l \in \{1, \ldots, n\} \) such that \( b_{j_{r_1}, b_{j_{r_2}}, \ldots, b_{j_{r_l}}, b_{k} \neq 0} \). Assume moreover that \( |a_{jj} - a_{kk}| \neq |a_{ll} - a_{mm}| \) if \( \{j, k\} \neq \{l, m\} \). Then the control system \( (\Sigma) : \dot{x} = uAx + Bx \) is controllable in \( S_n \) with piecewise constant controls with values in \( U \subset \mathbb{R} \), provided that \( U \) contains at least two points.

Third step. Given \( u : [0, T] \to \mathbb{R} \) let us represent the matrix \( M(t) = e^{-t((1/2)A+B)}e^{t((1/2)A+N)} \), where \( v(t) = \int_0^t u(\tau) d\tau \), in block form as follows

\[
M(t) = \begin{pmatrix} M^{(n,n)}(t) & M^{(n,N-n)}(t) \\ M^{(N-n,n)}(t) & M^{(N-n,N-n)}(t) \end{pmatrix}, \tag{8}
\]

where the superscripts indicate the dimensions of each block.

Claim 3.2: There exists a sequence of piecewise constant control functions \( u_k : [0, T] \to (0, \infty) \) such that the sequence of matrix-valued curves

\[
t \mapsto M_k(t) = e^{-v_k(t)(A+N)}B^{(N)}e^{v_k(t)(A+N)},
\]

where \( v_k(t) = \int_0^t u_k(\tau) d\tau \), converges uniformly w.r.t. to

\[
t \mapsto \bar{M}(t) = \begin{pmatrix} M^{(n,n)}(t) & 0_{n \times (N-n)} \\ 0_{(N-n) \times n} & M^{(N-n,N-n)}(t) \end{pmatrix},
\]

in the integral sense \( \int_0^t M_k(\tau) d\tau \overset{k \to \infty}{\longrightarrow} \int_0^t \bar{M}(\tau) d\tau \).

Fourth step. Let us fix two states \( \psi_0, \psi_1 \in \mathscr{H} \). Our aim is to approximately steer the system between \( \psi_0 \) and \( \psi_1 \). Let \( \varepsilon > 0 \). Let \( \Pi_n \) be the projection on the space generated by the first \( n \) eigenvectors of \( A \). Then, if we take \( n \) large enough and we apply Proposition 3.1 and Claim 3.2 with a suitable \( N \) it is possible to prove the following.

Claim 3.3: For \( k \) large enough,

\[
\|\Pi_n(e^{-v_k(T)}A\psi_k(t)) - \Pi_n(e^{-v(T)}A\psi_1(t))\| < \varepsilon \tag{9}
\]

where \( u(\cdot) \) is a control connecting \( \Pi_n(\psi_0) \) to \( \Pi_n(\psi_1) \) in the Galerkin approximation, \( v(t) = \int_0^t u(\tau) d\tau \), \( u_k(\cdot) \) is the trajectory associated to \( u_k(\cdot) \) starting from \( \psi_0 \).

Fifth step. The last step of the proof consists in showing that it is possible to get rid of the differences of phase in an arbitrary short time so that

\[
\|\Pi_n(\psi_k(T)) - \Pi_n(\psi_1(t))\| < \varepsilon.
\]

Finally the proof of Theorem 2.4 can be completed showing that the “queues” \( \|\psi_1 - \Pi_n(\psi_1)\| \) and \( \|\psi_k(T) - \Pi_n(\psi_k(T))\| \) can be made arbitrary small by choosing a suitably large \( n \) and a suitably small \( \varepsilon \).

IV. CONTROLLABILITY FOR DENSITY MATRICES

A. Physical motivations

A density matrix (sometimes called density operator) is a non-negative, self-adjoint operator of trace class [29, Vol. I] on a Hilbert space. The trace of a density matrix is normalized to one. As a consequence of the definition a density matrix is a compact operator (hence with discrete spectrum) and can always be written as a weighted sum of projectors,

\[
\rho = \sum_{j=1}^{\infty} P_j \varphi_j \varphi_j^*, \tag{10}
\]

where \( P_j \in [0,1], \sum_j P_j = 1, \) and \( \varphi_j \varphi_j^* \) is the orthogonal projector on the space spanned by \( \varphi_j \) with \( \varphi_j^* (\cdot) = \langle \varphi_j, \cdot \rangle \). Here \( \{\varphi_j\}_{j \in \mathbb{N}} \) is a set of normalized vectors not necessarily orthogonal.

The density matrix is used to describe the evolution of systems whose initial wave function is not known precisely, but only with a certain probability, or when one is dealing with an ensemble of identical systems that cannot be prepared precisely in the same state. More precisely (10) describes a system whose state is known to be \( \varphi_j \) with probability \( P_j, j \in \mathbb{N} \).

The time evolution of the density matrix is determined by the evolutions of the states \( \varphi_j \), namely

\[
\rho(t) = U(t)\rho(0)U^*(t) \tag{11}
\]

where \( U(t) \) is the operator of temporal evolution (the resolvent) and \( U^*(t) \) its adjoint. Notice that the spectrum of \( \rho(t) \) is constant along the motion.

B. Statement of the result

Fix \( \delta > 0 \) and let \( (A,B,(0,\delta)) \) be a skew-adjoint discrete-spectrum control system on a Hilbert space \( \mathscr{H}, \{\varphi_j\}_{j \in \mathbb{N}} \) an orthonormal basis of \( \mathscr{H} \) (not necessarily of eigenvectors of
A), \{P_j\}_{j \in \mathbb{N}} a sequence of non-negative numbers such that \(\sum_{j=1}^{\infty} P_j = 1\), and denote by \(\rho\) the density matrix
\[
\rho = \sum_{j=1}^{\infty} P_j \varphi_j \varphi_j^*.
\]

**Definition 4.1:** Two density matrices \(\rho_0\) and \(\rho_1\) are said to be unitarily equivalent if there exists a unitary transformation \(U\) of \(\mathcal{H}\) such that \(\rho_1 = U \rho_0 U^*\).

Obviously the controllability question for the evolution of the density matrix makes sense only for pairs \((\rho_0, \rho_1)\) of initial and final density matrices that are unitarily equivalent. Notice that this is a quite strong assumption, since it implies that the eigenvalues of \(\rho_0\) and \(\rho_1\) are the same. Controllability results in the case of density matrices that are not unitarily equivalent have been obtained in the case of open systems (i.e. systems evolving under a suitable nonunitary evolution) in the finite-dimensional case. See for instance [30].

For density matrices the following generalization of Theorem 2.4, proved in [23], holds.

**Theorem 4.2:** Let \(\rho_0\) and \(\rho_1\) be two unitarily equivalent density matrices. Then, under the hypotheses of Theorem 2.4, for every \(\varepsilon > 0\) there exists a piecewise constant control steering the density matrix from \(\rho_0\) \(\varepsilon\)-approximately to \(\rho_1\) i.e. there exist \(k \in \mathbb{N}\), \(t_1, \ldots, t_k > 0\) and \(u_1, \ldots, u_k \in (0, \delta)\) such that setting \(V = e^{t_k(A+u_k B)} \cdots \circ e^{t_1(A+u_1 B)}\), one has \(\|\rho_1 - V \rho_0 V^*\| < \varepsilon\), where \(\| \cdot \|\) denotes the operator norm on \(\mathcal{H}\).

**Remark 4.3:** As Corollary 2.5 is a particularization of Theorem 2.4 to the controlled Schrödinger equation, the hypotheses of Corollary 2.5 imply \(\varepsilon\)-approximate controllability of the corresponding density matrix.

V. EXAMPLES

**A. Perturbation of the spectrum**

The scope of Section V is to show how the general controllability results obtained in the previous sections can be applied in specific cases. In particular, we want to show how the conditions on the spectrum of the Schrödinger operator appearing in the hypotheses of Corollary 2.5 can be checked in practice.

Throughout this section we assume that one of the hypotheses (i) or (ii) of Corollary 2.5 holds true. Thus, \((A, B, U)\) is a well-defined controlled Schrödinger equation, where \(A = -i(-\Delta + V)\) and \(B = -iW\).

The study of the examples below is based on the simple idea that, even if the hypotheses of Corollary 2.5 are not satisfied by the operators \(A\) and \(B\), one can anyway ensure that they hold true for \(A_{\mu} = -i(-\Delta + V + \mu W)\) and \(B_{\mu} = -iW\) for some \(\mu\) in the interior of \(U\). This is enough to conclude that the system \(\dot{\psi} = A\psi + uB\psi\), \(u \in U\), is approximately controllable, since the replacement of \((A, B)\) by \((A_{\mu}, B_{\mu})\) corresponds to a reparameterization of \(U\) that sends \(u\) into a new control \(u - \mu \in U - \mu\) and \(V\) into \(V + \mu W\). Although the spectrum of \(A_{\mu}\) is not in general explicitly computable, we can nevertheless deduce some crucial properties about it by applying standard perturbation arguments. Theorem 5.1 recalls, in a simplified version suitable for our purposes, some classical perturbation results describing the dependence on \(\mu\) of the spectrum of \(-\Delta + V + \mu W\). (See [31, Chapter VII, Remark 4.22], [32, §II.10, Theorem 1] and also [33].)

**Theorem 5.1:** Let \(U\) be an open interval containing zero. Assume either that (i) \(\Omega\) is bounded, \(V, W\) belong to \(L^\infty(\Omega)\) or that (ii) \(\Omega = \mathbb{R}^d, V\) belongs to \(L^\infty(\mathbb{R}^d)\), \(W\) belongs to \(L^\infty(\mathbb{R}^d)\), \(\lim_{|x|\to\infty} V(x) = +\infty\) and \(\inf_{x \in \mathbb{R}^d} V(x) > -\infty\). In both cases (i) and (ii) assume that each eigenvalue of the Schrödinger operator \(-\Delta + V\) is simple. Denote by \((\Lambda_k)_{k \in \mathbb{N}}\) the sequence of eigenvalues of \(-\Delta + V\) and by \((\phi_k)_{k \in \mathbb{N}}\) the corresponding eigenfunctions. Then, for any \(k\) in \(\mathbb{N}\), there exist two analytic curves \(A_k : U \to \mathbb{C}\) and \(\Phi_k : U \to L^2(\Omega)\) such that:

- \(\Lambda_k(0) = \lambda_k\) and \(\Phi_k(0) = \phi_k\);
- for any \(\mu \in U\), \((A_k(\mu))_{k \in \mathbb{N}}\) is the family of eigenvalues of \(-\Delta + V + \mu W\) counted according to their multiplicities and \((\Phi_k(\mu))_{k \in \mathbb{N}}\) is an orthonormal basis of corresponding eigenfunctions;
- \(A'_k(0) = \int_{\Omega} W(x)|\phi_k(x)|^2 dx\).

We check below that if the derivatives \(A'_k(0)\) are \(Q\)-linearly independent then for almost every \(\mu \in U\) the eigenvalues of \(-\Delta + V + \mu W\) are \(Q\)-linearly independent. This fact is used in the following to apply Corollary 2.5 to situations in which the uncontrolled Schrödinger operator has a resonant spectrum.

Setting \(b_{jk} = \langle B \phi_j, \phi_k \rangle\) for any pair of integers \(j, k \in \mathbb{N}\), one has
\[
b_{jk} = \int_{\Omega} W(x)\phi_j(x)\phi_k(x) dx. \quad (12)
\]

In particular, \(A'_k(0) = \int_{\Omega} W(x)|\phi_k(x)|^2 dx\) is equal to \(b_{kk}\).

From the analytic dependence of the eigenvalues with respect to \(\mu\) it is possible to show the following result.

**Proposition 5.2:** Let \(U\) be an open interval containing zero and assume that \(\Omega, V\) and \(W\) satisfy one of the hypotheses (i) or (ii) of Theorem 5.1 and that the eigenvalues of \(-\Delta + V\) are simple. If the elements of the sequence \((b_{jk})_{k \in \mathbb{N}}\) are \(Q\)-linearly independent, then for almost every \(\mu \in U\) the elements of \((A_k(\mu))_{k \in \mathbb{N}}\) are \(Q\)-linearly independent.

The other crucial hypothesis of Corollary 2.5 is that \(b_{jj+1} \neq 0\) for every \(j \in \mathbb{N}\). Still by an analyticity argument one checks that either such hypothesis is always false or it is true for almost every \(\mu \in U\).

**Corollary 5.3:** Let \(U\) be an open interval containing zero and assume that \(\Omega, V\) and \(W\) satisfy one of the hypotheses (i) or (ii) of Theorem 5.1 and that the eigenvalues of \(-\Delta + V\) are simple. Assume moreover that the elements of the sequence \((b_{jk})_{k \in \mathbb{N}}\) are \(Q\)-linearly independent and that \(b_{jj+1} \neq 0\) for every \(j \in \mathbb{N}\). Then the controlled Schrödinger equation associated with \(\Omega, V, W\) and \(U\) is approximately controllable for every \(\hat{U} \subset U\) with nonempty interior.

**B. 1D harmonic oscillator**

In this section we study the Schrödinger equation describing the evolution of the controlled one-dimensional harmonic
oscillator,
\( \frac{\partial \psi}{\partial t}(t,x) = -\frac{\partial^2 \psi}{\partial x^2}(t,x) + \left(x^2 - u(t)W(x)\right)\psi(t,x), \) (13)
where \( \psi \) is the wave function depending on the time \( t \) and on a space variable \( x \in \mathbb{R} = \Omega \). Recall that \( u(\cdot) \) is a piecewise-continuous function with values in a subset \( U \) of \( \mathbb{R} \). Notice that the potential corresponding to the uncontrolled Schrödinger operator is \( V(x) = x^2 \). The control system (13) has been studied, among others, by Mirrahimi and Rouchon who proved its non-controllability in the case where \( W \) is the identity function (see [9]).

From classical results we know that the spectrum of \( -\Delta + V \) is discrete. Its explicit expression is
\[
\{\lambda_k = 2k + 1 \mid k \geq 0\},
\]
and therefore \( \lambda_{k+1} - \lambda_k \) are \( \mathbb{Q} \)-linearly dependent. Each \( \lambda_k \) is a simple eigenvalue whose corresponding eigenfunction is
\[
\phi_k(x) = \sigma_k e^{-x^2/2} H_k(x),
\]
where \( \sigma_k = 1/\sqrt{l_k!}\sqrt{\pi} \) and \( H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2} \) is the \( k \)-th Hermite polynomial.

In order to apply Corollary 5.3 we would like first of all to ensure that the elements
\[
b_{kk} = (-1)^k \sigma_k^2 \int_{\mathbb{R}} W(x) H_k(x) \frac{d^k}{dx^k} e^{-x^2} dx,
\]
are \( \mathbb{Q} \)-linearly independent. Notice that for \( W(x) = x \) (i.e., the non-controllable case pointed out by Mirrahimi and Rouchon), since each function \( \phi_k^2 \) is even, \( b_{kk} = \int \mathbb{R} \phi_k^2 = 0 \).

The existence of controlled potentials \( W \) for which the elements of \( \{b_{kk}\}_{k \in \mathbb{N}} \) are \( \mathbb{Q} \)-linearly independent can be easily inferred from the linear independence of the functions \( \phi_k^2 \). The proposition below provides some explicit \( W \) with such a property (and such that the corresponding Schrödinger equation is controllable). The potentials \( W \) will be chosen in \( L^\infty(\mathbb{R}) \) and it turns out that the corresponding solutions in the sense (4) coincide with mild or strong solutions, depending on the regularity of the initial condition.

**Proposition 5.4:** (1) If \( W \) is even, then system (13) is not approximately controllable. (2) If \( W \) has the form \( W(x) = e^{ax^2+bx+c} \), with \( a, b, c \in \mathbb{R} \) such that \( a < 0 \) and the two numbers \( \sqrt{-1-\alpha} \) and \( b \) are algebraically independent, then system (13) is approximately controllable, provided that \( U \) has nonempty interior.

**Proof:** Since each function \( \phi_k \) has the same parity as the integer \( k \), then \( \phi_k \) has the same parity as the integer \( j + k \). If \( W \) is even, then (12) shows that for every \( (j,k) \) such that \( j + k \) is odd, \( b_{jk} = 0 \). It turns out that the spaces spanned by the sets \( \{\phi_k \mid k \text{ even}\} \) and \( \{\phi_k \mid k \text{ odd}\} \) are invariant by the dynamics of system (13). In particular, there is no way to steer system (13) from \( \phi_1 \) to a point \( \varepsilon \)-close to \( \phi_2 \) if \( \varepsilon \) is smaller than \( \sqrt{2} \). This proves (1).

In order to prove (2) one needs to apply Corollary 5.3 (with \( U \) playing the role of \( \mathbb{U} \) and \( \mathbb{R} \) the role of \( U \)) with \( W \) having the special form \( W : x \mapsto e^{ax^2+bx+c} \).

After rather standard computations (see [23] for the details) it is possible to prove, if \( j + k \) is even, that
\[
b_{jk} = \sigma_j \sigma_k \sqrt{-\alpha} e^{-x^2/2} S_{jk}(b)
\]
where \( S_{jk} \) is a nonzero polynomial with coefficients in \( \mathbb{Q}(\sqrt{1-\alpha}) \) of degree exactly \( j + k \).

For the sake of simplicity, assume that \( \{b_{jk}\}_{j,k=0} \) is connected. Fix \( j, k \in \{0, \ldots, n\} \). We should prove the existence of a sequence \( r_1, \ldots, r_l \in \{0, \ldots, n\} \) such that \( b_{r_1 r_2} b_{r_1 r_3} \cdots b_{r_{l-1} r_l} b_{r_l} \neq 0 \). If \( j \) and \( k \) have the same parity then we are done since \( b_{jk} \neq 0 \). Otherwise, a simple computation shows that
\[
b_{01} = \frac{be^{-x^2/2}}{\sqrt{1-a^2}} \neq 0
\]
and we can conclude by taking \( \{r_1, r_2\} = \{0, 1\} \).

**C. 3D potential well**

Consider the Schrödinger equation
\[
\frac{i}{\partial t} \psi(t,x) = -\Delta \psi(t,x) + u(t)W(x)\psi(t,x),
\]
where \( \psi \) depends on the time \( t \) and on the space variable \( x = (x_1, x_2, x_3) \in \Omega = (0, l_1) \times (0, l_2) \times (0, l_3) \) and satisfies the Dirichlet boundary condition \( \psi|_{\partial \Omega} = 0 \). Notice that the potential corresponding to the uncontrolled Schrödinger operator is \( V(x) = 0 \). For every \( W \) measurable bounded, solutions in the sense (4) coincide with mild or strong solutions, depending on the regularity of the initial condition.

The spectrum of the Schrödinger operator is
\[
\left\{\lambda_{k_1, k_2, k_3} = \pi^2 \left(\frac{k_1^2}{l_1^2} + \frac{k_2^2}{l_2^2} + \frac{k_3^2}{l_3^2}\right) \mid k_1, k_2, k_3 \geq 1\right\}.
\]

For the sake of simplicity, assume that \( (l_1 l_2)^2, (l_1 l_3)^2, \) and \( (l_2 l_3)^2 \) are \( \mathbb{Q} \)-linearly independent, so that all the eigenvalues are simple and the perturbation result appearing in Theorem 5.1 can be applied. (The case of multiple eigenvalues can be treated similarly, applying a refined perturbation argument as the one used in [33].)

The normalized eigenfunction corresponding to \( \lambda_{k_1, k_2, k_3} \) is given, up to sign, by
\[
\phi_{k_1, k_2, k_3}(x_1, x_2, x_3) = \frac{2^2}{\sqrt{l_1 l_2 l_3}} \sin \left(\frac{k_1 x_1 \pi}{l_1}\right) \sin \left(\frac{k_2 x_2 \pi}{l_2}\right) \sin \left(\frac{k_3 x_3 \pi}{l_3}\right).
\]

**Proposition 5.5:** Let \( (l_1 l_2)^2, (l_1 l_3)^2, \) and \( (l_2 l_3)^2 \) be \( \mathbb{Q} \)-linearly independent and define \( W(x) = e^{ax^2} \) with \( a = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \). Assume that \( \alpha_1, \alpha_2, \alpha_3 \) are nonzero and that \( (\pi/\alpha_1 l_1)^2, (\pi/\alpha_2 l_2)^2, (\pi/\alpha_3 l_3)^2 \) are algebraically independent. Then the control system (16) is approximately controllable.
In order to prove Proposition 5.5 we will need the following technical result (whose proof can be found in [23]).

Lemma 5.6: Let $\beta$ be a real number transversal to a field $F$ with $Q \subset F \subset R$. Then the elements of the family $\left(\frac{1}{\alpha^{2}+1}\right)_{\alpha \in Q}$ are $F$-linearly independent.

Proof of Proposition 5.5. Theorem 5.1 and Fabini’s theorem imply that the eigenvalues $\lambda_{k_{1},k_{2},k_{3}}(\mu)$ of $-\Delta + \mu W$ on $\Omega$ for the Dirichlet boundary value problem satisfy

\[ \Lambda_{k_{1},k_{2},k_{3}}(0) = Ck_{1}^{2}k_{2}^{2}k_{3}^{2}\left(\frac{4\pi^{2}}{\alpha_{1}^{2}+1} + 1\right)\left(\frac{4\pi^{2}}{\alpha_{2}^{2}+1} + 1\right)\left(\frac{4\pi^{2}}{\alpha_{3}^{2}+1} + 1\right), \]

where

\[ C = \frac{64(e^{\alpha_{1}l_{1}} - 1)(e^{\alpha_{2}l_{2}} - 1)(e^{\alpha_{3}l_{3}} - 1)\pi^{6}}{(\alpha_{1}l_{1}^{2}\alpha_{2}l_{2}^{2}\alpha_{3}l_{3}^{2})^{1/3}}. \]

Let $\beta_{j} = 4\pi^{2}/(\alpha_{j}^{2}l_{j}^{2})$, $j = 1, 2, 3$. The $Q$-linear independence of the elements of $\left(\Lambda_{k_{1},k_{2},k_{3}}(0)\right)_{k_{1},k_{2},k_{3}\in N}$ is obtained from the expression above thanks to three nested applications of Lemma 5.6 with $F = Q(\beta_{1},\beta_{2})$ and $\beta = \beta_{3}$, $F = Q(\beta_{3})$ and $\beta = \beta_{2}$, and $F = Q$ and $\beta = \beta_{1}$. In order to complete the proof, let us check that every matrix $B^{(n)}$ is connected. (The conclusion then follows from Corollary 5.3.)

A straightforward computation shows that for every triple of positive integers $(k_{1},k_{2},k_{3})$ and $(h_{1},h_{2},h_{3})$ the integral

\[ \int_{\Omega} e^{\alpha_{2}x^{2}} \phi_{k_{1},k_{2},k_{3}}(x) \phi_{h_{1},h_{2},h_{3}}(x) \, dx \]

is different from zero, i.e., every element of $B^{(n)}$ is nonzero.

\[ \square \]

REFERENCES


[31] T. Kato, Perturbation theory for linear operators, ser. Die Grundl...