Constrained Linear System under Disturbance Feedback: Convergence with Probability One

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Abstract—This paper considers a control parametrization under Model Predictive Control framework for constrained linear discrete time systems with bounded additive disturbances. Like the control parametrization in recent literature, the proposed parametrization uses affine disturbance feedback but includes an additional term. As a result, the parametrization has the same representative ability but has a different closed-loop convergence property. More exactly, the state of the closed-loop system converges to the minimal invariant set with probability one. Deterministic convergence to the same set is also possible if a less intuitive cost function is utilized. Numerical experiments are provided that validate the results.

I. INTRODUCTION

This paper considers the system:

\[ x_{t+1} = Ax_t + Bu_t + w_t, \quad (x_t, u_t) \in Y, \quad w_t \in W, \quad \forall \ t \geq 0 \]

where \( x_t \in R^n \), \( u_t \in R^m \) and \( w_t \in W \subset R^n \) are the state, control and disturbance acting on the system at time \( t \), respectively. The set \( Y \) represents the joint constraint on \( x \) and \( u \) of the system. The study of such a system under the Model Predictive Control (MPC) framework has been an active area of research in the past few years [1], [2], [3], [4], [5]. One important issue is the choice of control parametrization within the control horizon. Several choices have been proposed in the literature [2], [3], [5], [6], [7], [8], [9] and a popular choice is \( u_t = Kx_t + c_t \) [2] where \( K \) is a fixed feedback gain and \( c_t \) is the new variable. However, such a choice is known to be conservative and its use will result in a relatively small domain of attraction.

In an effort to reduce conservatism, control parametrization based on affine function of disturbances have been proposed [6], [8], [9], [10]. Löfberg [6] proposes the control parametrization of

\[ u_t^W = \sum_{j=1}^{i} M_{t,j}^i w_{t-j} + v_i, \quad i = 0, \ldots , N-1 \]  

where \( M_{t,j}^i \) and \( v_i \) are the optimization variables and \( N \) is the length of the horizon used in MPC. Goulart et al. [8] show that parametrization (3) is equivalent to that of time-varying affine state feedback in terms of sets of states that are reachable within the horizon. They also showed that, under mild assumptions, the origin of the closed-loop system is input-to-state stable (ISS) under the MPC control law derived using time-varying affine state feedback law. Recently, Wang et al. [9], [11] propose an extended disturbance feedback parametrization

\[ u_t^W = K_f x_t + c_i + \sum_{j=1}^{N-1} C^i_j w_{t-j}, \ i = 0, \ldots , N-1 \]  

where \( K_f \) is a fixed feedback gain such that \( \Phi := A + BK_f \) is strictly stable. They show that parametrization (4) under the MPC framework has the same domain of attraction as that using (3) but has a stronger stability result in that the state of closed-loop system converges to the minimal disturbance invariance set, \( F_{\infty} \) [12], of the system \( x_{t+1} = \Phi x_t + w_t \). Unlike (3), it is possible that \( i < j \) for \( w_{t-j} \) in (4). When this happens, \( w_{t-j} \) refers to past realized disturbances. This also means that the resulting MPC control law derived from (4) is a dynamic compensator, requiring the values of \( x_t \) and \( w_{t-1}, \ldots , w_{t-N+1} \) for its evaluation at time \( t \). On the other hand, the parametrization proposed in this paper results in a state feedback MPC control law, requiring only the knowledge of \( x_t \) for its evaluation. Correspondingly, a weaker convergence result is obtained : the closed-loop system state converges to \( F_{\infty} \) with probability one. Additionally, deterministic convergence to the same set is also possible if a less intuitive cost is used.

The rest of this paper is organized as follows. This section ends with notations used, assumptions needed and a brief review of standard results. Section II gives the proposed control parametrization and the finite horizon (FH) optimization problem including the choice of the cost function. The result of probabilistic convergence of the closed-loop system state is given in section III. Section IV shows a formulation that strengthens the result under weaker assumptions. Numerical examples and discussions are the contents of section V. The last section concludes the paper.

The following notations are used. \( Z_k \) denotes the integer set \( \{0, 1, \ldots , k\} \) and \( Z_k^+ \) denotes \( \{1, \ldots , k\} \); given matrices \( A \in R^{n \times m} \) and \( B \in R^{p \times q} \), \( A \otimes B \) is the Kronecker product of \( A \) and \( B \); \( \text{vec}(A) = [A^T \cdots A^T]^T \in R^{nm} \) is the stacked vector of columns of \( A \) and \( \|A\| := \sqrt{\lambda_{\text{max}}(A^T A)} \) is the induced norm of matrix \( A \). \( A \succeq (\succeq) 0 \) means that square matrix \( A \) is positive definite (semi-definite). For any...
\[ A > 0, \|x\|_A^2 = x^T A x, \] 1 \text{ is a } r \text{-vector with all elements being 1 and } I_n \text{ is the } n \times n \text{ identity matrix. For any set } X, Y \subset R^n, X + Y := \{x + y : x \in X, y \in Y\} \text{ is the Minkowski sum of } X \text{ and } Y.

The system (1)-(2) is assumed to satisfy the following assumptions:

(A1) system \((A, B)\) is stabilizable;

(A2) the set
\[
Y := \{(x, u) \mid Y_x x + Y_u u \leq 1_q\} \subset R^{n+m}
\]
is compact and contains the origin;

(A3) the disturbance \(w_i, t \geq 0\) are independent and identically distributed (i.i.d.) with zero mean and \(W \subset R^n\) is convex and compact;

(A4) a constant feedback gain \(K_f \in R^{m \times n}\) is given such that \(\Phi := A + BK_f\) has a spectral radius \(\rho(\Phi) < 1\).

One other technical assumption is also needed and is discussed in section II. Assumption (A1) is standard. The characterization of \(Y\) in (A2) is made out of the need for a concrete computational representation. Assumption (A3) is mild and can be satisfied by many disturbance models. Additionally, the zero mean and i.i.d. condition can be relaxed and this will be discussed in details in section IV. Assumption (A4) is easily satisfied under (A1) and is made for convenience. Under (A1)-(A4) and the results in [12], [13] show that, for sufficiently small \(W\), a constraint-admissible maximal disturbance invariant set,
\[
X_f := \{x \mid G x \leq 1_g\},
\]
exists in the sense that \(\Phi x + w \in X_f, (x, K_f x) \in Y\) for all \(x \in X_f\) and for all \(w \in W\). It is also known [12] that the state of the system \(x_{i+1} = \Phi x_i + w_i\) converges to the minimal disturbance invariant set, \(F_\infty\), given by
\[
F_\infty = W + \Phi W + \Phi^2 W + \cdots
\]
and that \(F_\infty\) is compact.

II. CONTROL PARAMETRIZATION

MPC formulation solves an \(N\)-stage finite horizon (FH) optimization problem. Let \(x_i\) and \(u_i, i \in Z_{N-1}\) denote the predicted state and predicted control at the \(i\)th stage, respectively, within the horizon. The proposed control parametrization within the FH optimization problem takes the form
\[
u_i = K_f x_i + d_i + \sum_{j=1}^{i} D^j w_{i-j} ,\ i \in Z_{N-1} \tag{8}
\]
where \(d_i \in R^m, D^j_i \in R^{m \times n}, j = Z^+_i, i \in Z_{N-1}\) are the variables of the FH problem and \(K_f\) is the feedback gain in (A4). Since \(i-j \geq 0\), \(w_{i-j}\) is the \((i-j)^{\text{th}}\) predicted disturbance at each stage \(i\). In this regard, (8) is similar to (3) in that only predicted disturbances are used in the parametrization. However, in terms of the family of functions that can be represented, \(u_i\) is equivalent to \(u^W_{i}\) and \(w_{i}\), the respective parameterizations of Löfberg [6] (or Goulart et. al. [8]) and Wang et. al. [9]. To see this, set \(C^j_i = 0\) for all \(j > i\) in (4) and it follows that \(u_i\) is a special case of \(u^W_{i}\).

To show the converse, let
\[
\begin{align*}
d_i &= c_i + \sum_{j=i+1}^{N-1} C^j_i w_{i-j} ,\ i \in Z_{N-1} \\
D^j_i &= C^j_i \\
n &\leq j, i \in Z_{N-1} 
\end{align*}
\]
for any \(c_i\), \(C^j_i\) that defines \(u^W_{i}\). This establishes the equivalence of \(u_i\) and \(u^W_{i}\).

Remark 1: The equivalence of \(u^L_{i}\) and \(u^W_{i}\), in terms of family of functions that can be represented, has already been established in [9]. With the above result, the representabilities of \(u_i\), \(u^L_{i}\) and \(u^W_{i}\) are all equivalent.

Let the design variables within the control horizon \(N\) in (8) be collected in
\[
D := (D_1, D_2, D_3, \ldots, D_{N-1}),
\]
d := \((d_0, d_1, \ldots, d_{N-1})\)
then the FH optimization problem of (8), referred hereafter as \(P_N(x_i)\), is
\[
\begin{align*}
\min_{d, D} J(d, D) \\
\text{s.t.} \ x_0 = x_t \\
x_{i+1} = Ax_i + Bu_i + w_i, \ i \in Z_{N-1} \\
u_i = K_f x_i + d_i + \sum_{j=1}^{i} D^j_i w_{i-j} ,\ i \in Z_{N-1} \\
(x_i, u_i) \in Y, \ \forall w_i \in W, \ i \in Z_{N-1} \\
x_N \in X_f, \ \forall w_i \in W, \ i \in Z_{N-1}
\end{align*}
\]
The above is a standard FH optimization problem for MPC with horizon \(N\) with \(X_f\) being the maximal disturbance invariant set of (6). The cost function \(J(d, D)\) takes the form
\[
J(d, D) := \sum_{i=0}^{N-1} \left[ \|d_i\|_\Psi^2 + \sum_{j=1}^{i} \|\text{vec}(D^j_i)\|_A^2 \right]
\]
for any choice of \(\Psi\) and \(\Lambda\) that satisfy
\[(A5)\ \ \Phi > 0 \text{ and } \Lambda \geq \Sigma_w \otimes \Psi \text{ where } \Sigma_w \text{ is the covariance matrix of } w_i.\]

The technical conditions of (A5) are needed to ensure the convergence property of the closed-loop system and its role will become clear in the proof of Theorem 2. However, some comments on the ease of verification of (A5) is appropriate.

Remark 2: Since \(\Lambda - \Sigma_w \otimes \Psi\) has to be positive definite, (A5) can be easily satisfied even when the covariance matrix \(\Sigma_w\) is unknown. For example, let \(\Lambda = \alpha^2 I_n \otimes \Psi\) where \(\alpha = \max_{w \in W} \|w\|_2\). Then it follows that \(\Lambda \geq \Sigma_w \otimes \Psi\) because \(\alpha^2 I_n \geq E[w w^T]\) for all \(w \in W\) which implies that \(\alpha^2 \geq E[w w^T] \otimes \Psi\).

Remark 3: Although Remark 2 implies that (A5) will be satisfied as long as eigenvalues of \(\Lambda\) are large enough, an over-large \(\Lambda\) will degrade the performance of the resulting MPC controller. This will be verified in the numerical examples and discussed further in section V.

Remark 4: Give matrices \(Q \succeq 0, R > 0\) and \(P \succ 0\) satisfying algebraic Riccati equation, it is shown [9, [14],
Consider the FH optimization problem under

\[
E \left[ \sum_{i=0}^{N-1} (\|x_i\|_Q^2 + \|u_i\|_R^2) + \|x_N\|_P^2 \right] = x_0^T P x_0 + \text{N trace}(\Sigma w P) + J(d, D).
\]

if \( \Psi = R + B^T P B \) and \( \Lambda = \Sigma w \otimes \Psi \) and \( K_f = -(R + B^T P B)^{-1} B^T P A \). Hence, cost function (16) can be related to expected value of standard LQ cost.

From (12) and (13), it is obvious that \( x_i \) and \( u_i \) are affine functions of \( w_i, i \in \mathbb{Z}_{N-1} \). Correspondingly, constraints (14) and (15) under assumptions (A2) and expression of \( x_f \) in (6) are affine in \( w_i \in W, i \in \mathbb{Z}_{N-1} \). Since \( w_i, i \in \mathbb{Z}_{N-1} \) are predicted disturbances within the horizon and have not been realized at time \( t \), \( \mathcal{P}_N(x_t) \) is a quadratic programming problem with linear constraints in its constraints. Its numerical solution is obtained from the deterministic equivalent of \( \mathcal{P}_N(x_t) \). This process is done using the dual variables of the constraints and is a standard procedure in robust optimization [10]. The exact procedure has been discussed in [8], [9] for the case where \( W \) is a polytope and will not be elaborated here. It is also possible to formulate the deterministic equivalence when \( W \) is a conic or second-order cone representable set [16], [17].

Let the feasible set of optimization problem \( \mathcal{P}_N(x_t) \) be

\[
\Pi_N(x_t) := \{ (d, D) | \mathcal{P}_N(x_t) \text{ is feasible} \}
\]

and the set of admissible initial states be

\[
\mathcal{X}_N := \{ x | \Pi_N(x) \neq \emptyset \}.
\]

Remark 5: Consider the FH optimization problem under different control parameterizations, it follows from Remark 1 that the same admissible set \( \mathcal{X}_N \) is achieved for the case where (3) or (4) replaces (13).

The rest of the MPC formulation is standard: \( \mathcal{P}_N(x_t) \) is solved at each time \( t \) to obtain the optimizer \((\hat{d}_t^*, \hat{D}_t^*) := (\hat{d}(x_t), \hat{D}(x_t))\) and the corresponding \( u_{0|t}^* := u_0^*(x_t) \) is applied to system (1) resulting in the MPC control law,

\[
u_t = u_{0|t}^* = K_f x_t + d_{0|t}^*
\]

III. FEASIBILITY AND STABILITY

The feasibility of \( \mathcal{P}_N(x_t) \) at different time instants and stability of the closed-loop system under the feedback law (19) are addressed in this section.

Theorem 1: If \( \mathcal{P}_N(x_{t+1}) \) admits an optimal solution, so does \( \mathcal{P}_N(x_{t+1}) \) under the feedback law (19) for all possible \( w_t \in W \).

Proof: The proof is standard, but the details are given for their relevance to Theorem 2. For clarity, additional subscripts \( |t|, |t|+1 \) are used to denote the variables at the different times. Let \((\hat{d}_t^*, \hat{D}_t^*) \) denote the optimal solution of \( \mathcal{P}_N(x_t) \). At time \( t+1 \) when \( w_t \) is realized, choose \((\hat{d}_{t+1}, \hat{D}_{t+1})\) by letting

\[
\hat{d}_{i|t+1} = \begin{cases} 
  d_{i+1}|t + (D_{i+1}|t)^* w_t & i \in \mathbb{Z}_{N-2} \\
  0 & i = N - 1
\end{cases}
\]

and it is feasible to \( \mathcal{P}_N(x_{t+1}) \) for all possible \( w_t \in W \) due to the disturbance invariance of \( x_f \) for system (1) under control law \( u_t = K_f x_t \). It is clear that \( \Pi_N(x) \) is compact for all \( x \in \mathcal{X}_N \). Since \( W \) is bounded and \( J \) is a norm function, \( \max_w J(d_{t+1}, D_{t+1}) < \infty \) and the set \( \{ (d, D) \in \mathcal{X}_N(x_{t+1}) | J(d, D) \leq \max_w J(d_{t+1}, D_{t+1}) \} \) is compact. Hence, the optimum of \( \mathcal{P}_N(x_{t+1}) \) exists, following the Weierstrass’ theorem.

The main result of probabilistic convergence of the closed-loop system is stated in the next theorem.

Theorem 2: Suppose \( x_0 \in \mathcal{X}_N \) and (A1)-(A5) are satisfied. System (1) under MPC control law (19) has the following properties: (i) \((x_t, u_t) \in Y \) for all \( t \geq 0 \), (ii) \( x_t \rightarrow F_\infty(K_f) \) with probability one as \( t \rightarrow \infty \).

Proof: (i) The stated result follows directly from Theorem 1. (ii) Let \( J_t^* := J(d_t^*, D_t^*) \) and \( J_{t+1}(w_t) := J(d_{t+1}(w_t), D_{t+1}) \) where \( (d_{t+1}(w_t), D_{t+1}) \) are given by (20)-(21). Then it follows that

\[
J_t^* - J_{t+1}(w_t) = \sum_{i=0}^{N-1} (\|d_{i|t}^*\|_\Psi^2 - \|\hat{d}_{i|t+1}\|_\Psi^2) + \sum_{i=1}^{N-1} \sum_{i=1}^{N-1} \|\text{vec}(D_{i|t})^*\|_\Lambda^2
\]

where

\[
g(w_t) := \sum_{i=0}^{N-1} (\|\text{vec}(D_{i|t})^*\|_\Lambda^2 - \|D_{i|t}\|_\Psi^2 - 2(d_{i|t})^T \text{vec}(D_{i|t})^* w_t
\]

Taking the expectation of (22) over \( w_t \), it follows that

\[
J_t^* - \|d_{0|t}^*\|_\Psi^2 = E_{w_t} [J_{t+1}(w_t)] + E_{w_t}[g(w_t)]
\]

\[
\geq E_{w_t} [J_{t+1}(w_t)]
\]

\[
geq E_{w_t} [J_{t+1}(w_t)] = E_t [J_{t+1}(w_t)].
\]

where \( E_t \) is the expectation taken over \( w_t, i \geq t \). Inequality (24) follows from the fact that \( E_{w_t}[g(w_t)] \geq 0 \). This is true because by taking the expectation of (23), one gets

\[
E_{w_t}[g(w_t)] = \sum_{i=1}^{N-1} (\|\text{vec}(D_{i|t})^*\|_\Lambda^2 - \|\text{vec}(D_{i|t})^*\|_\Psi^2).
\]
where the last term is zero due to (A3) and the rest is non-negative due to (A5).

Inequality (25) follows from the fact that \( \frac{\delta^*}{\delta t}(x_{t+1}) \geq J_{t+1}(w_t) \) for every \( w_t \in W \) which implies that 

\[
E_{w_t} [J_{t+1}(w_t)] \geq E_{w_t} [J_{t+1}^*(w_t)].
\]

The last equality of (25) follows from the fact that \( J_{t+1}^*(w_t) \) depends on \( w_t \) only and not on any \( w_i, i > t \).

Repeating the inequality of (25) for increasing \( t \), one gets,

\[
J_{t+1}(x_{t+1}) - \|d_0^*|_{t+1}(x_{t+1})\|^2 \geq E_{w_{t+1}} [J_{t+1}^*(w_{t+1}, w_{t+1})]
\]

where the dependence of the various quantities on \( x_{t+1} \) are added for clarity. Since \( x_{t+1} \) depends on \( x_t \) and \( w_t \), the above can be equivalently written as

\[
J_{t+1}(w_t) - \|d_0^*|_{t+1}(w_t)\|^2 \geq E_{w_{t+1}} [J_{t+1}^*(w_t, w_{t+1})].
\]

The above inequality holds true for all possible \( w_t \), hence

\[
E_{w_t} [J_{t+1}^*(w_t)] - E_{w_t} [\|d_0^*|_{t+1}(w_t)\|^2] \geq E_{t} [J_{t+1}^*(w_t, w_{t+1})]
\]

or

\[
E_{t} [J_{t+1}^*(w_t)] - E_{t} [\|d_0^*|_{t+1}(w_t)\|^2] \geq E_{t} [J_{t+1}^*(w_t, w_{t+1})].
\]

The equality in (27) follows from assumption (A3), particularly,

\[
E_{w_t} [E_{w_{t+1}} [J_{t+1}^*(w_t, w_{t+1})]]
\]

\[
= E_{w_t} \left[ \int J_{t+1}^*(w_t, w_{t+1}) f_{w_{t+1}}(w_{t+1}) dw_{t+1} \right]
\]

\[
= \int J_{t+1}^*(w_t, w_{t+1}) f_{w_{t+1}}(w_{t+1}) dw_{t+1} f_{w_t}(w_t) dw_t
\]

\[
= E_{w_t, w_{t+1}} [J_{t+1}^*(w_t, w_{t+1})] = E_{t} [J_{t+1}^*(w_t, w_{t+1})]
\]

where \( f_{w_{t+1}}(\cdot) \), \( f_{w_t, w_{t+1}}(\cdot, \cdot) \) and \( f_{w_t, w_{t+1}}(\cdot, \cdot) \) are density functions of \( w_{t+1} \), \( w_t \) and \( w_{t+1} \) respectively, and \( f_{w_t, w_{t+1}}(\cdot, \cdot) = f_{w_t}(\cdot) f_{w_{t+1}}(\cdot) \) from assumption (A3). Summing (25) and (28) leads to

\[
J^*_t \geq \|d_0^*|_{t}\|^2 + E_{t} [\|d_0^*|_{t+1}(w_t)\|^2] + E_{t} [J_{t+1}^*(w_t, w_{t+1})]
\]

Repeating the above procedure infinite times leads to

\[
\infty > J^*_t \geq \sum_{i=t}^{\infty} E_t [\|d_0^*|_{i}\|^2 \geq \epsilon]
\]

By applying Markov bound (given non-negative random variable \( R \) and any \( \epsilon \geq 0 \), \( E[R] \geq \epsilon P(R \geq \epsilon) \), we have

\[
\infty > \epsilon \sum_{i=t}^{\infty} P_t (|d_0^*|_{i}\|^2 \geq \epsilon)
\]

for any arbitrary small \( \epsilon > 0 \). From the First Borel-Cantelli Lemma [18], this implies that \( \lim_{t \to \infty} P_t (|d_0^*|_{i}\|^2 \geq \epsilon) = 0 \). Hence \( d_0^*|_{i} \) approaches zero with probability one as \( t \) increases. Consequently, the MPC control law (19) converges to \( K_f x_t \) with probability one. When this happens, the closed-loop system converges to \( x_{t+1} = \Phi x_t + w_t \) and, hence, \( x_t \) converges to \( F_\infty (K_f) \) with probability one.

IV. DETERMINISTIC CONVERGENCE

While the assumption of \( W \) being a convex compact set is reasonable, the assumption of \( w_t \) being zero mean and i.i.d. is harder to verify in practice. This section is concerned with the relaxation of assumption (A3) while achieving a stronger convergence result than that of Theorem 2. Consider

(A3a) \( w_t \in W \) and \( W \) is convex and compact.

and define the cost function

\[
V(d,D) := \sum_{i=0}^{N-1} \left[ \|d_i\|_F^2 + \sum_{j=1}^{i} (\gamma_1 \|\text{vec}(D_j^\gamma)\|^2 + \gamma_2 \|\text{vec}(D_j^\gamma)\|^2) \right]
\]

for some constants \( \gamma_1 \) and \( \gamma_2 \) satisfying

(A5a) \( \gamma_1 \geq 2\alpha \|\Psi\|, \gamma_2 \geq 2\alpha \beta \|\Psi\| \).

where \( \beta := \max_{x,d} \|d_t(x,d)\|_{T_N} \), \( T_N \) is the set of \( (x,d) \) defined by (11)-(15) and \( \alpha := \max_{w \in W} \|w\|_F \). The existence of \( \alpha \) and \( \beta \) are guaranteed by compactness of the \( W \) and \( T_N \) sets.

Theorem 3: Suppose \( x_0 \in \mathcal{X}_N \) and (A1-A2), (A3a), (A4) and (A5a) are satisfied and \( J(d, D) \) is replaced by \( V(d, D) \) in \( \mathcal{P}_N(x) \), then system (1) under the MPC control law (19) satisfies (i) \( x_t, u_t \in Y \) for all \( t \geq 0 \), (ii) \( x_t \to F_\infty (K_f) \) as \( t \to \infty \).

Proof: (i) The replacement of cost function \( J(d, D) \) by \( V(d, D) \) does not affect the feasibility of problem \( \mathcal{P}_N(x) \). This means that part (i) of Theorem 2 remains valid. (ii) Let \( V^*_t \) and \( J^*_t+1 \) be defined in the same manner as \( J^*_t \) and \( J^*_t+1 \) in the statement of proofs of Theorem 2. Following the same reasoning as in (22), it can be shown that

\[
V^*_t - J^*_t+1 = \|d^*_0\|^2 + p(w_t)
\]

where

\[
p(w_t) = \sum_{i=1}^{N-1} (\gamma_1 \|\text{vec}(D_i^\gamma)\|^2 + \gamma_2 \|\text{vec}(D_i^\gamma)\|^2)
\]

Hence

\[
p(w_t) \geq \sum_{i=1}^{N-1} (\gamma_1 \|\text{vec}(D_i^\gamma)\|^2 + \gamma_2 \|\text{vec}(D_i^\gamma)\|^2)
\]

\[
- 2\|\text{vec}(D_i^\gamma)\|_F \|w_t\|_F \|\text{vec}(D_i^\gamma)\| - \|\Psi\|_F \|w_t\|_F \|\text{vec}(D_i^\gamma)\|_F^2
\]

\[
\geq \sum_{i=1}^{N-1} (\gamma_1 \|\text{vec}(D_i^\gamma)\|^2 + \gamma_2 \|\text{vec}(D_i^\gamma)\|^2)
\]

\[
- 2\alpha \beta \|\Psi\| \|\text{vec}(D_i^\gamma)\| - \alpha \beta \|\Psi\| \|\text{vec}(D_i^\gamma)\|_F^2
\]

\[
= \sum_{i=1}^{N-1} ((\gamma_1 - \alpha \beta \|\Psi\|) \|\text{vec}(D_i^\gamma)\|^2
\]

\[
+ (\gamma_2 - \alpha \beta \|\Psi\|) \|\text{vec}(D_i^\gamma)\|_F^2
\]

\[
\geq 0
\]

where the fact \( \|\text{vec}(D_i^\gamma)\|^2 \leq \|\text{vec}(D_i^\gamma)\|_F^2 \), i.e. 2-norm of a matrix is less than its Frobenius norm, is used. Hence,
$p(w_t) \geq 0$ under (A5a). As a consequence, equation (31) implies

$V_t^* - \|d_{0t}^*\|_\Psi^2 \geq V_{t+1}^* \geq 0$ (33)

Hence, $\{V_t^*\}$ is a monotonic non-increasing sequence and is bounded from below by zero. This means that $V_\infty := \lim_{t \to \infty} V_t^* \geq 0$ exists. Repeating (33) for $t$ from 0 to $\infty$ and summing them up, it follows that

$\infty > V_0^* - V_\infty \geq \sum_{t=0}^{\infty} \|d_{0t}^*\|_\Psi^2$ (34)

Since $\Psi$ is positive definite, this implies that $\lim_{t \to \infty} d_{0t}^* = 0$ and $\lim_{t \to \infty} u_t = K_f x_t$. Therefore, the stated result follows.

**Remark 6:** Several choices of the cost function of (30) are possible. For example, the results of Theorem 3 remain true if $\|\text{vec}(D_t^f)\|$ is replaced by $\|D_t^f\|_\infty$. This may be more appealing as less conservative bounds on $\gamma_1$ and $\gamma_2$ can be found to ensure the non-negativity of $p(w_t)$. However, its use will result in a semi-definite programming problem for $\mathcal{P}_N(x)$ and is less desirable computationally. The use of $\|\text{vec}(D_t^f)\|$ results in a second-order cone programming for $\mathcal{P}_N(x)$ and is computationally more amiable.

**Remark 7:** The computation of $\beta$ can be simplified to $\beta = \max_{x,d,D} \|d_0\|_\infty$, see Appendix for details. Note that any upper bound of $\beta$ can be used to guarantee the results of Theorem 3. One such upper bound is $\beta = \max (\sigma)$ where $\sigma_i := \max_{x,d,D} \|d_0(i)\|$ and $d_0(i)$ is the $i$th element of $d_0$.

V. NUMERICAL EXAMPLES AND DISCUSSIONS

The performance of the proposed MPC control law is illustrated on an example having $n = 2$ and $m = 1$. The system parameters and constraints are:

$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $K_f = [-0.7434 -1.0922]$, $Y = \{x, u\} | u | \leq 1, \|x\|_\infty \leq 8 \}$, $W = \{w\} | w(1) = \hat{w} - 0.2 \hat{\bar{w}}, w(2) = \hat{w} - 0.2 \leq \hat{\bar{w}}, \hat{\bar{w}} \leq 0.2 \}$.

where $\hat{w}$ and $\hat{\bar{w}}$ are random variables uniformly distributed over $[-0.2, 0.2]$. Terminal set $X_f$ is the corresponding maximal constraint-admissible disturbance invariant set of (1) under $u_t = K_f x_t$. The weight matrices in the cost function (16) are chosen to be

$\Psi = 1$, $\Lambda = \Sigma_w \otimes \Psi = \begin{bmatrix} 0.0139 & -0.0027 \\ -0.0027 & 0.0133 \end{bmatrix}$

The proposed algorithm is simulated with $N = 8$ and $x_0 = [-4 \ 2]_0^T$ over 15 realizations of disturbance sequences and resulting trajectories are shown in Fig. 1 to 4 by solid lines. $\hat{F}_\infty$ in Fig. 1 is a tight outer bound of $F_\infty$ obtained using procedures given in [19].

It is clear from Fig. 1 and 3 that both the state and control constraints are satisfied by all trajectories, in accordance to property (i) of Theorem 2. Figure 4 shows the convergence of $d_t = d_{0t}^*$ to zero as $t$ increases. Hence, the closed-loop state converges to $\hat{F}_\infty(K_f)$ as shown in Fig. 2 where $\text{dis}(x_t, F_\infty) = \min_{x \in \hat{F}_\infty} \|x - x_t\|$. 
Next experiment attempts to show the influence of the weight matrices on the performance of the MPC controller. Without loss of generality, only \( \Lambda \) is regulated instead of both \( \Psi \) and \( \Lambda \). In order to make the difference obvious, \( \Lambda \) is multiplied by 10000 and the system is simulated with same initial conditions and disturbance realizations as in the previous experiment. The results are shown in Fig. 1 to 4 by dash lines.

It can be observed that although the constraints are satisfied and the state converges to \( F_\infty \) set as well, the convergence is much slower this time as shown in Fig. 2 and 4. The reason is that by using a large \( \Lambda \) the time-varying disturbance feedback gains \( D_j^i \) becomes the dominating factors of the cost function and they are forced to vanish as fast as possible. As a consequence, the control law (8) is forced to be a fixed disturbance feedback control law as the one in [2]. Hence, the advantage of time-varying disturbance feedback is lost, leading to a degraded performance of the MPC controller. The results also verify the statement in Remark 3.

VI. CONCLUSIONS

A control parametrization is proposed for MPC of constrained linear systems with disturbances. This parametrization has the same feasible domain as that achieved by parametrization using affine time-varying state feedback law. Under the resultant controller, the closed-loop system state converges to the minimal robust invariant set \( F_\infty \) with probability one and this is achieved by minimizing a norm-like cost function. If a less intuitive cost is minimized, determinisitic convergence to the same set is also achievable.

REFERENCES


APPENDIX

\[
\beta := \max_{(\bar{x}, \bar{d}, \bar{D}) \in T_N} \|d_0\| = \max_{(x, d, D) \in T_N} \|d_0\|
\]

is due to the fact that for any \((x, d, D) \in T_N\) and integer \(i \in \mathbb{Z}_N^+\) a set of \((\bar{x}, \bar{d}, \bar{D}) \in T_N\) can be found such that \(d_0 = d_i\). Specifically, given \((x, d, D) \in T_N\) and let the correspondingly defined state and control sequence be \(\{x_0, \ldots, x_N\}\) and \(\{u_0, \ldots, u_{N-1}\}\). According to (15) \(x_N \in X_f\) for all possible disturbances. Then for any \(i \in \mathbb{Z}_N^+\), \((\bar{x}, \bar{d}, \bar{D})\) can be defined by

\[
\bar{x} = \Phi^i x + \sum_{j=0}^{i-1} \Phi^{i-1-j} B d_j, \quad \bar{d}_j = \begin{cases} d_{j+i} & j \in \mathbb{Z}_N^{i-1-i} \\ 0 & N-i \leq j \leq N-1 \end{cases}
\]

\[
\bar{D}_j^i = \begin{cases} D_{j+i}^k & j \in \mathbb{Z}_N^{i-1-i} \\ 0 & N-i \leq j \leq N-1 \end{cases}
\]

where \(\bar{x}\) is the nominal state of \(x_i\) defined by \((x, d, D)\) and \((\bar{x}, \bar{d}, \bar{D})\) define the control sequence \(\{u_0, \ldots, u_{N-1}\}, K_{f,x} N, K_{f,x} N+1\). According to (A4) under controller \(u_t = K_{f,x} x_t\) all the constraints are satisfied and \(x_t \in X_f\) for \(t \geq N\) since \(x_N \in X_f\). Therefore, \((\bar{x}, \bar{d}, \bar{D})\) satisfies (11)-(15), namely \((\bar{x}, \bar{d}, \bar{D}) \in T_N\). As a result, \(\max_{(\bar{x}, \bar{d}, \bar{D}) \in T_N} \|d_0\| \geq \max_{(x, d, D) \in T_N} \|d_i\|\), for any \(i \in \mathbb{Z}_N^+\) and \(\beta = \max_{(x, d, D) \in T_N} \|d_0\|\).