Abstract

The state of uniform fluidization is usually unstable to small disturbances, and this can lead to the formation of vertically traveling voidage waves. In inverse fluidization, when particle density is less than fluid density ($\rho_s < \rho_f$), particles fluidize in the direction of gravity when the drag force exerted by the fluid overcomes buoyancy. Inverse fluidization thus provides a unique parameter space, which augments the study of instability behavior in normal fluidization when $\rho_f < \rho_s$. Using continuum equations of continuity and motion, we compared the linear stability of normal and inverse bed modes to examine the effect of the Froude number ($Fr$) and fluid to solid density ratio ($\delta = \rho_f/\rho_s$). Making use of numerical bifurcation analysis and continuation, periodic solutions in the form of one-dimensional traveling waves (1D-TWs) were computed. Based on wave growth rates and bifurcation structure, we identified the $Fr$ as an important parameter for predicting instability strength. However, $\delta$ affects instability onset, or the point at which the base state is rendered unstable. In the case studies we examined, traveling waves were shown to propagate in the direction of fluidization, and asymmetrical, high amplitude 1D-TW profiles suggest fully developed bubble-like structures are oriented in the direction of fluidization.

1 Introduction

In inverse fluidization, low density particles become mobile, or fluidize, when the drag exerted by a heavier fluid flowing downwards through the column overcomes the buoyancy force on the particles (Göz, Glasser, Kevrekidis & Sundaresan [1]). Inverse fluidization is the reverse of what is considered to be normal fluidization, where heavier particles are fluidized by the upwards flow of a lighter gas or liquid. Fluidizing lightweight particles by a heavier medium is advantageous in many important industrial applications where enhanced multi-phase mixing can improve heat and mass transfer performance (see Muroyama & Fan [2]). For example, in biotechnology and catalytic chemical reaction engineering, inverse turbulent three-phase reaction systems have been investigated for improved selectivity and yield. In these systems, lightweight particles are fluidized by the countercurrent flow of liquid downwards and gas bubbles upwards (Fan, Muroyama & Chern [3]; Krishnaiah, Guru & Sekar [4]; Comte, Bastoul, Hebrard, Roustan & Lazarova [5]). In fluidized-bed dry particle coating, a high-density super critical fluidization medium (operating in inverse mode) may improve coating efficiency by affecting the frequency and impact value of particle-particle collisions. However, it is difficult to support the use of this mode as a viable alternative without a better understanding of how fluidization direction (relative to gravity) affects instability behavior in the bed.
In normal fluidized beds, it has been well-documented that the base-state of uniform fluidization is usually unstable to small disturbances, and this can lead to the formation and propagation of vertically traveling \textit{voidage waves}. When primary instabilities become spatially amplified in the bed, this can bring about complex bubbling and turbulent flow regimes, which completely alter the flow characteristics of the system (Gibilaro [6]). In gas-fluidized beds, voidage waves are in the form of \textit{bubbles}, where experimental evidence has shown that just beyond conditions of minimum fluidization, the solids tend to remain compacted as increasing volumes of gas pass through the condensed phase “much in the manner of a gas passing through an actual liquid” (Wilhelm & Kwauk [7]). This mode of fluidization is often referred to as \textit{aggregative}, and differs dramatically from flow behavior that is sometimes observed in liquid-fluidized beds, which expand uniformly and are generally more stable in operation (referred to as non-bubbling or \textit{particulate}).

In the fluidization research, \textit{two-phase} continuum models have been used to study the stability behavior of gas- and liquid-fluidized beds. This approach uses \textit{ensemble-} or volume-averaged equations of continuity and motion to describe the behavior of the fluid and particle phases using constitutive relationships or closure laws to express the various force terms as functions of locally averaged variables. Researchers have generally adopted closures based on empirical correlations (Pigford & Baron [8]; Murray [9]; Anderson & Jackson [10]; Garg & Pritchett [11]), but constitutive terms have also been theoretically derived using physical arguments (Batchelor [12]), and from first principles (Koch & Sangani [13]). Anderson, Sundaresan & Jackson [14] successfully demonstrated that these equations do capture the physics necessary to distinguish between bubbling and non-bubbling systems. Recently, Duru, Nicolas, Hinch & Guazzelli [15] tested this approach experimentally by relating the physical properties of saturated voidage waves to the particle phase pressure and viscosity terms. Their results confirmed that the model was satisfactory for describing the behavior of one-dimensional voidage waves within the experimental parameter range investigated (see also Duru & Guazzelli [16]).

In the experimental work of Wilhelm & Kwauk [7], solid-air (or aggregative) systems were found to be separable from solid-water (or particulate) systems on the basis of the dimensionless Froude number evaluated at minimum fluidization velocity, for a wide range of particle species. Experimental evidence of such distinct flow behavior has prompted its investigation by linear stability analysis of the uniform fluidization state. In a stability analysis of gas- and liquid-fluidized beds, Göz [17, 18] analyzed primary bifurcations of two-dimensional vertically and oblique traveling waves from the base-state, and found only minor differences between gas- and liquid-fluidized beds. Göz [19] also found similar bifurcation structure exhibited in gas- and liquid-fluidized beds having small $Fr$ approximations. Göz & Sundaresan [20] extended a previous analysis performed by Göz [21], to examine the stability of one-dimensional periodic waves to two-dimensional perturbations of large transverse wavelength in liquid-fluidized beds by considering the effects of fluid phase inertia and viscosity. These authors demonstrated that the instability mechanism is the same for both gas- and liquid-fluidized beds, and concluded that scaling differences play an important role in distinguishing the difference in gas- and liquid-fluidized bed behavior, \textit{viz.} the $Fr$ number group.

Linear stability analyses of the base state have since led to the computation of fully-developed, one and two-dimensional traveling wave solutions using numerical simulation techniques and bifurcation theory (Glasser, Kevrekidis & Sundaresan [22]). These authors found that for both gas- and liquid-fluidized beds, two-dimensional traveling waves were subsequently born out of one dimensional traveling wave solutions emerging through Hopf bifurcations of the steady state solution. Glasser, Kevrekidis & Sundaresan [23] proposed that a distinction between bubbling and non-bubbling flow behavior can be made based upon an examination of the particle-phase velocity
field in high-amplitude two-dimensional traveling wave solutions. By examining a wide range of parameter values, these authors demonstrated that the potential for bubbling is dictated by the dimensionless quantity \( \Omega \) where \( \Omega^2 \) is shown to be equivalent to \( Fr \) by adopting a natural scale for the particle phase viscosity.

In this paper, a comparative linear stability analysis of the uniform fluidization state is carried out in inverse and normal systems to determine if the role of the \( Fr \) in distinguishing bubbling from non-bubbling bed behavior is consistent with the ideas put forth previously by other researchers. The inverse liquid bed proves to be an important case study because it introduces an additional dimensionless parameter set having values, which do not exist within the set defined by normal fluidization. Moreover, the role of fluidization direction (with respect to gravity) can be critically examined. We compared instability behavior in normal and inverse liquid beds for systems having comparable \( Fr \) numbers and for systems having \( Fr \) numbers, which differed by a factor of 4. Linear stability is analyzed at the marginal stability point, or point at which both systems are rendered unstable at an expanded bed volume defined by some critical solids volume fraction \( \phi_c \). High amplitude one-dimensional traveling wave solutions are used to compare the structure and propagation behavior of wave forms in the two beds.

Volume averaged equations of continuity and motion from the theory of Anderson & Jackson [10] are presented and discussed in Section 2. In Section 3, the linear stability of the uniformly fluidized base state is examined in normal and inverse bed modes, and one-dimensional traveling wave solutions are computed in Section 4 using a derivation from the work of Needham & Merkin [24]. Results are presented in Section 5 for water-fluidized systems using the two-fluid model to examine the effect of the dimensionless \( Fr \) number and \( \delta \). Conclusions are discussed in Section 6.

## 2 Equations of motion

We begin with a description of the volume-averaged equations of continuity and motion for a two-phase system consisting of a fluid and solid phase [10]. These equations have been written in a moving frame of reference at constant velocity \( \omega \) (Göz [25]), and take the form:

\[
\frac{\partial \epsilon}{\partial t} + \nabla \cdot [\epsilon \mathbf{u} - \omega \mathbf{k}] = 0
\]  
\[\text{(1)}\]

\[
\frac{\partial \phi}{\partial t} + \nabla \cdot [\phi \mathbf{v} - \omega \mathbf{k}] = 0
\]  
\[\text{(2)}\]

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} - \omega \mathbf{k}) \cdot \nabla \mathbf{u} = -\epsilon \nabla \cdot \mathbf{\sigma}_f - \tilde{F} + \epsilon \rho_f \mathbf{g}
\]  
\[\text{(3)}\]

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} - \omega \mathbf{k}) \cdot \nabla \mathbf{v} = -\phi \nabla \cdot \mathbf{\sigma}_s - \nabla \cdot \mathbf{\sigma}_s + \tilde{F} + \phi \rho_s \mathbf{g}
\]  
\[\text{(4)}\]

where \( \phi \) is the local mean solids volume fraction, \( \epsilon \) is the local mean bed voidage (\( \epsilon = 1 - \phi \)), and \( \rho_f \) and \( \rho_s \) are the fluid and solid phase densities respectively. The locally averaged interstitial fluid velocity and particle phase velocity vectors are written in the laboratory frame of reference as \( \mathbf{u} \) and \( \mathbf{v} \) respectively. The fluid and solid phase stress tensors (defined in a compressive sense) are represented by \( \mathbf{\sigma}_f \) and \( \mathbf{\sigma}_s \). The gravity force vector is \( \mathbf{g} \), and \( \mathbf{k} \) is the unit vector pointing in the positive vertical direction against gravity. \( \tilde{F} \) represents the fluid-particle interactive force per unit
of bed volume, which results from the relative motion of the fluid and particle phases. Writing the equations in this way introduces the wavespeed $\omega$, which is used in this analysis as a bifurcation parameter.

The fluid phase stress tensor is represented by $\sigma_f$ and is, in general, a function of the rate of deformation of the fluid phase. A form analogous to a Newtonian fluid will be assumed for the fluid phase stress tensor [10]:

$$\sigma_f = P I - \mu \left[ \nabla u + \nabla (u)^T - \left( \frac{2}{3} - \frac{\lambda}{\mu} \right) (\nabla \cdot u) I \right]$$ (5)

where $P$ is the fluid pressure, and $\mu$ and $\lambda$ are the fluid shear and bulk viscosities respectively. For the particle phase, continuum mechanics arguments provide a constitutive relation for $\sigma_s$ in terms of the rate of deformation of the particle phase (Anderson & Jackson [26]). The particle phase stress tensor takes the form:

$$\sigma_s = P_s I - \mu_s \left[ \nabla v + \nabla (v)^T - \left( \frac{2}{3} - \frac{\lambda_s}{\mu_s} \right) (\nabla \cdot v) I \right]$$ (6)

where $P_s$ is the solid phase pressure, and $\mu_s$ and $\lambda_s$ are the effective shear and bulk viscosities respectively. In this study, we have adopted a closure from the work of Johnson & Jackson [27] for expressing $P_s$ as a monotonically increasing function with respect to solids volume fraction $\phi$:

$$P_s = \frac{g_0 \phi^{m_1}}{(\phi_p - \phi)^{m_2}}$$ (7)

where $g_0$ is a constant, and $\phi_p$ represents the solids volume fraction under close-packed conditions ($\phi_p = 0.65$, Berryman [30]). We have considered both a linear form for $P_s$ ($m_1 = 1$ and $m_2 = 0$) and a non-linear form ($m_1 = 1$ and $m_2 = 2$) [22]. The shear viscosity of the solid $\mu_s$ is expected to be a monotonically increasing function with respect to $\phi$ [22]:

$$\mu_s = \frac{R \phi}{1 - (\phi/\phi_p)^{1/3}}$$ (8)

where the value of parameter $R$ is selected to yield a shear viscosity within a range suggested by experiments. The bulk viscosity is assumed to be zero ($\lambda_s = 0$) in this study [26].

The force due to the relative motion of the fluid and solid ($\tilde{F}_i$) consists of a frictional or “drag” force in the direction of fluid flow, which is a function of slip velocity between the particles and fluid, and a force of virtual mass, which is a function of the acceleration reaction of fluidized particles induced by a change in fluid phase momentum. We have adopted a general closure from [26] to express this force term:

$$\tilde{F}_i = \epsilon \beta (u - v) + \phi C_{\rho_f} \frac{d}{dt} (u - v).$$ (9)

The first term on the right hand side of equation 9 represents the drag on the particles due to the flow of fluid where $\beta$ is the drag coefficient. A convenient form for a monocomponent bed is the Richardson & Zaki relation [28], which expresses the interstitial fluid velocity in the vertical $z$ direction ($u_z$) as a function of bed expansion $\phi$:

$$u_z = v_t (1 - \phi)^{(n-1)}$$ (10)
In this expression, \( v_t \) is the terminal settling velocity (normal bed mode) or rising velocity (inverse bed mode) of a single particle in an infinite fluid medium. The empirical correlation index \( n \) is a function of the local voidage and particle Reynolds number \( Re \) computed at \( v_t \) [28]. The settling (or rising) velocity is computed under equilibrium conditions when a single sphere is allowed to settle by gravity (when \( \rho_s > \rho_f \)), or rise by buoyancy (when \( \rho_s < \rho_f \)) in a viscous fluid at constant velocity. The drag force coefficient (in expression 9), derived from the Richardson–Zaki relation for a uniformly fluidized bed, is written as,

\[
\beta = \frac{\phi (\rho_s - \rho_f) g_z}{v_t (1 - \phi)^{(n-1)}} \tag{11}
\]

where \( g_z \) is the standard acceleration of gravity. Using the particle Reynolds number defined as,

\[
Re = \frac{2r_p v_t \rho_f}{\mu} \tag{12}
\]

\( v_t \) is computed using the equilibrium force balance relationship:

\[
F_D + F_B = F_G \tag{13}
\]

where \( F \) is the force per unit volume exerted on a single spherical particle of radius \( r_p \), and subscripts \( D, B \) and \( G \) represent ‘drag’, ‘buoyancy’, and ‘gravity’ respectively. The individual force expressions are written as,

\[
F_D = \frac{\pi}{2} \rho_f v_t^2 r_p^2 \beta_D \tag{14}
\]

\[
F_B = + \frac{4\pi}{3} \rho_f r_p^3 g_z \tag{15}
\]

\[
F_G = - \frac{4\pi}{3} \rho_s r_p^3 g_z \tag{16}
\]

The case studies examined in this work fall within the intermediate flow regime defined by \( 1 \leq Re \leq 10^3 \) where the drag coefficient \( \beta_D \) in equation 14 is estimated by \( \beta_D \approx 18 Re^{-0.6} \) (Denn [29]). Using the force balance relationship (equation 13), \( v_t \) can be estimated by,

\[
v_t \approx \left[ \frac{2g_z}{27} \left( \frac{\rho_s}{\rho_f} - 1 \right) \right]^{5/7} (2r_p)^{8/7} \left( \frac{\rho_f}{\mu} \right)^{3/7} \tag{17}
\]

Hence the terminal velocity (normal mode) is positive, and the rising velocity (inverse mode) is negative due to the sign of the term \( (\rho_s/\rho_f - 1) \).

The second term on the right hand side of equation 9 represents the force of virtual mass, which is considered to be important in liquid-fluidized beds. \( C \) is the virtual mass coefficient. The relative acceleration rate takes the following form [26]:

\[
\frac{d(u - v)}{dt} = \frac{\partial}{\partial t} (u - v) + v \cdot \nabla (u - v)
\]
3 Linear stability analysis

The simplest solution to the model equations 1 through 4 represents that of the uniform fluidization state where; the local mean particle velocity vector, \( \mathbf{v} \) is zero; the local mean fluid velocity vector, \( \mathbf{u} \) is constant in space and time and directed in either the positive or negative vertical direction depending on the term \((\rho_s/\rho_f - 1)\) in equation 17; and the local mean solids volume fraction, \( \phi \) is spatially uniform and constant in time. Under these conditions,

\[
\phi = \phi_o \tag{18}
\]

\[
\epsilon = \epsilon_o = (1 - \phi_o) \tag{19}
\]

\[
u = \pm ku_o \tag{20}
\]

\[
v = 0 \tag{21}
\]

where the subscript ‘o’ is used to indicate that a quantity is evaluated at conditions corresponding to the uniform base state. In the absence of velocity gradients, all inertial and viscous force terms reduce to zero. Moreover, the bed is considered homogeneous with respect to the locally averaged particle concentration \( \rho_s \phi_o \). As a result, the gradient of the isotropic compressive force term \( \nabla P_s \) also goes to zero, and the pressure gradient across the bed is due only to the dynamic fluid pressure in the direction of fluid velocity. Since we are interested in the stability of the uniform base state, we impose a perturbation in the form of a localized voidage disturbance having small amplitude, and rewrite the equations in terms of perturbation variables \( \phi', \epsilon', u', v' \) and \( P' \), which are defined as,

\[
\phi = \phi_o + \phi' \tag{22}
\]

\[
u = \pm ku_o + u' \tag{23}
\]

\[
v = 0 + v' \tag{24}
\]

\[
P = P_o + P' \tag{25}
\]

The equations are linearized about the uniform base state by substituting the above expressions for \( \phi, \epsilon, u, v \) and \( P \) (equations 22 through 25) into equations 1 through 4, and performing a Taylor series expansion about the steady state solution. Neglecting terms in the series involving powers greater than one, and eliminating products of perturbation variables results in a system of linearized partial differential equations (PDEs) written in one-dimension \( z \) as,

\[
\frac{\partial \epsilon'}{\partial t} + \epsilon_o \nabla \cdot \mathbf{u} + (u_o - \omega)k \frac{\partial \epsilon'}{\partial z} = 0 \tag{26}
\]

\[
\frac{\partial \phi'}{\partial t} + \phi_o \nabla \cdot \mathbf{v} - \omega k \frac{\partial \phi'}{\partial z} = 0 \tag{27}
\]

\[
\rho_f \left(1 + \frac{\phi_o C_o}{(1 - \phi_o)}\right) \left[ \frac{\partial \mathbf{u}}{\partial t} + (u_o - \omega)k \frac{\partial \mathbf{u}}{\partial z} \right] - \phi_o C_o \rho_f \left(1 - \phi_o\right) \left[ \frac{\partial \mathbf{v}}{\partial t} - \omega k \frac{\partial \mathbf{v}}{\partial z} \right] = \rho_f \left(1 + \frac{\phi_o C_o}{(1 - \phi_o)}\right) \frac{\partial \mathbf{u}}{\partial t} - \rho_f \phi_o \left(1 + \frac{C_o}{(1 - \phi_o)}\right) \frac{\partial \mathbf{v}}{\partial t} + (u_o - \omega)k \frac{\partial \mathbf{u}}{\partial z} \tag{28}
\]

\[
-\nabla \cdot P' + (\lambda_o + 1/3\mu_o) \nabla (\nabla \cdot \mathbf{v}) + \mu_o \nabla^2 \mathbf{v} - \beta_o (\mathbf{u} - \mathbf{v}) - \beta_o \phi_o u_o k \frac{\partial \mathbf{v}}{\partial t} \tag{29}
\]

\[
\mathbf{P}_{so} \nabla \cdot \phi' + (\lambda_{so} + 1/3\mu_{so}) \nabla (\nabla \cdot \mathbf{v}) + \mu_{so} \nabla^2 \mathbf{v} + \beta_o (\mathbf{u} - \mathbf{v}) + \beta_o \phi_o u_o k - \phi' (\rho_s - \rho_f) g_z k
\]
where the terms $\beta_o$ and $Pt_o$ are used to represent the following derivatives evaluated under conditions of uniform fluidization:

$$\beta_o = \left( \frac{\partial \beta}{\partial \phi} \right)_{\phi = \phi_o} \quad Pt_o = \left( \frac{\partial P_o}{\partial \phi} \right)_{\phi = \phi_o}$$

These equations have been made non-dimensional by taking the particle radius $r_p$ as a length scale, and the interstitial fluid velocity at uniform fluidization in the axial direction $\pm(u_{zo})$ as a velocity scale computed using the Richardson & Zaki form. Hence, in normal mode operation, fluid velocity is in the positive vertical $z$ direction ($+u_{zo}$) and in the direction of gravity ($-u_{zo}$), in the inverse mode. The fluid to solid density ratio is defined as $\delta = \rho_f/\rho_s$; time is scaled with $\pm(r_p/u_{zo})$; and $\beta$ is scaled with $\pm(\rho_s u_{zo}/r_p)$. The particle phase pressure $P_o$ and constant $g_o$ are scaled with $\rho_s(u_{zo})^2$, and therefore always positive. Scaling results in two dimensionless groups: the Froude number ($Fr$) defined as $Fr = u_{zo}^2/g_z r_p$, and the particle Reynolds number defined as $Re_p = (\rho_s r_p | u_{zo} |) / \mu_s$. $Re_p$ is computed at $u_{zo}$, and differs from $Re$ computed at $v_t$, which is used to determine the Richardson-Zaki correlation index $n$ (see equation 10). $Re_p$ only considers fluid velocity magnitude $| u_{zo} |$ so that the sign of $Re_p$ is always positive. This way, Reynolds number effects in the two bed modes can be compared without regard to flow direction. Using a factorization method [26], the velocity terms can be eliminated from equation 29 by substituting expressions for $\nabla \cdot \mathbf{u}$ and $\nabla \cdot \mathbf{v}$ from 26 and 27 into the divergence of equation 29 to obtain a single scalar equation, which is linear in the perturbation variable $\phi t$ [20]:

$$L\phi t = 0 \quad (30)$$

where the linear operator $L$ is defined as

$$L = A\partial^2_t + 2\left( \bar{C} - A\bar{\omega} \right) \partial_t \partial_z - M\Delta + \left( A\omega^2 - 2\bar{C}\bar{\omega} + \bar{C} \right) \partial_z^2$$

$$+ E\partial_t + (D - E\bar{\omega}) \partial_z + (J\bar{\omega} - H)\delta \partial_t - J\delta \partial_t$$

The overbar designates scaled variables, and $\Delta$ is the Laplacian operator. Substituting in the drag force coefficient $\beta_o$, evaluated under steady state conditions from equation 11, the coefficients in equation 30 are defined as

$$A = Fr \left[ \frac{(1 - \phi_o)}{\delta \phi_o} + 1 + \frac{C_o}{\phi_o (1 - \phi_o)} \right] \quad \bar{C} = Fr \left( 1 + \frac{C_o}{(1 - \phi_o)} \right)$$

$$D = \frac{(1 - \delta)n}{\delta} \quad E = \frac{(1 - \delta)}{\delta \phi_o} \quad H = 0$$

$$J = \frac{Fr}{Re_p \phi_o} \frac{(1 - \phi_o)}{\delta \phi_o} \quad M = \frac{Fr Pt_o (1 - \phi_o)}{\delta \phi_o}$$

When the base state of uniform fluidization is unstable, the fastest growing disturbance takes the form of a one-dimensional vertically traveling wave having no transverse structure. Thus, we seek a solution to equation 30 in the form of a plane wave disturbance $\phi t = \tilde{\phi} \exp(st) \exp(k \cdot x)$ having a complex amplitude $\tilde{\phi}$, and wavenumber vector $\mathbf{k}$. The position vector is denoted by $\mathbf{x}$, and $s$ represents a complex conjugate $s = \zeta \pm i\kappa$. Substituting in the expression for $\phi t$ and its derivatives into 30, we obtain the following dispersion relation expressing wave velocity $\omega$ as a function of wavenumber $\mathbf{k}$ (in one dimension) in the traveling wave frame:

$$A(s - ik_z \omega)^2 + (s - ik_z \omega) \left( E + 2i\bar{C}k_z + Jk_z^2 \right) + iDk_z + \left( M - \bar{C} \right) k_z^2 + iHk_z^3 = 0 \quad (31)$$
The quantities \( s, \omega \) and \( \kappa_z \) are scaled quantities unless specified otherwise. The overbars have been omitted for convenience. Eigenvalues \( \sigma = s - i\kappa_z \) can thus be obtained from this dispersion relation in any traveling wave frame. In the laboratory frame \( (\omega = 0) \), complex eigenvalues \( s_{1,2} = \zeta \pm i\chi \) describe one-dimensional periodic wave solutions satisfying the system of linearized equations 26 through 29 where the real part of \( s \ (\zeta) \) determines the growth (or decay) rate of the disturbance, and the imaginary part \( (\chi) \) determines the propagation velocity of the wave. In the laboratory frame of reference, it has been shown by Göz [25] and Göz et al. [1] that the base state is linearly stable to disturbances of small amplitude if the following two conditions are met:

\[
f(d) \geq 0; \quad f(h) \geq 0
\]

where \( f(d) \) and \( f(h) \) are not independent of one another, and

\[
f(d) = m - c + 2cd - d^2, \quad f(h) = m - c + 2ch - h^2
\] (32)

\[
m = M/A, \quad c = \tilde{C}/A, \quad d = D/E, \quad h = H/J
\]

This criterion is based upon the behavior of \( \zeta \), the real part of \( s \) at small and large wavenumbers. In each system this work examines, it can be shown that \( d \) is always greater than \( h \) and \( f(h) \) is always greater than \( f(d) \). Hence for all conditions \( f(d) < 0 \), the base state is unstable to small disturbances, and at \( f(d) = 0 \) the base state is marginally stable at some value \( \phi_c \). This condition allows us to calculate a minimum value for the particle pressure derivative \( P'_{so} \) evaluated at \( \phi_o = \phi_c \) at which point we might suspect the state of uniform fluidization loses stability. As recognized by Garg & Pritchett [11], the contribution of a force term in the momentum balance equations, proportional to the gradient of \( \phi \), and monotonically increasing with respect to \( \phi \), is necessary to stabilize the bed. The particle phase isotropic compressive force, \( P_s \) provides such a force term. It can be shown in this work that the stability of the uniform state is extremely sensitive to the Taylor series expansion of this term, and \( P'_{so} \) is thus regarded as a valuable measure of bed stability in the neighborhood of the uniform base state. In this study we examined closures for \( P_s \) defined by equation 7. We see from the derivative of \( P_s \) that the value of the constant \( g_o \) is useful for comparing the stability of two fluidized systems at \( \phi_c \) under identical operating conditions.

We know that a Hopf bifurcation point is possible when the vector field, which has been linearized about the base state, has a set of purely imaginary eigenvalues with all remaining eigenvalues having non-zero real parts. It is at these values that one-dimensional traveling waves bifurcate from the steady state solution. Göz & Sundaresan [20] show that by setting \( s = 0 \) in equation 31, we can obtain the propagation velocity of the wave \( (\omega = \omega_{crit}) \), and the critical wavelenth \( (\kappa_z = \kappa_{crit}) \) at the Hopf bifurcation point as,

\[
\omega_{crit} = c + \left( c^2 + m - c \right)^{1/2} \quad \kappa_{crit} = \left[ \frac{E (d - \omega_{crit})}{J (\omega_{crit} - h)} \right]^{(1/2)}
\]

These results provide criteria for comparing the bifurcation structure of traveling wave solutions in the vicinity of \( \phi_c \).

4 Quasi-steady periodic solutions

In this section, we compute quasi-steady periodic solutions in the traveling wave frame (of reference). The derivation which follows is based upon that performed by Needham & Merkin [24] using two-phase continuum equations of continuity and motion describing a single-component gas-fluidized
bed. The equations we derive consider the viscous and inertial effects of the fluid phase (including virtual mass), which were considered negligible by these authors in their analysis of gas systems. We consider one-dimensional vertical flow in normal and inverse fluidized beds for which equations 1 through 4 apply. We simplify matters by adding the two continuity equations 1 and 2 in the laboratory frame of reference \((ω = 0)\), and then integrate with respect to \(z\) to obtain the following equation in one-dimension \(z\):

\[
(1 - φ)u_z + φv_z = \bar{M}(t)
\]  

(33)

where \(\bar{M}(t)\) is some function of time (constant with respect to space), and \(u_z\) and \(v_z\) are used to represent the locally averaged fluid and solid phase velocities respectively in the axial direction \(z\). Equation 33 replaces equation 1 in this analysis. The non-dimensional equations can be written in one-dimension in the laboratory frame of reference as,

\[
(1 - φ)\bar{u}_z + φ\bar{v}_z = \bar{M}(t)
\]  

(34)

\[
\frac{∂φ}{∂t} + \frac{∂(φ\bar{v}_z)}{∂z} = 0
\]  

(35)

\[
\phi \left[ \frac{∂\bar{v}_z}{∂t} + \bar{v}_z \frac{∂\bar{v}_z}{∂z} \right] - δφ \left[ \frac{∂\bar{u}_z}{∂t} + \bar{u}_z \frac{∂\bar{u}_z}{∂z} \right] = \frac{4η}{3Re_p} \frac{∂^2\bar{v}_z}{∂z^2} - \frac{∂P_s}{∂φ} \frac{∂φ}{∂z} - \frac{φ}{Fr} (1 - δ) + \frac{φ (1 - δ) (1 - φ_o)^n}{Fr} \left( \bar{u}_z - \bar{v}_z \right) + \frac{φ}{(1 - φ)^n} C_δ \left[ \left( \frac{∂\bar{u}_z}{∂t} + \bar{u}_z \frac{∂u_z}{∂z} \right) - \left( \frac{∂\bar{v}_z}{∂t} + \bar{v}_z \frac{∂v_z}{∂z} \right) \right]
\]

(36)

where all scaled quantities are represented with an overbar. The solid phase viscosity \(μ_s\) is scaled with the viscosity of the particle assembly at uniform fluidization condition, \(μ_{so}\), evaluated at uniform solids volume fraction \(φ_o\). This dimensionless quantity is defined as \(η = μ_s/μ_{so}\). We assumed that the fluid and solid phase bulk viscosities \(λ\) and \(λ_s\) both equal zero, and the particle shear viscosity \(μ_s\) takes the form of expression 8. The drag coefficient \(β\) has been replaced by

\[
\bar{β} = \frac{φ (1 - δ) (1 - φ_o)^n}{Fr} \frac{(1 - φ)^n}{(1 - φ_o)^n}
\]

For simplicity, we assumed a linear form for \(P_s\) defined by equation 7 where \(m_1 = 1\) and \(m_2 = 0\).

If the base state of uniform fluidization is unstable to small amplitude disturbances in voidage, a bifurcation to a family of traveling waves may be possible. We seek quasi-stationary periodic solutions to equations 34, 35 and 36 by transforming these equations to a frame of reference, which moves at the same velocity as the wave. We first introduce the moving coordinate system \((Y = z - ωt)\), where the wave velocity \((ω)\) is a constant, and serves as the bifurcation parameter. We then transform the equations by incorporating the dimensionless derivatives \(∂_z = ∂_Y\) and \(∂_t = -ω∂_Y\) where:

\[
ω = ±ωu_{zo}
\]

\[
Y = \bar{z} - ω\bar{t}
\]

At uniform fluidization, we know from equations 20 and 21 that \(\bar{v}_z = 0\) and \(\bar{u}_z = 1\). From equation 34, \(\bar{M} = (1 - φ_o)\) and

\[
\bar{u}_z = \frac{(1 - φ_o)}{(1 - φ_o)} - \frac{φ}{(1 - φ_o)} \bar{v}_z
\]

(37)

Transforming equation 35 to the traveling wave frame,

\[
-\bar{ω} \frac{∂φ}{∂Y} + \frac{∂[φ\bar{v}_z]}{∂Y} = 0
\]

(38)
and integrating with respect to $Y$ yields $\phi(\bar{v}_z - \bar{\omega}) = \bar{N}$ where $\bar{N}$ is a constant. Using conditions at uniform fluidization, we find $\bar{N} = -\phi_o\bar{\omega}$, and

$$\bar{v}_z = \bar{\omega}\left(1 - \frac{\phi_o}{\phi}\right)$$  \hspace{1cm} (39)

Substituting the expressions for $\bar{u}_z$ and $\bar{v}_z$ (from 37, 39) and their derivatives with respect to $Y$ into equation 36 yields a single second order equation in $\phi$. Since we seek periodic solutions to equation 36, we work in the phase plane $(\phi, \Omega)$, where $\Omega = d\phi/dY$, and write two first order differential equations,

$$f_1 = \frac{d\phi}{dY} = \Omega$$  \hspace{1cm} (40)

$$f_2 = \frac{d\Omega}{dY} = \frac{2\Omega^2}{\phi} + B\phi^2 \left[ \bar{g}_o - \frac{\phi(1-\phi_o)^2(1-\bar{\omega})^2\delta}{(1-\phi)^3} \left(1 + \frac{C}{(1-\phi)}\right) - \left(\frac{\bar{\omega}\phi_o}{\phi}\right)^2 \left(1 + \frac{C\delta}{(1-\phi)}\right) \right] \Omega$$  \hspace{1cm} (41)

where the coefficient $B$ is defined as,

$$B = \left(\frac{3Re_p}{4n\bar{\omega}\phi_o}\right)$$

The simplest solution to equations 40 and 41 is that which represents the uniform fluidized state, $\phi = \phi_o$ and $\Omega = 0$. Periodic solutions corresponding to traveling waves are closed orbits, which surround the equilibrium state $(\phi_o, 0)$ in the phase plane $(\phi, \Omega)$. Such solutions are found by determining the two eigenvalues $s_{1,2} = \zeta \pm i\chi$ of the linearized equations $f_1$ and $f_2$, which are

$$s_{1,2} = \frac{1}{2} \left[ tr\bar{J} \pm \sqrt{(tr\bar{J})^2 - 4 |\bar{J}|} \right]$$

where $\bar{J}$ is the Jacobian matrix.

5 Results

5.1 Linear stability analysis

In this section, we examine instabilities of fluidized beds operating in normal and inverse mode. The beds have been uniformly expanded with water to reach a marginally stable and spatially uniform steady state defined by a critical solids volume fraction $\phi_c = \phi_o$. We investigated the stability of this base state against perturbations to the flow distribution in the form of localized voidage disturbances of small amplitude. In order to investigate the effect of the $Fr$ number on overall bed stability, we considered examples of normal and inverse beds having comparable $Fr$ numbers and $Fr$ numbers, which differed by as much as a 4:1 ratio. We focused on this dimensionless group because of its identified importance in distinguishing instability behavior in gas- and liquid-fluidized beds as reported in the theoretical and experimental literature.

In the experimental work of Wilhelm & Kwauk [7], bed behavior is differentiated as being either particulate (having greater operational stability), or aggregative (exhibiting more complex bubbling behavior) based primarily on the Fr number. These authors predict that the higher the
$Fr$, the more likely the bed is to bubble, and exhibit aggregative behavior. Attempts to confirm the empirical significance of the $Fr$ number have successfully been made by other researchers. For example, Anderson et al. [14] and Glasser et al. [22] show qualitative differences in the structure of two-dimensional traveling wave forms, which bifurcate from the uniform fluidization state in a two-dimensional stability analysis of gas- and liquid-fluidized beds. Göz & Sundaresan [20] show similar results in a low amplitude analysis. However, $Fr$ numbers in gas- and liquid-fluidized systems can vary by several orders of magnitude. The inverse mode of operation provides a unique opportunity to take a closer look at the effect of fluidization direction and $Fr$ number in uniformly fluidized beds described by two sets of subtly varying dimensionless groups.

**Case I: Comparable $Fr$ number**

At this stage, it is useful to consider some specific examples of water-fluidized beds whose particle properties are shown in Table 1. We have chosen two systems, *Case I* and *Case II*, which are realistic so that future experimental work might be possible. We first consider the example of *Case I*: the bed operating in normal mode consists of 775µm water saturated carbon char particles ($\rho_s = 1500$ kg/m$^3$), and the inverse bed consists of 1000µm plastic particles ($\rho_s = 666.7$ kg/m$^3$). The particles are considered to be spherical, and the wall-effects of the fluidization column are not considered in the calculation of rising and settling velocities. Particle size and density were selected, such that the terminal settling velocity in normal mode would be exactly equivalent to the terminal rising velocity in inverse mode for an individual particle in an infinite volume of fluid. As a result, the $Fr$ numbers are comparable in magnitude and always positive because of the squared $u_{zo}$ term; however, they are not identical. The diameter of the low density plastic particles is 25% larger than the carbon in order to obtain equivalent rising and settling velocities. We have not specified the plastic material, only its density and size. However, we have assumed that the plastic has a non-porous surface, although it may be impregnated with air. As a basis for comparison,

<table>
<thead>
<tr>
<th>Case</th>
<th>Bed Type</th>
<th>Material</th>
<th>$d_p$ (µm)</th>
<th>$\rho_s$ (kg/m$^3$)</th>
<th>$\Uparrow v_t$ (mm/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>normal</td>
<td>carbon-char*</td>
<td>775</td>
<td>1500**</td>
<td>50.4</td>
</tr>
<tr>
<td></td>
<td>inverse</td>
<td>plastic</td>
<td>1000</td>
<td>666.7</td>
<td>-50.4</td>
</tr>
<tr>
<td>II</td>
<td>normal</td>
<td>glass beads</td>
<td>1000</td>
<td>2200</td>
<td>126.0</td>
</tr>
<tr>
<td></td>
<td>inverse</td>
<td>plastic</td>
<td>1000</td>
<td>454.5</td>
<td>-71.7</td>
</tr>
</tbody>
</table>

* water impregnated  
** mean density of water saturated hollow char  
$\Uparrow$ velocity in the intermediate flow regime calculated using Denn (1980).
the bed is marginally stable at $\phi_c$ using the non-linear closure for $P_s$ taken from equation 7 with $m_1 = 1$ and $m_2 = 2$, and the stability condition from [20] (see equation 32). The normal bed has a fluid to solid density ratio $\delta_{\text{norm}}$, which is reciprocal to that of the inverse bed $\delta_{\text{inv}} = 1/\delta_{\text{norm}}$. All dimensionless quantities are based on a length scale of $r_p$ and velocity scale $\pm(u_{zo})$. Based on linear

stability results of primary instabilities in gas- and liquid-fluidized beds [20], one might expect that the limiting value of $g_o(min)$ would be greater in the bed having the higher $Fr$ number (in this case, the normal bed). This reasoning is consistent with the criterion established by Wilhelm & Kwauk [7]. In Case I however, $g_o(min)$ is greater in the inverse bed of plastic particles even though the $Fr$ number is slightly lower than in the normal bed of carbon particles. In fact, it can be shown from equation 32 that at the point of marginal stability ($f_d = 0$ at $\phi_o = \phi_c$), the constant $g_o(min)$ is independent of $Fr$ and a function only of the fluid to particle density ratio $\delta$. We now move on to investigate the effect of $Fr$ number on relative instability strength, which we have measured using both the magnitude of the maximum dimensionless growth rate $|\zeta_{\text{max}}|$ and the critical wavenumber $\kappa_{\text{crit}}$ at which saturated one-dimensional traveling waves bifurcate from the steady state solution.

The linear stability of the uniform state against one-dimensional disturbances is illustrated in figures 1(a) and 1(b) for the Case I normal and inverse beds, respectively, whose properties and dimensionless parameters are described in Tables 1 and 2. In these figures, the real part ($\zeta$) of the complex growth rate ($s$) of a one-dimensional, vertically traveling disturbance is plotted versus the wavenumber $\kappa_z$ for a range of $\phi_o (\leq \phi_c)$ values. The plotted quantities are dimensionless, and since the units for $\zeta$ are reciprocal seconds ($s^{-1}$), the growth rate is scaled with $-(u_{zo}/r_p)$ in the inverse bed. We have thus plotted $-\zeta$ versus $\kappa_z$ in figure 1(b) so that actual growth (or decay) of the wave is obvious to the reader. We have examined the linear stability of the base state for various $\phi_o$ values, which are indicated in the figure captions. Computed results for the Case I stability analysis are tabulated in Table 3 for direct comparison. Figures 1(a) and 1(b) show that both beds are stable at all vertical wavenumbers for $\phi_o \geq \phi_c$ since the (dimensional) real part of $s$ is always less than zero in this range. At $\phi_c = 0.576$, the bed is considered to be marginally stable. For $\phi_o < \phi_c$, the uniform state is unstable for a finite range of $\kappa_z$ values. As $\phi_o$ decreases, the bed becomes more unstable at higher values of $\kappa_z$, and there is a corresponding increase in the maximum (dimensionless) growth rate of the wave (indicated by $|\zeta_{\text{max}}|$) as shown in Table 3. In this table, we have also included the dimensional maximum growth rate $\zeta_{\text{max}}^*$ to compare the relative magnitudes of the growth rates in

<table>
<thead>
<tr>
<th>Case</th>
<th>Bed Type</th>
<th>$\delta_{\text{norm}}$</th>
<th>$Re_p \times 10^3$</th>
<th>$Fr \times 10^2$</th>
<th>$g_o(min)$</th>
<th>$\pm u_{zo}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I normal</td>
<td>1/1.5</td>
<td>6.7</td>
<td>1.9</td>
<td>2.3</td>
<td>8.4</td>
<td></td>
</tr>
<tr>
<td>inverse</td>
<td>1.5/1</td>
<td>4.1</td>
<td>1.7</td>
<td>3.5</td>
<td>-9.0</td>
<td></td>
</tr>
<tr>
<td>II normal</td>
<td>1/2.2</td>
<td>42.6</td>
<td>16.2</td>
<td>1.4</td>
<td>28.2</td>
<td></td>
</tr>
<tr>
<td>inverse</td>
<td>2.2/1</td>
<td>6.4</td>
<td>4.0</td>
<td>4.3</td>
<td>-14.0</td>
<td></td>
</tr>
</tbody>
</table>

$\dagger$ fluidization velocity at $\phi_c = 0.576$ in water.
Table 3: Linear stability of water fluidized beds at values $\phi_o < \phi_c$: Case I & II

<table>
<thead>
<tr>
<th>Linear Stability Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>I</td>
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<td></td>
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<td></td>
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</tbody>
</table>

* dimensional maximum growth rate evaluated at $\phi_o$ having units of $s^{-1}$.
† evaluated at the Hopf bifurcation point

We can see from these figures and from the tabulated data that both $\zeta^{**}_{max}$ and $|\zeta_{max}|$ corresponding to the higher $Fr$ (in this case, the normal bed of carbon) are always greater than those for the lower $Fr$ (inverse bed). These data suggest that instabilities occurring in the neighborhood of the base state in normal fluidization grow at a much faster rate than in the inverse bed, even though the inverse bed has a higher propensity to become unstable as previously recognized. We have thus considered two measures of bed stability; viz. the propensity of the bed to become unstable, as measured by the relative value of $g_o(min)$ computed at $\phi_c$; and the strength of the instability, as measured by the maximum growth rates $\zeta^{**}_{max}$ and $|\zeta_{max}|$ of traveling wave solutions. These results are consistent with the results of [20] for gas and liquid systems when one considers their dimensional predictions of maximum growth rate. These authors found that dimensional growth rates in the air-fluidized system, having a $Fr$ four orders of magnitude greater than the water-fluidized system, were considerably larger than the water-fluidized bed, which we know to be less unstable than the air-fluidized bed.

In figures 1(a) and 1(b), let us choose the curve representing $\phi_o = 0.54$. Beginning at the far right hand side of the x axis, or highest value of $\kappa_z$, and moving to the left, we encounter a critical wavenumber value $\kappa_{crit}$ at point A where $\zeta = 0$, and the two eigenvalues become purely imaginary. This point is a Hopf bifurcation point, and signals the birth of a family of one-dimensional traveling wave solutions. Each traveling wave solution moves at a dimensionless wave velocity $\omega$ relative to the laboratory frame of reference, which can be determined as part of the solution. If one were to travel in a moving frame of reference at velocity $\omega$, the solution would appear to be a steady state. The Hopf bifurcation points for the values of $\phi_o = 0.54$ and $\phi_o = 0.57$ are labeled A and B.
respectively, and the values of $\kappa_{\text{crit}}$ and $\omega_{\text{crit}}$ are tabulated in Table 3 for these points and at other $\phi_o$ conditions. Note that the $\kappa_{\text{crit}}$ values are consistently higher in the normal bed, especially at the lower $\phi_o$ values, and that these results are consistent with higher reported values of $|\zeta_{\text{max}}|$. The effect of $Fr$ number and the fluid to solid density ratio $\delta$ on instability strength are further examined in Case II to follow where beds are selected having $Fr$ numbers differing by a factor of 4 to 1 respectively.

**Case II: Fr number differing by 4:1**

We attempted to add to our understanding by considering another case (Case II) of water-fluidized normal and inverse beds having $\delta_{\text{norm}}$ and $\delta_{\text{inv}}$, which are further from unity than in the Case I systems. In this case, 1000 $\mu$m heavy glass beads ($\rho_s = 2200 \text{ kg/m}^3$) and 1000 $\mu$m light plastic particles ($\rho_s = 454.5 \text{ kg/m}^3$) are fluidized with water under $Fr$ number conditions differing by a factor of 4 to 1 respectively. The large particle density difference of the glass and plastic contributes significantly to the variation in $Fr$ because of the difference in terminal rising and settling velocities of the equi-sized particles. Particle properties for the Case II systems are shown in Table 1; the dimensionless parameters are shown in Table 2 at equilibrium conditions $\phi_c = 0.576$. In an experimental system, these fluidized beds would visually appear identical if the particles were the same color and the beds were both uniformly stable at constant bed voidage. Note in Table 2 that the computed value for $g_o(\text{min})$ in the inverse plastic bed is three times that for the normal glass bed despite the four fold $Fr$ number difference. Based upon the imposed stability criteria, these results confirm previous findings that the propensity of the bed to destabilize is a function of $\delta$ and independent of $u_{zo}^2 (1/r_p$ is the same in this case). The uniformly fluidized inverse bed thus appears to be less stable to perturbations than the normal bed at $\phi_o$ values close to $\phi_c$ due to the ratio of fluid to solid density alone.

The results of a linear stability analysis of the uniformly fluidized systems in Case II are reported in Table 3 for various values of $\phi_o$ close to $\phi_c$. We can see from these data that the growth rates of the disturbances also follow the same trend as observed in Case I, i.e., $|\zeta_{\text{max}}|$ and $\zeta_{\text{max}}^*$ are much greater in the glass system having the higher $Fr$ number. However, the variation in growth rate is more dramatic in the glass & plastic beds suggesting a strong dependency on the square of the fluid velocity term. Moreover in Case II, we see in Table 3 that the Hopf bifurcation points of the uniform base state in the glass bed occur at higher wavenumber values $\kappa_{\text{crit}}$, and corresponding lower $\omega_{\text{crit}}$ values than in the plastic bed for every $\phi_o$ value we examined. This means that when the normal bed becomes unstable, one-dimensional traveling waves in the glass bed grow at a faster rate and propagate through the bed at a slower velocity than waves moving through the inverse bed. These results are consistent with growth rate predictions suggesting the relative strength of unstable waveforms within the $\delta$ range investigated can be predicted based primarily on $Fr$ number with some dependency on $\delta$ as shown in Case I. This conclusion is consistent with ideas put forth previously.

In summary, the $Fr$ number appears to be an important parameter with respect to predicting instability strength, but has no effect on the propensity of the bed to destabilize, since we have shown that the inverse bed is significantly less stable to perturbations regardless of $Fr$ in all the case studies examined. These results suggest that $\delta$ controls the onset of an instability, and that the strength of the instability is strongly influenced by $Fr$ and to a lesser extent, $\delta$. The experimental and theoretical literature clearly show a correlation between $Fr$ number and the likelihood of gas- and liquid-fluidized systems to exhibit bubbling behavior when the inertial and viscous effects of
the gas system are neglected. It is important to point out that Fr number variation in gas- and liquid-fluidized systems is quite significant (varying by orders of magnitude) compared to the subtle variations observed in the water-fluidized systems examined here. Moreover, liquid fluidized beds are far less likely to “bubble”.

So far, we have examined the linear stability of the uniform base state using the bed voidage $\phi_o$ as the basis for comparison. This seemed reasonable, since the particle pressure term plays a dominant role in bed stabilization, and we have represented it as an increasing function of solids volume fraction. We now move to the traveling wave frame to compute one-dimensional traveling wave solutions (1D-TW’s) emanating from Hopf bifurcations of the steady state solution. We compare the bifurcation diagrams and high amplitude wave profiles of 1D-TW’s, which can suggest the structure of fully developed wave forms.

### 5.2 One-dimensional traveling waves

Periodic solutions describing a family of one-dimensional traveling waves were computed numerically using a continuation technique from the software package AUTO (Doedel [31]). This software was used to compute branches of periodic solutions satisfying the ordinary differential equations 40 and 41 in the phase plane $(\phi, \Omega)$. We made use of a continuation scheme, which starts at a Hopf bifurcation of the uniform fluidization state, and uses $\omega$ as the continuation or bifurcation parameter. We present results for the glass bead and plastic beds constituting the Case II systems, which were previously discussed and whose particle properties and dimensionless parameters are shown in Tables 1 and 2 respectively. The developed wave structures and amplitudes are much more dramatic in this case than in the carbon and plastic Case I analysis. The results for the carbon and plastic beds are qualitatively similar however, and the same conclusions were arrived at in both cases regarding the behavior of one-dimensional traveling wave structures in inverse and normal beds.

Figure 2(a) is a bifurcation diagram of one-dimensional periodic wave solutions, which were numerically computed for the normal bed of water-fluidized glass beads. In this figure, the $l_2$-norm of $\Omega$, $|| \Omega ||$ is plotted as a function of (dimensionless) wavespeed $\omega$, where $|| \Omega ||$ gives a measure of the amplitude of the solution with respect to the uniform state. A Hopf bifurcation of the steady state solution ($\phi_o = 0.54, \Omega = 0$) is represented by point A in this figure, and is the starting point of the continuation scheme. We see that $|| \Omega ||$ increases with increasing $\omega$, eventually reaching a maximum point at $\omega \approx 1$ corresponding to point Q. Beyond this point, $|| \Omega ||$ decreases steadily with increased wavespeed. Here, waves become steeper and have greater amplitude as the fluctuation of volume fraction increases about $\phi_o$. In numerical simulations of bubbling behavior in normal mode gas-fluidized beds, Anderson et al. [14] show that high amplitude, two-dimensional traveling wave solutions have bubble-like holes with fluid traveling upwards through the center of the hole. Particles accelerate downwards through the ‘roof’ (or top) of the bubble, but then begin to decelerate as they move downwards though the bubble, and exit through the bubble ‘floor’. The structure developed by this velocity field exhibits a high voidage fluid floor with a rounded roof of higher particle concentration. These authors show that the asymmetry exhibited in one-dimensional solutions at high amplitude is indicative of this structure formation.

Periodic solutions describing high amplitude traveling waves, having increased steepness and a more defined structure, were numerically computed at increased wavespeeds for the system of glass beads in water. For very steep waves, the computational scheme failed, and further con-
continued could not be carried out. Traveling wave profiles computed in the vicinity of point \( S \) on the bifurcation diagram figure 2(a) are shown in figure 3(a) where (dimensionless) \( Y = z - \omega t \) is plotted versus \( \phi \). In this figure, the asymmetry of the wave structure becomes apparent. Notice that, as one moves up the \( y \) axis from the origin, \( \phi \) transitions abruptly in the area labelled [1], and decreases rapidly to a minimum \( \phi \) value. Volume fraction then transitions back to baseline in a more gradual manner in the area labelled [2]. The asymmetry exhibited by the one-dimensional structure is described by a sharp flattened ‘floor’ represented by transition [1], and rounded ‘roof’ represented by transition [2], where ‘top’ and ‘bottom’ are defined with respect to the positive vertical axis \( +z \).

The bifurcation diagram for the complementary inverse bed of plastic particles in water is shown in figure 2(b). Although the bifurcation structure is similar in both bed modes, the bifurcation occurs at a higher \( \omega_{crit} \) value (labelled point \( A \)) in the inverse bed, and waves have slightly lower amplitudes. This is because instabilities were found to be weaker in the inverse bed, as measured by comparatively higher \( \omega_{crit} \) and lower \( \kappa_{crit} \) values. Based on the bifurcation diagrams alone, instability behavior in the two bed modes cannot be distinguished on a qualitative basis. However, high amplitude wave solutions computed in the vicinity of point \( R \) in figure 2(b) are illustrated in figure 3(b), and describe structures, which are distinct from those computed in the normal bed of glass beads (curve \( S \), figure 3(a)). In particular, high amplitude one-dimensional wave profiles in the inverse bed develop very steep, shock-like fronts as one moves down the \( y \) axis. Abruptly, there is a step change in volume fraction \( \phi \) located at point [1] in figure 3(b). With an incremental decrease in \( Y \), the system returns to constant \( \phi \) in a more gradual way (area labelled [2]). The asymmetry suggests a bubble ‘floor’ at point [1] and bubble ‘roof’ located below it with respect to the vertical axis \( +z \). The one-dimensional wave structure would appear to be “flipped” over relative to a wave in the normal bed propagating upwards in the column in the \( +z \) direction. Results also show that 1D-TW’s in the inverse bed always propagate in the direction of gravity for all the case studies examined.

In summary, we have shown that high amplitude one-dimensional traveling wave solutions computed from Hopf bifurcation points using a continuation scheme in the bifurcation parameter \( \omega \) become steep and highly asymmetric, and that high amplitude 1D-TW’s become shock-like in the inverse beds we examined. Moreover, the asymmetry of 1D-TW’s of high amplitude is reversed about the vertical axis suggesting that fully developed bubble-like structures (indicated by asymmetrical one-dimensional solutions) are flipped over in the two bed modes. This suggests that the orientation of a (fluid filled) bubble ‘floor’ and (particle filled) bubble ‘roof’ are reversed with respect to the axial dimension \( z \) and that the direction of wave propagation in the two bed modes is consistent with these findings.

6 Conclusions

The inverse fluidized bed has provided an opportunity to examine unstable flow behavior in beds, which are described by a range of dimensionless groups not physically realized in normal fluidization mode. In a linear stability analysis of various sets of uniformly fluidized normal & inverse beds, the dimensionless \( Fr \) number and (to a lesser extent) fluid to solid density ratio \( \delta \) were shown to be indicators of instability strength, based upon bifurcation structure and growth rates of one-dimensional traveling wave solutions. The effect of \( Fr \) and \( \delta \) on instability strength was confirmed in three case studies of water-fluidized normal & inverse beds having reciprocal fluid to solid density
ratios and operating under identical, similar and differing $Fr$ number conditions. These results are consistent with experimental and theoretical evidence showing a correlation between $Fr$ number and the likelihood of gas- and liquid-fluidized systems to exhibit bubbling behavior. The fluid to solid density ratio $\delta$ was shown to be significant with respect to instability onset, defined by the conditions under which the uniformly fluidized bed is rendered marginally stable. In particular, it has been shown that, at the point of marginal stability, the particle pressure constant $g_o(min)$ is independent of $Fr$ and a function only of the fluid to particle density ratio $\delta$

In all of the case studies examined, the computed traveling wavespeed $\omega$ was shown to be in the direction of fluid flow. We know from the experimental literature that voidage waves travel upwards through the bed in normal mode fluidization, but since there is no experimental evidence of inverse bed behavior, we can only presume that disturbances propagate downwards through the bed, and perturbations in the positive vertical direction (against fluid flow) are damped out. Moreover, high amplitude, one-dimensional traveling wave solutions were steep and highly asymmetrical about the horizontal axis where the asymmetry was reversed or “flipped over” in the two bed modes. This suggests that fully developed bubble-like structures are orientated in the direction of fluid flow with respect to a particle rich bubble ‘roof’ and fluid-filled bubble ‘floor’. Results from these analyses naturally suggest a comprehensive experimental study be carried out to bring forth further qualitative differences in the unstable flow regime.

In a case study involving normal and inverse beds operating under equivalent $Fr$ number conditions, results show eigenvalue structure to be identical in the two modes, and therefore independent of the quantity $\delta$. Although the bifurcation structure of the steady state solution is qualitatively similar in the two bed modes, Hopf bifurcation points differ quantitatively in a way that is consistent with linear stability results of beds with differing $Fr$ numbers. This confirms that $\delta$ along with $Fr$ number affects the strength of one-dimensional waves as measured by the critical wavenumber at the Hopf bifurcation.

7 References

Figure 1: Case I: Linear stability of the uniform state. Real part of the (complex) growth rate \( s = \zeta + i\chi \) as a function of \( \kappa_z \) for various values of \( \phi_o \leq \phi_c = 0.576 \). Hopf bifurcation points located at A and B. All quantities are dimensionless.

(a) Normal bed of 775 \( \mu m \) carbon particles in water. Points A (\( \kappa_{crit} = 0.400 \), \( \phi_o = 0.54 \)) and B (\( \kappa_{crit} = 0.138 \), \( \phi_o = 0.57 \)).

(b) Inverse bed of 1000 \( \mu m \) plastic particles in water. Points A (\( \kappa_{crit} = 0.389 \), \( \phi_o = 0.54 \)) and B (\( \kappa_{crit} = 0.134 \), \( \phi_o = 0.57 \)).
Figure 2: Case II: Bifurcation diagram for one-dimensional traveling waves. The $l_2$ norm $||\Omega||$ versus bifurcation parameter $\omega$ in water fluidized beds. Hopf bifurcation (point A) at equilibrium point (0.54,0). Point Q corresponding to $||\Omega||_{\max}$. Continuation scheme fails in the vicinity of points S and R. All quantities are dimensionless.

(a) Normal bed of 1000 $\mu$m glass beads in water. Point A at $\omega_{crit} = 0.76$, $\kappa_{crit} = 0.463$.

(b) Inverse bed of 1000 $\mu$m plastic particles in water. Point A at $\omega_{crit} = 0.90$, $\kappa_{crit} = 0.396$. 
Figure 3: Case II: Asymmetrical traveling wave profiles corresponding to high amplitude one-dimensional traveling wave solutions computed in the vicinity of points S and R in figures 2(a) and 2(b) respectively. (Dimensionless) $Y=\frac{z-\omega t}{\phi}$ versus solids volume fraction $\phi$.

(a) Normal bed of 1000 $\mu m$ glass beads in water.
(b) Inverse bed of 1000 $\mu m$ plastic particles in water.