Verification of the relationship between desired angle ratio and control performance in the links of a gymnastic based controller for an underactuated robot

Masaki Akiyama¹, Tomohiro Henmi² and Toru Yamamoto³

Abstract—In this paper, the influence on control performance with a changing desired angle ratio are discussed, for a gymnastic based controller of an underactuated robot. An original evaluation function, which uses input torques and an amount of energy change, is applied for a verification of the control performance.

I. INTRODUCTION

The three-link underactuated robot is a simple robot modeled on a horizontal bar gymnast, which comprises an underactuated system. The first joint of the three-link underactuated robot doesn’t have an actuator, i.e. it imitates the hand of the gymnast. On the other hand, the second and third joints, which each have an actuator, imitate the shoulder and waist of the gymnast. In a similar study, development of the control law has been conducted based on the stability theory of Lyapunov using the law of conservation of energy[1][2]. In addition, as a solution to the control problem of the underactuated robot, a motion controller based on the ECM (Equivalent Center of Mass) was applied to the control problem of the underactuated robot in order to prove that the swing-up motion of the gymnast is inspected[3]. However, out of all the existing pairs of second and third joint angle, the swing-up control was performed only when the second and third joint angle were equal based on the EMC of a three-link underactuated robot in this control law. Thus, the conditions of the ECGM-based control law were extended, and the swing-up performance was inspected by various evaluation functions defined by the mechanical energy and input torque of every joint.

II. THE THREE-LINK UNDERACTUATED ROBOT

A. Dynamics of the three-link underactuated robot

Fig. 1 shows the model of the three-link underactuated robot driven by control input torques, τ₂ and τ₃ for actuators with the second joint and the third joint. The dynamics of the three-link underactuated robot are shown as

$$\begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\
\ddot{q}_2 \\
\ddot{q}_3
\end{bmatrix} +
\begin{bmatrix}
h_1 \\
h_2 \\
h_3
\end{bmatrix} +
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{bmatrix} +
\begin{bmatrix}
\mu_1 \dot{q}_1 \\
\mu_2 \dot{q}_2 \\
\mu_3 \dot{q}_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
\tau_2 \\
\tau_3
\end{bmatrix}
$$

where, matrix $M$ shows the inertial force and vector $h$, $\phi$, and $\mu$ show the centrifugal and Coriolis force, the gravity, and the friction force, respectively. The parameter functions of the dynamics are shown as reference [4] and the parameters of the three-link underactuated robot are defined in Table I.

In this study, the performance of the swing-up motion of a horizontal bar gymnast, which is an underactuated skill, is inspected[4]. However, out of all the existing pairs of second and third joint angle, the swing-up control was performed only when the second and third joint angle were equal based on the EMC of a three-link underactuated robot in this control law. Thus, the conditions of the ECGM-based control law were extended, and the swing-up performance was inspected by various evaluation functions defined by the mechanical energy and input torque of every joint.

Fig. 1. the three-link underactuated robot

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B. ECM of the three-link underactuated robot

In order to apply a technique of the gymnast to a controller of the three-link underactuated robot, whose number of links are fewer than those of the gymnast, we used a controller designed based on ECMG[5], which is the center of mass of the whole system. By using this method, the behavior of the ECM of the three-link underactuated robot(ECMR) and the ECMG can be treated as the same system i.e. a variable length single pendulum, shown in Fig. 1. Therefore it is easy to apply the technique of the gymnast to the controller of the three-link underactuated robot. Moreover, since the ECM of all serial linkage systems can be shown by the variable length single pendulum, applying this method to other underactuated serial linkage systems can be expected. Therefore, this method does not only apply to three-link robots, but can be also applied to robots with any number of links.

Coordinate data \((x_g, y_g)\) of the ECMR shown in Fig. 1 is shown by

\[
x_g = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \tag{2}
\]

\[
y_g = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \tag{3}
\]

where,

\[
x_1, y_1 = (l_{c1} \sin q_1, l_{c1} \cos q_1) \tag{4}
\]

\[
x_2, y_2 = (l_1 \sin q_1 + l_{c2} \sin (q_1 + q_2), l_1 \cos q_1 + l_{c2} \cos (q_1 + q_2)) \tag{5}
\]

\[
x_3, y_3 = (l_1 \sin q_1 + l_2 \sin (q_1 + q_2) + l_{c3} \sin (q_1 + q_2 + q_3),
\]

\[
l_1 \cos q_1 + l_2 \cos (q_1 + q_2) + l_{c3} \cos (q_1 + q_2 + q_3))
\]

the x-axis and y-axis are the horizontal line and the vertical-line shown in Fig. 1 respectively. From coordinate data of the ECMR, the angular position \(q_g\) of the ECMR and the length \(l_g\) from the root to the ECMR are obtained as

\[
q_g = \arctan \frac{x_g}{y_g} \tag{6}
\]

\[
l_g = \sqrt{x_g^2 + y_g^2} = \sqrt{f_1 + f_2 \cos q_2 + f_3 \cos(q_2 + q_3) + f_4 \cos q_3} \tag{7}
\]

where

\[
f_1 = \frac{2m_1 l_1 m_2 l_1 + m_2 l_2 (m_2 l_2 + m_3 l_2)^2}{(m_1 + m_2 + m_3)^2} + f_5
\]

\[
f_2 = \frac{m_2 l_2 (m_1 l_1 + m_2 l_1 + m_1 l_2 + m_2 l_2 + m_3 l_2 + m_3^2 l_2)^2}{(m_1 + m_2 + m_3)^2}
\]

\[
f_3 = \frac{2m_2 l_2 (m_1 l_1 + m_2 l_1 + m_1 l_2 + m_2 l_2 + m_3 l_2 + m_3^2 l_2)^2}{(m_1 + m_2 + m_3)^2}
\]

\[
f_4 = \frac{2m_2 l_2 (m_2 l_1 + m_3 l_1 + m_3 l_2 + m_3^2 l_2 + m_2 l_2 + m_3 l_2 + m_3^2 l_2)^2}{(m_1 + m_2 + m_3)^2}
\]

\[
f_5 = \frac{m_1 l_1 + m_2 l_1 + m_3 l_1 + m_1 l_2 + m_2 l_2 + m_3 l_2 + m_3^2 l_2 + m_2 l_2 + m_3 l_2 + m_3^2 l_2)^2}{(m_1 + m_2 + m_3)^2}
\]

III. ANALYSIS OF THE GYMNAST

This section discusses numerical analysis of the technique of the gymnast. In this study, a motion of a KIP, a typical basic swing-up technique of the gymnast is analyzed by using the motion capture technique.

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<table>
<thead>
<tr>
<th>(m_i)</th>
<th>Mass of the i-th link [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_i)</td>
<td>Length of the i-th link [m]</td>
</tr>
<tr>
<td>(l_{c_i})</td>
<td>Length to the center of mass of the i-th link [m]</td>
</tr>
<tr>
<td>(I_i)</td>
<td>Moment of inertia of the i-th link [kgm²]</td>
</tr>
<tr>
<td>(\tau_i)</td>
<td>Torque which acts on the i-th link [Nm]</td>
</tr>
<tr>
<td>(q_i)</td>
<td>Angle of the i-th link [rad]</td>
</tr>
<tr>
<td>(\mu_i)</td>
<td>Coefficient of friction which acts on the i-th joint [N/m/rad]</td>
</tr>
</tbody>
</table>

A. Obtaining motion data of the ECM of the gymnast

At first, we take a motion picture of light emitting markers which were attached to each joint of the horizontal bar gymnast while moving. Markers were attached to the vertex, the point of antilobium, the elbows, the anteroposterior lower end of the costal bone, the great trochanter, the knee joints, the ankle joints and bar instead of the wrist fig 2. Refer to reference [5] for the detailed data.

B. Analyzing the behaviors of the ECMG of the kip motion

The behaviors of the ECMG of the swing-up motion are shown in Fig. 3 which illustrates the behaviors of \(q_{hg}\), \(l_{hg}\), \(U_h\) and \(T_h\). To analyse the behaviors of the ECMG in the swing-up motion, we focus on the motion of a initial swing-up action (Frame=35-70 shown by square (a) in Fig.3) and a motion (Frame=140-210 shown by the square (b) in Fig.3) when \(q_{hg}\) has large values. We regard these simple motions as the most efficient motion for swinging-up and the relationship between \(q_{hg}\) and \(l_{hg}\) in these motions are formulated in order to apply these efficient motions to the swing-up control of the three-link underactuated robot. An approximate length \(l_{hg}^n\) of the ECM, which corresponds to angular position \(q_{hg}\) of ECM in swing-up motion, can be obtained as the following simple low order polynomial equations.

\[
l_{hg}^n(q_{hg}) = \begin{cases} 
  l_{h1} & \text{if } q_{hg} < 0 \\
  l_{h2} & \text{if } 0 \leq q_{hg} < 0.20 \\
  l_{h3} & \text{if } 0.20 \leq q_{hg} < 0.80 \\
  l_{h4} & \text{if } 0.80 \leq q_{hg} < 1.30 \\
  l_{h5} & \text{if } 1.30 \leq q_{hg} < 2.00 \\
  l_{h6} & \text{if } 2.00 \leq q_{hg} < 3.00 \\
  l_{h7} & \text{if } 3.00 \leq q_{hg} < 4.00 \\
  l_{h8} & \text{if } 4.00 \leq q_{hg} < 5.00 \\
  l_{h9} & \text{if } 5.00 \leq q_{hg} < 6.00 \\
  l_{h0} & \text{if } 6.00 \leq q_{hg} < 7.00 \\
\end{cases}
\]
IV. CONTROLLER BASED ON TECHNIQUE OF THE GYMNAST

In this section, the structure of a proposed controller is discussed in detail. The controller (depicted in Fig. 4) is designed focusing on the ECMR and the ECMG and has three sections.

1. Deriving a relationship of the ECM by connecting the ECM of the gymnast with that of the robot.
2. Transform $l^r_g$ based on the results of analysis of section III to the desired angular position $q^r_2$ of the second link and the angular position $q^r_3$ of the third link.
3. Tracking controller using partial linearization method in order to track $q_2$ to $q^r_2$ and $q_3$ to $q^r_3$.

A. Deriving the relationship between ECMR and ECMG

A desired length of the KIP reproducing ECMR is calculated based on results of analysis of $l^g_r$ and $l^g_2$ shown by eq. (5). Due to the desired lengths of the ECMG, $l^g_2$ corresponding to $q_2$ and the size of the gymnast and the three-link underactuated robot not being the same, $l^r_g$ is transformed to $l^r_g$ so as to match the size of the Acrobat by using the following transform equation.

$$l^r_g = l^m_{\text{min}} + (l^r_{hg})(q_2 - l^m_{hg}) + l^m_{\text{max}} - l^m_{\text{min}}$$

where $l^m_{\text{max}}(l^m_{\text{min}})$ means maximum(minimum) length of the ECMR and $l^m_{\text{max}}(l^m_{\text{min}})$ means maximum(minimum) length of the ECM, respectively.

B. Transform $l^r_g$ to $q^r_2$ and $q^r_3$

The desired angular positions of the second link and the third link, which realize the desired length $l^r_g$ of the ECM, are derived by using eq. (5). However, in the three-link underactuated robot, there exist multiple patterns of links which attain the center of gravity position of the robot. Thus, in this study, we chose the angular position Fig. 5-(b) to decide the ECMR.

1) In the case of the desired angle ratio $1:n(q_2^* = q_3^*)$: By substituting $l^r_g$ to $l_g$ in eq. (5), the desired angular position of the second link $q^r_2$ and the third link $q^r_3$ can be obtained by

$$q^r_2 = q^r_3 = \arccos \left( \frac{-(f_2 + f_3) + f}{4f_3} \right)$$

where $f = \sqrt{(f_2 + f_3)^2 - 8f_3(f_1 - f_3 - l_g^r)}$ and $f_3 = 4f_3 \cos q_2 + 2f_2 \cos^2 q_2 + (f_4 - 3f_3) \cos q_2 + (f_1 - f_2 - l_g^r) = 0$.

Fig. 6 shows the solution trajectory of these formulas. The formula of the desired angle was defined as a function of the frame number.
having the same solution trajectory as that of a human joint’s range of motion. From Fig 6, it is defined as the approximated formula eq 10, which combines two trajectory solutions into one.

\[
\cos q_2 = 4.4208l_g^3 - 4.5851l_g^2 + 3.7273l_g + 0.1572 \tag{10}
\]

Thus, the desired angle formula of the second and the third joint \(q_2^r\) are similarly eq 11 and eq 12.

\[
q_2^r = \arccos(4.4208l_g^3 - 4.5851l_g^2 + 3.7273l_g + 0.1572) \tag{11}
\]

\[
q_3^r = 2q_2^r \tag{12}
\]

By a similar technique, the desired angle formula of each joint with angle ratio 1:3 is similarly given by eq 14 and eq 15, and each desired joint angle formula with angle ratio 1:4 by eq 15 and eq 16.

\[
q_2^r = \arccos(12.3662l_g^3 - 10.6451l_g^2 + 4.3023l_g + 0.3072) \tag{13}
\]

\[
q_3^r = 3q_2^r \tag{14}
\]

\[
q_4^r = \arccos(1.3084l_g^3 - 1.464l_g^2 + 1.2098l_g + 0.728) \tag{15}
\]

\[
q_5^r = 4q_2^r \tag{16}
\]

C. Tracking controller of \(q_2\) for \(q_2^r\) and \(q_3\) for \(q_3^r\)

The controller to track the angular positions of the second link \(q_2\) and the third link \(q_3\) to the desired angular positions \(q_2^r\) and \(q_3^r\) is made by using the partial linearization method [6].

Now, the following nonlinear feedback is designed as the control input.

\[
\tau_2 = \frac{M_2(\phi_1 + h_1) + (M_{31}M_{23}' - M_{33}'M_{22}) \phi_2 + M_2h_3 + M_2}{M_{22}' - M_{23}'M_{32}'} \tag{17}
\]

\[
\tau_3 = \frac{M_3(\phi_1 + h_1) + (M_{21}M_{33}' - M_{23}'M_{31}) \phi_2 + M_3h_2 + M_3}{M_{22}' - M_{23}'M_{32}'} \tag{18}
\]

where, \(|M|\) is a determinant of the inertial matrix \(M\) in eq. (1) and \(M_{ij}'\) are elements of the inverse inertial matrix \(M^{-1}\). Moreover, \(v_2\) and \(v_3\) are the equivalent control inputs, which can be set to nonlinear feedback eq. (17), eq. (18), producing

\[
\ddot{q}_2 = v_2 \tag{19}
\]

\[
\ddot{q}_3 = v_3 \tag{20}
\]

In order to track \(q_2\) to \(q_2^r\) and \(q_3\) to \(q_3^r\), equivalent control \(v_i (i = 2, 3)\) is designed as

\[
v_i = \ddot{q}_i = -k_{pi}(q_i - q_i^r) - k_{di} \dot{q}_i, \quad (k_{pi} > 0, k_{di} > 0)
\]

then the linearized subsystem is

\[
\ddot{q}_i + k_{pi}(q_i - q_i^r) + k_{di}(q_i - q_i^r) = 0, \quad (i = 2, 3)
\]

Since eq. 22 is a stable polynomial equation, it is possible to track the angular positions of the second link \(q_2\) and the third link \(q_3\) to the desired angular positions \(q_2^r\) that is,

\[
\lim_{t \to \infty} q_2 = q_2^r, \lim_{t \to \infty} q_3 = q_3^r.
\]

As a result, tracking the length of the ECMR \(l_g\) to the desired length \(l_g^r\) can be achieved, and the ECMR operates at the most efficient motion for the swing-up to that of the ECMG.

V. NUMERICAL SIMULATIONS

In this Section, in order to verify the effectiveness of any proposed controller, a numerical simulation of swing-up control of the three-link underactuated robot is performed by using MATLAB/Simulink.

A. Condition of simulation

The parameters of the three-link underactuated robot used in the numerical simulation are shown in table II and the control parameters used by the swing-up control for the desired angle ratio 1:1-1:4 were shown in table III. \(k_{pi}^h\) and \(k_{di}^h\) are similarly proportional to the difference gain in swing-up. \(T_s\) is the simulation time taken to shift into the giant-swing. In addition, the \(q_0\) at 125\(^\circ\) is a necessary condition for the swing-up to shift into the giant-swing.

B. Swing-up performance

The swing-up performance was compared and inspected according to eq. 23 to compare the swing-up performance when changing the desired angle ratio. The evaluation function eq 23 calculates the mechanical energy in the input torques of every joint \((\tau_2, \tau_3)\) per unit time. Thus, if the value of eq 23 is large, the swing-up performance is interpreted as being good. \(\tau_{max}\) calculates the momentary maximum of input torque of every joint. Thus if the value of \(\tau_{max}\) is small, the swing-up motion is performed without needing momentary large power. In other words, the swing-up performance is interpreted...
TABLE II
Parameters of the three-link underactuated robot

<table>
<thead>
<tr>
<th>Parameters</th>
<th>1st link</th>
<th>2nd link</th>
<th>3rd link</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_i$ [kg]</td>
<td>0.100</td>
<td>0.0750</td>
<td>0.006</td>
</tr>
<tr>
<td>$l_{ci}$ [m]</td>
<td>0.200</td>
<td>0.0750</td>
<td>0.006</td>
</tr>
<tr>
<td>$I_{ci}$ [kgm$^2$]</td>
<td>3.333 x 10$^{-4}$</td>
<td>1.406 x 10$^{-3}$</td>
<td>7.333 x 10$^{-4}$</td>
</tr>
</tbody>
</table>

TABLE III
Parameters of conditions used in the simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>1:1</th>
<th>1:2</th>
<th>1:3</th>
<th>1:4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{h1}$</td>
<td>250</td>
<td>700</td>
<td>800</td>
<td>700</td>
</tr>
<tr>
<td>$k_{h2}$</td>
<td>445</td>
<td>410</td>
<td>400</td>
<td>900</td>
</tr>
<tr>
<td>$k_{h3}$</td>
<td>16</td>
<td>16</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>$T_s$</td>
<td>15.19</td>
<td>15.13</td>
<td>15.36</td>
<td>15.92</td>
</tr>
</tbody>
</table>

as being good. $J = \sum_{i=0}^{n} \frac{\sqrt{(U_i+T_i)^2}}{T_s}$ calculates the sum of the input torque while swing-up motion. Thus if value of $\sum \sqrt{q_i^2 + \tau_i^2}$ is small, the swing-up motion is performed without needing total input torque required for swing-up. In other words, the swing-up performance is interpreted as being good.

\[ J = \sum_{i=0}^{n} \frac{\sqrt{(U_i+T_i)^2}}{T_s} \]  \hspace{1cm} (23)

C. The numerical simulation result and consideration

The Simulation result, for when the swing-up of the desired angle ratio is 1:1, is shown in fig 7. When the second and the third joint angles are $q_2$ and $q_3$, the angular position and the length of the ECM are similarly $q_{hg}$ and $l_{hg}$, and the potential and the kinetic energy are similarly $U_h$ and $T_h$, respectively and the input torques of the second and the third joint is $\tau_2$ and $\tau_3$ respectively. Similarly, the simulation result, for when the swing-up of the desired angle ratio is 1:2, is shown in fig 8. Also, the simulation results for when the swing-up of the desired angle ratio is 1:3 and 1:4 are similarly shown in fig 8. As a result, by applying the behavior of the ECMG to the three-link underactuated robot, the kinetic energy increases and the robot can achieve the swing-up motion based on the ECMG. The value of the evaluation function eq23 for each desired angle ratio is shown in table IV.

As seen in table IV, the swing-up performance was greatest when the desired angle ratio is 1:2. Moreover, the amplification factor, which is the average of $q_{hg}$, the input torque of every joint used over the simulation time and $\tau_{max}$ defined as the maximum of $|\tau_2|$ and $|\tau_3|$ are shown in table V. As shown in table V, even though the amplification factor's of the 1:2 and 1:3 swing-up are approximately equal, the sum of the input torque is small in 1:2, thus the swing-up with 1:2 is performed efficiently. However, when we consider it from the viewpoint of momentary maximum torque, the angle ratio 1:3 is better than angle ratio 1:2. Therefore, when we look from the viewpoint of the input efficiency and the energy efficiency, it is thought that the angle ratio of 1:2 with swing-up’s characteristic of small load and momentary large torque is good. On the other hand, when we look from the viewpoint of the input value, it is thought that the angle ratio of 1:3 with the swing-up's characteristic of small torque is good.

TABLE IV
Evaluation function in link's desired angle ratio

<table>
<thead>
<tr>
<th>Angle ratio</th>
<th>1:1</th>
<th>1:2</th>
<th>1:3</th>
<th>1:4</th>
</tr>
</thead>
<tbody>
<tr>
<td>evaluation function</td>
<td>567.42</td>
<td>788.56</td>
<td>437.38</td>
<td>552.42</td>
</tr>
</tbody>
</table>

TABLE V
Evaluation function in link's desired angle ratio

<table>
<thead>
<tr>
<th>Angle ratio</th>
<th>1:1</th>
<th>1:2</th>
<th>1:3</th>
<th>1:4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{max}$</td>
<td>6.8866</td>
<td>7.3185</td>
<td>4.7355</td>
<td>9.4154</td>
</tr>
<tr>
<td>$\sum \sqrt{q_i^2 + \tau_i^2}$</td>
<td>809.89</td>
<td>788.56</td>
<td>922.02</td>
<td>949.6</td>
</tr>
<tr>
<td>$\sum \sqrt{q_i^2 + \tau_i^2}$</td>
<td>1.547</td>
<td>1.5051</td>
<td>1.426</td>
<td>1.455</td>
</tr>
</tbody>
</table>

VI. CONCLUSION

In this study, the swing-up control were performed with the angle ratios 1:2, 1:1, and 1:4, as well as the angle ratio 1:1. As a result, the swing-up was confirmed to change the angle ratio. Also, when the input torque, the swing-up performance, and the amplification factor, the average of $q_{hg}$ in the swing-up motion, are compared, the value of the input torque and the energy efficiency differ depending on the desired angle.

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References

Fig. 7. As a result of swing-up control in desired angle ratio 1:1

Fig. 8. As a result of swing-up control in desired angle ratio 1:2