Identification of FIR models using basis models of first-order plus time delays*

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Abstract—In this article, a new methodology is presented for identification of finite-impulse-response (FIR) models. The central idea is to use a series of stable models of first-order plus time delays (FOPTD) to approximate the dynamics of the process. A quantitative analysis of the modelling error is provided. Compared to the existing basis model approaches, the advantages of the proposed method lie in the following aspects: (i) a simple model structure with a small number of basis models is sufficient to obtain satisfactory approximations, thereby reducing the model structural risks; and (ii) basis models could be determined with limited prior process information, which is beneficial for practical implementations. Representative simulation examples are provided to illustrate the superiority of our method over the existing identification approaches based on basis models.

Keywords: System identification, FIR models, basis models

1. INTRODUCTION

System identification plays a central role in practical implementation of industrial control systems, and its primary objective is to capture the dynamic relations between controlled variables (CV) and manipulated variables (MV) or disturbance variables (DV). In the process industries, most processes have complicated mechanisms, and developing rigorous first-principle models is usually a tough task. Alternatively, the data-driven system identification technique has drawn considerable attentions from researchers, which aims to establish simplified dynamic models using collected data samples (Ljung, 1999). In the field of system identification, nonparametric models, such as the finite-impulse-response (FIR) models and step-response models, are widely applied. These models are "input-only" model, indicating that the model output is a function of the past inputs. Both FIR models and step-response models can reveal the dynamics, and a significant number of model predictive control algorithms are based on these nonparametric models, such as the dynamic matrix control (DMC) (Qin and Badgwell, 2003).

A direct approach to identification of the nonparametric models is to directly resort to step tests as they are rather common and easy to implement in the process industries. For chemical processes, it has been recognized that the dynamics may in general be restricted to simple process models (Latour et al., 1967), and first-order plus time delay (FOPTD) types or second-order plus time delay (SOPTD) types are usually adopted, which can further be estimated by the area approaches (Åström and Hägglund, 1995) or the two-point method (Normey-Rico, 2007). A robust identification approach was proposed by Wang et al. (2001), which enables satisfactory estimations even in the presence of large measurement noise. For other types of test input signals, the test-step based methods cannot be applied directly, and the FIR models and step-response models should be estimated via prediction-error methods firstly. Nevertheless, a practical problem is the high dimensionality, which may introduce high variance in the presence of measurement noise (Ricker, 1988). A straightforward approach to this issue is to rely on the regularization regression. The basic idea of regularization is to obtain a more robust estimation with smaller variances at the cost of a possible bias from actual model parameters, where a penalty term about the model complexity is added to the cost function (Chen et al., 2012). Hence the regularization matrix should be carefully designed to yield a robust estimation with a tolerable model bias. Determination of a proper regulatory matrix can be done within the Bayesian framework, yet a non-convex optimization is involved to estimate related hyper parameters and may result in high computational burden.

The basis-model based approach is a pragmatic approach to model reduction of both FIR models and step-response models. The number of basis models is usually much smaller than the length of the FIR sequence and step-response sequence, and hence the model complexity is reduced. The frequency-sampling-filter (FSF) basis model was proposed by Wang and Cluett (1993), and a major advantage is that this approach can well capture the input delay and does not need to pre-specify the time delay. Nevertheless, the FSF basis models have significant underdamped properties, yet a significant portion of industrial processes exhibit evident overdamped properties. Even though this issue can be compensated by introducing more basis models, the model complexity inevitably increases. Alternatively, the Laguerre model is a better choice in such scenarios (Wahlberg, 1991) and also a typical approach for nonlinear system identification (Reddy and Saha, 2016). However, the choice of poles of the Laguerre basis models requires prior information about the poles of the process, which is not always available in practice. Another drawback is that processes with several scattered dominant poles can hardly be well approximated by Laguerre basis models.

In this article, a FOPTD basis model based identification method is presented. The idea of using FOPTD basis models was firstly proposed to measure process nonlinearities (Lee et al., 1997) and is adopted for system identification in this work. An analytical analysis on the modelling error is provided,

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which reveals that the dynamics of overdamped processes can be well captured using a small number of basis models. The choice of the basis models entails only limited process information: the range of the poles of the basis models should cover that of the process for modelling. This requirement can usually be guaranteed without much difficulty in practice, and hence improved applicability of the proposed method can be expected.

The layout of this article is organized as follows. In Section 2, the proposed FOPTD-basis model identification method is presented, and the quantitative analysis of the modelling error is given. In Section 3, some implementation issues are discussed. In Section 4, case studies are illustrated to show the efficiency of the propose method. In the final section, conclusion remarks are provided.

2. FOPTD BASIS MODEL IDENTIFICATION APPROACH

2.1 Approximating process dynamics using a series of FOPTD basis models

Consider a stable process with all the poles being real and different, which exhibits significant underdamped properties and is rather common in the process industries. This process can be decomposed as:

\[
G(s) = \sum_{i=1}^{n} K_i F_i(s) = \sum_{i=1}^{n} \frac{K_i}{s + \alpha_i} e^{-\alpha_i s}
\]

where \(n\) is the order of \(G(s)\), \(L\) is the input delay and \(F_i(s)\) can be viewed as the modes of \(G(s)\) with the pole being \(\alpha_i\). For the transfer functions with multiple roots, some amendments should be made to the choice of basis models, which are not discussed in this article due to page limitation. In practice the mode \(F_i(s)\) is unknown, yet it can be approached by a span of pre-specified FOPTD basis models:

\[
F_i(s) = \sum_{k=1}^{L} c_{ik} B_k(s) + E_i(s)
\]

\[
B_k(s) = \frac{1}{s^2 + \alpha_k} e^{-\alpha_k s}
\]

where \(\{B_k(s)\}\) are the FOPTD basis models with the poles \(\{\tau_k\}\) being pre-specified, and \(E_i(s)\) stands for the modelling error. Since \(\{\tau_k\}\) may be different from \(\alpha_i\), the modelling error \(E_i(s)\) is inevitable. However, if the range of \(\{\tau_k\}\) can cover \(\alpha_i\), the modelling error can be reduced and tend to zero by increasing the number of basis models. These contents would be discussed in detail shortly. Therefore \(G(s)\) can be well approximated by the span of \(\{B_i(s)\}\) if the number of basis models is properly selected and the poles \(\{\tau_k\}\) of \(\{B_k(s)\}\) are distributed reasonably:

\[
G(s) = \sum_{i=1}^{n} m_i B_i(s) + E(s)
\]

\[
m_i = \sum_{k=1}^{L} K_i c_{ik} E_i(s) = \sum_{k=1}^{L} E_i(s)
\]

The coefficients \(m_i\) of the basis models can be determined by minimizing the following cost function:

\[
V_m = \sum_{i=1}^{L} \left(y(t) - \sum_{i=1}^{n} m_i x_i(t) \right)^2
\]

\[
x_i(t) = B_i(s) u(t), m = [m_1 \cdots m_p]^T
\]

where \(u(t)\) and \(y(t)\) are the input and output signals respectively, \(x_i(t)\) is obtained by feeding \(B_i(s)\) with \(u(t)\), and \(N\) is the number of samples. The optimal coefficient vector \(m\) is estimated by the least-squares regression (provided that \(N>>p\)):

\[
m = \left( X_m^T X_m \right)^{-1} X_m Y
\]

\[
Y = [y(1) \cdots y(N)]^T
\]

\[
X_{m,a} = \begin{bmatrix} x_1(1) & \cdots & x_p(1) \\ \vdots & \ddots & \vdots \\ x_1(N) & \cdots & x_p(N) \end{bmatrix}
\]

The impulse response \(g(t)\) of \(G(s)\) can then be estimated:

\[
g(t) = \hat{g}(t) = \sum_{k=1}^{p} m_k g_k(t)
\]

where \(g_k(t)\) is the impulse response of \(B_k(s)\).

2.2 Quantitative analysis of the modelling error

The impulse response \(g(t)\) of a FOPTD model is an exponential function with respect to time:

\[
g(t) = Ka e^{-\alpha t}, t > L
\]

where \(K\), \(\alpha\) and \(L\) are the gain, the opposite number of the pole (inverse of the time constant) and delay respectively. In the rest of this work, the term \(Ka\) is assumed to be unity, yet this simplification would not cause any loss of generality. The impulse response \(g(t)\) can be approximated by a series of FOPTD basis models with the same delay:

\[
g(t) = \sum_{k=1}^{p} m_k g_k(t) + \varepsilon(t)
\]

where \(g_k(t)\) is the impulse response of \(B_k(s)\), and \(\varepsilon(t)\) is the modelling error. It is further assumed that the poles of the basis models are different from each other. The modelling error is measured by the \(l_2\) norm of a signal, and the optimal estimate \(\hat{g}(t)\) with the smallest modelling error can be determined by solving the following optimization problem:

\[
\min_{\{m\}} \left\{ l_2 \left[ \left( \hat{g}(t) - g(t) \right) \right]^2 \right\}
\]

\[
\varepsilon(t) = e^{-\alpha t} - \sum_{k=1}^{p} m_k e^{-\alpha_k t}
\]

\[
l_2 \left[ \varepsilon(t) \right]^2 = \int_{t=L}^{\infty} \left[ \varepsilon(t) \right]^2 dt
\]

The optimization problem (9) is actually a quadric-programming problem:

\[
\min \left\{ \frac{1}{2 \alpha} + m^T A m - b^T m \right\}
\]

where

\[
A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \end{bmatrix}
\]

(10a)
The optimal estimate $\hat{g}(t)$ can then be derived as:

$$\hat{g}(t) = \sum_{i=1}^{p} m_i \hat{g}_i(t)$$

(11)

In addition, the optimal coefficient vector $m^{*}$ has an analytical form and is formulated as follows:

$$m^{*} = A^{-1} b, \quad \hat{g}^{*}(t) = \hat{g}(t) - g(t)$$

(10b)

The optimal estimate $\hat{m}^{*}$ can then be derived as:

$$\hat{m}^{*} = A^{-1} b, \quad \hat{g}^{*}(t) = \hat{g}(t) - g(t)$$

(10b)

Obviously, $rio_{\max}(r, p)$ is a function of $r$ and $p$. According to the analysis in Section 2.2, $rio_{\max}(r, p)$ is a decreasing function of $p$ since the modelling error can be reduced by increasing the number of basis models. The functional relationship between $rio_{\max}(r, p)$ and $r$ for different values of $p$ is shown in Fig. 1, and it can be concluded that $rio_{\max}(r, p)$ can be reduced by using a smaller $r$. Based on (13), it can be concluded that a smaller error can be achieved if the poles of the basis models are closer to those of the process for modelling. By using a smaller $r$, the range of poles is partitioned more compactly, and hence the poles of the basis models can be closer to those actual poles of the process for modelling. Nevertheless, more basis models are required to cover a fixed pole range for a smaller $r$, which would increase the model complexity. A reasonable choice of the parameter $r$ can be 2. If $r$ is set to be 2, the $rio_{\max}$ is about 0.01 by using only 4 basis models, which indicates that satisfactory approximations of the process dynamics can be guaranteed using a small number of basis models. In addition, a wide range of poles can be covered by a small number of basis models, and very little prior process information is required to determine the poles of the basis models. There are no restrictions on $r_1$ but the poles of the process for modelling should be between $r_1$ and $2^n r_1$, which is not difficult to fulfil by adopting a small $r_1$ and a proper $p$.

For the estimate of the delay, a simple yet effective approach is to search the delay that can yield the smallest $V_{\lambda}(m)$ defined in (4). Noting that one trial of solving the optimization (5) needs very little computational burden, the estimate of the delay can hence be accomplished in a cost efficient way.

3. IMPLEMENTATION ISSUES

3.1 Choice of FOPTD basis models

Even though the prior information about the exact locations of poles is hardly available in practice, it is usually not difficult to acquire the rough ranges of the poles. A reasonable choice of basis models is to enable the range of $\{r_k\}$ cover the pole of the process:

$$\min \{r_k\} \leq \alpha \leq \max \{r_k\}$$

(14)

In this article, the basis models with geometrically growing poles are adopted:

$$r_k = r r_{k-1}, r > 1$$

(15)

By such approach, a wide range of poles can be covered using a small number of basis models. The index $rio$ defined in (13) can hence be rewritten as:

$$rio(c| r, p) = \prod_{k=1}^{p} \left| c - r^{p-1} \right| \prod_{k=1}^{p} \left| c + r^{p-1} \right|$$

(16)

where $c = \alpha r_1$.

The index $rio_{\max}(r, p)$, which is the maximum $rio$ for all FOPTD models satisfying (14), is defined as follows:

$$rio_{\max}(r, p) = \max_{c| r, p} rio(c| r, p)$$

(17)

![Fig. 1. The relationship between $rio_{\max}(r, p)$ and $r$ for different values of $p$](image)

In addition, the optimal coefficients vector $m^{*}$ has an analytical form and is formulated as follows:

$$m^{*} = A^{-1} b, \quad \hat{g}^{*}(t) = \hat{g}(t) - g(t)$$

(10b)

The optimal estimate $\hat{m}^{*}$ can then be derived as:

$$\hat{m}^{*} = A^{-1} b, \quad \hat{g}^{*}(t) = \hat{g}(t) - g(t)$$

(10b)
a large divergence from that of overdamped processes. To compensate this issue, a large number of basis models should be utilized, which would lead to an overly complicated model structure and may increase the model structural risk.

The Laguerre models are formulated as follows:

$$L_n(s, \gamma) = \frac{s^{n+1} - \gamma}{s^{n+1} + \gamma}$$  \hspace{1cm} (20)

where $\gamma$ is the pole of the Laguerre model and should be specified by the users. A truncated series is usually sufficient to capture the features of stable linear processes. Nevertheless, the selection of $\gamma$ should be considered with great care. The value of $\gamma$ should be chosen to be close to the dominant pole of the process for modelling, otherwise an unreasonable identification result would be obtained. The prior knowledge about the locations of pole is not always available in practice, which may increase the difficulty for practical implementation. Furthermore, Laguerre basis models have the same poles, and hence processes with several scattered poles may not be well approached by a truncated Laguerre series.

4. SIMULATION CASE STUDIES

In this section, several examples are provided to illustrate the superiority of the proposed method over the existing basis-model ones, including the Laguerre method and the FSF method. The modelling errors between the approximated FIRs and the actual FIR are investigated as a key performance index. Furthermore, the identification accuracies of different approaches for general input signals are compared through the Monte Carlo method.

4.1 Example 1: Investigation of the modelling accuracy of FIR models

The following example is used to investigate the modelling accuracy of the FIR for different basis-model based approaches. The process to be identified is given as follows:

$$G(s) = \frac{-16704s^3 - 247s^2 + 50s + 1}{(116s + 1)(33s + 1)(25s + 1)(18s + 1)(0.1s + 1)}$$  \hspace{1cm} (21)

which can further be decomposed as:

$$\frac{1.181}{(116s + 1)} + \frac{1.003}{(33s + 1)} + \frac{1.004}{(25s + 1)} + \frac{0.181}{(18s + 1)} + \frac{0.001}{(0.1s + 1)}$$  \hspace{1cm} (22)

Notice that the poles of $G(s)$ distribute in a rather wide range and each pole corresponds to an inertia unit in (22). According to the classical control theory, for cascade systems the farther the pole is from the origin, the relevant dynamic takes smaller proportion in the step response of the process. The farthest pole of $G(s)$ is -10 corresponding to the smallest time constant 0.1 and is much farther from the origin than the others. Such poles have little impact on the process dynamic and need not to be considered in choosing basis models.

For the choice of the basis models, a basic requirement is that the range of the poles of the basis models should cover that of the process for modelling, yet this requirement needs very little prior information and is not difficult to guarantee in practical applications. To validate this, three different series of basis models are selected and given as follows:

- **Series 1**: $\{r_k\} = \{1/128, 1/64, 1/32, 1/16, 1/8\}$
- **Series 2**: $\{r_k\} = \{1/208, 1/104, 1/52, 1/26, 1/13\}$
- **Series 3**: $\{r_k\} = \{1/304, 1/152, 1/76, 1/38, 1/19\}$

For each series, the poles of the basis models double. In this way, a wide range of poles can be covered using a small number of basis models. The actual FIR and the approximated FIRs by different series of basis modes are drawn in Fig. 2, and the modelling errors are listed in Table 1.

<table>
<thead>
<tr>
<th>$r_k$</th>
<th>FOPTD</th>
<th>FSF</th>
<th>Laguerre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Series 1</td>
<td>0.0062</td>
<td>0.0167</td>
<td>0.0027</td>
</tr>
<tr>
<td>Series 2</td>
<td>0.0032</td>
<td>0.0099</td>
<td>0.3388</td>
</tr>
<tr>
<td>Series 3</td>
<td>0.0027</td>
<td>0.0039</td>
<td>0.6276</td>
</tr>
</tbody>
</table>

For each series, the poles of the basis models double. In this way, a wide range of poles can be covered using a small number of basis models. The actual FIR and the approximated FIRs by different series of basis modes are drawn in Fig. 2, and the modelling errors are listed in Table 1.

All these series of basis models give pretty good approximations which are almost impossible to be distinguished from the true system. Notice the pole of the last term in (22) is not covered by the poles of the basis models, but the gain of the corresponding term is sufficiently small compared to the other terms, thus the effect of such a pole can be ignored. Therefore, the FOPTD basis model approach can
achieve an acceptable modelling error using a small number of basis models. Furthermore, model parameters can be selected with ease and little prior process information is required.

For comparison purposes, the FIR of \( G_r(s) \) is approximated by the FSF and the Laguerre approaches. For FSF, the only parameter to be specified is the number of models.

Three different specifications, namely, \( p=5, 41 \) and \( 101 \), are adopted. The results are shown in Fig. 3. It can be observed that evident oscillation occurs when \( p=5 \). As \( p \) increases, the oscillation becomes slighter and when \( p=101 \), the approximated model seems acceptable. Since the FSF models exhibit evident underdamped features, a sufficient number of models should be used to guarantee satisfactory approximation for overdamped processes, yet increased model complexity is the side-effect. From Table 1, the modelling error is still much larger than that of the FOPTD basis model approach even if a larger number of FSF models are used.

For the Laguerre approach, the number of basis models can be truncated in comparison with FSF model approach. It is set to be five and is identical to that of the FOPTD basis models. The pole \( \gamma \) of Laguerre models should be chosen with care to yield reasonable approximation. The results for different values of \( \gamma \) are shown in Fig. 4. The optimal approximation can be achieved when \( \gamma = 0.0213 \), yet the modelling error is still larger than that of the FOPTD models according to Table 1. This is because the Laguerre method approximates the process with scattered poles with the same pole. Other non-optimal values of \( \gamma \) are adopted. Even though the values of \( \gamma \) just have a small derivation from the optimal pole, the modelling accuracy deteriorates significantly as shown in Fig. 4. Hence the Laguerre model is more sensitive to the model parameters compared to the proposed method.

In order to further illustrate the proposed method, more examples in addition to \( G_r(s) \) are considered:

\[
G_2(s) = \frac{400s^2 + 62s + 1}{(30s + 1)(50s + 1)(120s + 1)} \quad (26)
\]

\[
G_3(s) = \frac{60s + 1}{(24s + 1)(82s + 1)(100s + 1)} \quad (27)
\]

\[
G_4(s) = \frac{(160s + 3)(45s + 1)(201s + 1)}{(110s + 1)(53s + 1)(30s + 1)(84s + 1)} \quad (28)
\]

For the FOPTD basis model approach, the same series formulated in (23) - (25) are used, and for the Laguerre model approach the optimal \( \gamma \) is chosen. Results are listed in Table 1, and it can be seen that the proposed method can achieve superior modelling accuracy.

As a conclusion, for underdamped processes with scattered poles, the proposed FOPTD basis model based approach outperforms the FSF and Laguerre methods. A smaller modelling error can be achieved, and the choice of basis models requires very little prior information and can be determined with ease. These features make the proposed method an attractive approach to identification of industrial processes in practice.

4.2 Example 2: Comparison of the methods with different basis models

In this section, identification accuracy of different basis model methods for common input signals is compared. The process \( G_r(s) \) in (21) is studied, and a coloured noise is added to the output, yielding the following model:

\[
y(t) = G_r(s)u(t) + \frac{1}{50s + 1} e(t) \quad (29)
\]

where \( e(t) \) is a white Gaussian noise sequence with zero mean and unit variance. The step-response model will be estimated and user-specified parameters of the three methods are set to minimize the modelling error as Example 1. In order to compare the influence of different inputs, two common signals, namely, the step signal and the pseudo random binary sequence (PRBS), are used. The noise to signal ratio (NSR) is 10% for the step signal input and 20% for the PRBS input. To figure out the effect of noise on the identification result, 200 Monte Carlo simulations are carried out. Related input and output data of a trial of simulation are shown in Fig. 5.

| Table 2 Average modelling errors for different approaches |
|-------------------|---------------|---------------|---------------|
| Input signal      | FOPTD         | Laguerre      | FSF           |
| step signal       | 0.0200        | 0.0258        | 0.0644        |
| PRBS              | 0.0220        | 0.0359        | 0.1379        |
The identification results for the two input signals are shown in Fig. 6 and Fig. 7 respectively. For the FSF approach, the model structure is complex with a large number of basis models and would lead to high variance problems in the presence of noise. Nevertheless, the problems are not obvious for the other two approaches because a small number of basis models are used. Table 2 lists the average modelling errors (mean(rio)) for the three approaches. The average modelling error of the FOPTD basis model approach is smaller than that of the Laguerre model approach due to the inherent approximation accuracy.

5. CONCLUSION

In this article, a basis-model based method is proposed for identification of underdamped processes based on FOPTD models. It requires the poles of the basis models cover the true system. Due to the geometrically growing values of poles, a wide range of poles can be covered with a small number of basis models, and thus little prior process information is needed to ensure the real poles falling in the range. In addition, the proposed method is more robust and accurate compared to the existing basis-model based approaches. These features make the proposed method an attractive approach to identification of underdamped processes in practice.

APPENDIX

Derivation of (13)

Define:

\[ I = \begin{bmatrix} e^0 & e^{-\alpha} & \cdots & e^{-\alpha(n-1)} \end{bmatrix}^T \]

\[ I_k = \begin{bmatrix} e^0 & e^{-\tau_k} & \cdots & e^{-\tau_k(n-1)} \end{bmatrix}^T \]

\[ L = \begin{bmatrix} I_1 & I_2 & \cdots & I_p \end{bmatrix} \]

where \( I \) and \( I_k \) are the impulse response sequences of the original model and the basis model respectively. Notice \( A \) and \( b \) in (10) can be expressed as:

\[ A = L^T L \text{ as } n \to \infty \]  

(33)

The approximated impulse response sequence can then be calculated as:

\[ I' = L(L^T L)^{-1}L^T I \]  

(34)

Then

\[ \text{rio} = \frac{I' I}{(I' - I)(I' - I)} = \frac{(A^T b)^T b - 2(A^T b)^T b + b^T I}{I^T I} \]  

(35)

Next the terms in (35) will be calculated. For the convenience of expression, define:

\[ c = 1 - e^{-\alpha}, q_k = \frac{(1 - e^{-\tau_k})}{c} \]  

(36)

Substitute (36) and (37) into (10a) and (10b), then:

\[ A = \frac{1}{c} \begin{bmatrix} 1/2q_1 & 1/(q_1 + q_2) & \cdots & 1/(q_1 + q_p) \\ 1/(q_1 + q_1) & 1/2q_2 & \cdots & 1/(q_2 + q_p) \\ \vdots & \vdots & \ddots & \vdots \\ 1/(q_1 + q_p) & 1/(q_p + q_1) & \cdots & 1/2q_p \end{bmatrix} \]  

(37)

\[ b = \begin{bmatrix} 1/(q_1 + 1) & \cdots & 1/(q_p + 1) \end{bmatrix}^T / c \]  

(38)

The inverse of \( A \) can be calculated as:

\[ A^{-1} = 2c \begin{bmatrix} H_{i1} & H_{i2} & \cdots & H_{ip} \\ H_{i2} & H_{i3} & \cdots & H_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ H_{ip} & \cdots & \cdots & H_{ip} \end{bmatrix} \]

\[ H_i = q_i \sum_{j=1}^p \frac{(q_i + q_j)^2}{(q_i - q_j)^2} \]  

(39)

\[ H_q = 2q_i \sum_{j=1}^p \frac{(q_i + q_j)}{(q_i - q_j)^2} - i \neq j \]

Then:

\[ A^{-1} b = (-1)^{p+1} \begin{bmatrix} W_1 & W_2 & \cdots & W_p \end{bmatrix}^T \]

(40)

It is not difficult to obtain:

\[ I' I = 1/2c \]  

(41)

Substitute (37) – (41) into (35), then:

\[ \text{rio} = \frac{\sum_{j=1}^p (q_i - q_j)^2}{\sum_{j=1}^p (q_i + 1)^2} \]  

(42)

Due to the poles far away from the origin need not to be considered in the approximation, namely:

\[ \alpha \ll 1, \tau_k \ll 1 \]  

(43)

Then:

\[ c \approx q_1, q_k \approx \tau_k / \alpha \]  

(44)

Substitute (44) into (43), then (13) can be obtained.

REFERENCES