Extension of a Multi-rate Control Law Independently of Both Reference and Disturbance Responses

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Abstract—This study discusses a design method for sampled-data multi-rate control systems, in which the sampling interval of the plant output is an integer multiple of the hold interval of the control input. In this study, a desired response in discrete time is assumed to be obtained using a multi-rate control law, and then the multi-rate control law is extended such that an existing discrete-time disturbance and reference responses are simultaneously maintained. In the proposed multi-rate control system, the intersample response can be improved independently of the discrete-time response. Finally, the effectiveness of the proposed method is shown through numerical examples.

I. INTRODUCTION

In sampled-data control systems [1], [2], a continuous-time plant is controlled using a digital controller, in which the continuous-time plant output is sampled and converted into a continuous-time signal. In such systems, the hold and sampling intervals are not always equivalent due to hardware restrictions and so on, and multi-rate control systems are obtained [3], [4], [5]. This study discusses multi-rate systems, in which the sampling interval is longer than the hold interval of the control input because the hold interval is not limited but the sampling interval is restricted in process control [6], [7], [8] and in mechanical system [9], [10], [11], [12].

In the multi-rate control systems, the intersample output might oscillate even if the sampled output response is stabilized and converged because the control input can be changed between sampled outputs [6]. Tangirala et al. have shown the condition that the intersample ripples are eliminated in the steady state [13]. On the one hand, this problem can be also resolved using the generalized holder [10], [14]. On the other hand, when the sampled response is obtained as an ideal behavior, it should be maintained. However, the ideal sampled response is changed or deteriorated using the generalized holder because sampled and intersample responses are simultaneously designed and changed such that both responses are optimized.

A new design method for a sampled-data multi-rate control system has been proposed [15]. In this method, a multi-rate control law can be extended independently of the sampled response. As a result, the intersample ripples can be eliminated without changing an existing sampled response. This design method has been extended to a state-space model [16] and to a multi-variable systems [17]. However, in the conventional methods, the sampled response from the reference input to the plant output can be maintained in the ideal situation, where there is no disturbance because the sampled response from the disturbance to the plant output is not maintained. Therefore, in this study, the conventional design method [15] is extended such that it can be applied to actual systems. In particular, this study proposes an extension method of a multi-rate control law such that the sampled response from the disturbance to the plant output is maintained. As a result, the intersample response is improved independently of both the sampled reference and disturbance responses.

This paper is organized as follows. Section II describes a plant model. In Section III, a conventional multi-rate control law is given. In Section IV, the multi-rate control law is extended such that the intersample response from the disturbance to the plant output can be redesigned without changing the discrete-time disturbance response. In Section V, the effectiveness of the proposed method is shown through numerical examples. Concluding remarks are presented in the last section.

II. CONTROLLED PLANT

The actual controlled plant is a continuous-time system, but a control system is designed using a digital controller. Hence, the designed control system is a sampled-data control system, in which the continuous-time plant output is sampled and the discrete-time control input is updated and converged into a continuous-time signal. This study discusses a design method for a sampled-data control system restricted under the following assumptions:

Assumption 2.1:

- Control input is update every step
- Plant output is sampled every \(l\) steps, where \(l\) is an integer

Assumption 2.2:

- Continuous-time model is not obtained
- Multi-rate model is unknown using an identification method based on control input-output data

Because the sampling interval of the plant output differs from the update interval of the control input, a multi-rate control system is designed in this study.
Consider the following multi-rate model:

\[ y(k) = \frac{B[z^{-1}]^T}{A[z^{-1}]} u(k - l) + \xi(k) \]  

(1)

\[ A[z^{-1}] = 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \]

\[ B[z^{-1}] = \begin{bmatrix} B_1[z^{-1}] \\
B_2[z^{-1}] \\
\vdots \\
B_l[z^{-1}] \end{bmatrix} \]

\[ B_j[z^{-1}] = b_{j,0} + b_{j,1} z^{-1} + \cdots + b_{j,m} z^{-m} \]

\[ u(k) = \begin{bmatrix} u(k) \\
\vdots \\
u(k + l - 1) \end{bmatrix} \]

(2)

where \( y(k) \), \( u(k) \) and \( \xi(k) \) are the plant output, the control input and the disturbance, respectively. This is the output error model shown in Fig. 1, and the plant output is directly disturbed by the disturbance.

In this paper, a control system is designed based on this multi-rate model. For simplicity of description, this paper deals with the case of \( l = 2 \).

### III. MULTI-RATE CONTROL LAW

A control system is designed using the following multi-rate control law:

\[ Y[z^{-1}] u(k) = K[z^{-1}] r(k) - X[z^{-1}] y(k) \]

(3)

\[ Y[z^{-1}] = \begin{bmatrix} Y_1[z^{-1}] & 0 \\
0 & Y_2[z^{-1}] \end{bmatrix} \]

\[ K[z^{-1}] = \begin{bmatrix} K_1[z^{-1}] \\
K_2[z^{-1}] \end{bmatrix} \]

\[ X[z^{-1}] = \begin{bmatrix} X_1[z^{-1}] \\
X_2[z^{-1}] \end{bmatrix} \]

where \( r(k) \) is the reference input, and \( Y_i[z^{-1}] \), \( K_i[z^{-1}] \) and \( X_i[z^{-1}] \) (\( i = 1, 2 \)) are polynomials. Also, \( Y[z^{-1}] \) is non-singular, and a closed-loop system is assumed to be stabilized by using the multi-rate control law.

Using this control law, the closed-loop system is given as follows:

\[ y(k) = \frac{z^{-1} Y_B[z^{-1}] K[z^{-1}]}{T[z^{-1}]} r(k) \]

(4)

\[ T[z^{-1}] = A[z^{-1}] Y[z^{-1}] \]

(5)

\[ Y_B[z^{-1}] = \begin{bmatrix} B_1[z^{-1}] Y_2[z^{-1}] - B_2[z^{-1}] Y_1[z^{-1}] \\
Y_1[z^{-1}] Y_2[z^{-1}] \end{bmatrix} \]

(6)

where \( |Y[z^{-1}]| \) denotes the determinant of \( Y[z^{-1}] \).

Assumption 3.1:

- Discrete-time closed-loop system given as Eq. (3) is ideal

Because of this assumption, the discrete-time response has to be maintained, and the intersample response is improved independently of the discrete-time sampled response.

### IV. EXTENSION OF MULTI-RATE CONTROL LAW

Using the extension method [15], the multi-rate control law is extended as follows:

\[ Y_c[z^{-1}] u(k) = K_c[z^{-1}] r(k) - X_c[z^{-1}] y(k) \]

(7)

\[ Y_c[z^{-1}] = Y[z^{-1}] - z^{-1} U_u[z^{-1}] B[z^{-1}] \]

\[ X_c[z^{-1}] = X[z^{-1}] + U_y[z^{-1}] A[z^{-1}] \]

\[ Y_B[z^{-1}] = \begin{bmatrix} Y_1[z^{-1}] Y_2[z^{-1}] - B_2[z^{-1}] Y_1[z^{-1}] \\
Y_1[z^{-1}] Y_2[z^{-1}] \end{bmatrix} \]

(8)

where \( U_u[z^{-1}] \) and \( U_y[z^{-1}] \) are design polynomial vectors and are given as follows:

\[ U_u[z^{-1}] = \begin{bmatrix} U_{u1}[z^{-1}] \\
U_{u2}[z^{-1}] \end{bmatrix} \]

\[ U_y[z^{-1}] = \begin{bmatrix} U_{y1}[z^{-1}] \\
U_{y2}[z^{-1}] \end{bmatrix} \]

In the conventional method [15], \( U_u[z^{-1}] \) and \( U_y[z^{-1}] \) are designed as follows:

\[ U_u[z^{-1}] = \begin{bmatrix} U_{u1}[z^{-1}] B_2[z^{-1}] Y_1[z^{-1}] \\
U_{u2}[z^{-1}] B_1[z^{-1}] Y_2[z^{-1}] \end{bmatrix} \]

\[ U_y[z^{-1}] = \begin{bmatrix} U_{y1}[z^{-1}] B_2[z^{-1}] Y_1[z^{-1}] \\
U_{y2}[z^{-1}] B_1[z^{-1}] Y_2[z^{-1}] \end{bmatrix} \]

(9)

where \( U_1[z^{-1}] \) and \( U_2[z^{-1}] \) are design polynomials. In this case, the closed-loop system is calculated as follows:

\[ y(k) = \frac{z^{-1} Y_B[z^{-1}] K_c[z^{-1}]}{T_c[z^{-1}]} r(k) \]

(10)

\[ T_c[z^{-1}] = A[z^{-1}] Y_c[z^{-1}] \]

(11)

\[ Y_c[z^{-1}] = Y[z^{-1}] - z^{-1} U_u[z^{-1}] B[z^{-1}] X_c[z^{-1}] \]

(12)

where \( |Y_c[z^{-1}]| \) denotes the determinant of \( Y_c[z^{-1}] \). From this equation, the closed-loop system from the reference input to the plant output is the same as the original closed-loop system Eq. (3). Moreover, because \( U_1[z^{-1}] \) and \( U_2[z^{-1}] \) can be designed independently of the discrete-time reference response from the reference input to the plant output, the intersample response can be redesigned independently of the discrete-time reference response using \( U_1[z^{-1}] \) and \( U_2[z^{-1}] \).
On the other hand, the closed-loop system using the extended control law from the disturbance to the plant output is not the same as the original one, but Eq. (12) shows that the disturbance response is also the same as the original one if $|Y_d[z_2^{-1}]| = |Y[z_2^{-1}]|$. $|Y_e[z_2^{-1}]|$ is calculated as follows:

$$
|Y_e[z_2^{-1}]| = (Y_1[z_2^{-1}] - z_2^{-1}U_{u1}[z_2^{-1}]B_1[z_2^{-1}])
\times (Y_2[z_2^{-1}] - z_2^{-1}U_{u2}[z_2^{-1}]B_2[z_2^{-1}])
- z_2^{-1}U_{u1}[z_2^{-1}]B_1[z_2^{-1}]z_2^{-1}U_{u2}[z_2^{-1}]B_2[z_2^{-1}]
$$

(13)

Because $U_u[z_2^{-1}]$ and $U_y[z_2^{-1}]$ are decided using Eq. (10) and Eq. (11), $|Y_e[z_2^{-1}]|$ is not $|Y[z_2^{-1}]|$ at the present moment, but it is calculated as follows:

$$
|Y_e[z_2^{-1}]| = Y_1[z_2^{-1}]Y_2[z_2^{-1}](1 - z_2^{-1}(U_1[z_2^{-1}] + U_2[z_2^{-1}]))
\times B_1[z_2^{-1}]B_2[z_2^{-1}]
$$

(14)

From this equation, $|Y_e[z_2^{-1}]|$ can be made to be $|Y[z_2^{-1}]|$ if the following equation is satisfied:

$$
U_1[z_2^{-1}] = -U_2[z_2^{-1}]
$$

(16)

Therefore, the intersample response can be improved independently of the sampled response because the disturbance response is maintained independently of the selection of $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ when Eq. (16) is satisfied. In this study, these polynomials are decided in order to eliminate the steady-state intersample ripples due to the input vibration between sampled outputs. To this end, the gains from the reference input to the control input must be equivalent in the steady state [13]. In the proposed control system, the gains from $r(k)$ to $u(k)$ are calculated as follows:

$$
C[z_2^{-1}] = \begin{bmatrix}
C_1[z_2^{-1}] \\
C_2[z_2^{-1}]
\end{bmatrix}
= (A[z_2^{-1}]Y_e[z_2^{-1}] + z_2^{-1}X_e[z_2^{-1}][X](z_2^{-1})^{-1})
\times A[z_2^{-1}]K[z_2^{-1}]
$$

(17)

Therefore, using Eq. (16), $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ are decided so that $C_1[1] = C_2[1]$ is satisfied.

V. NUMERICAL EXAMPLE

Consider a continuous-time controlled plant shown by the following transfer function:

$$
G(s) = \frac{1}{(0.5s + 1)(2s + 1)}
$$

(18)

where the impulse-type disturbance with amplitude 0.3 is added from 18s to 20s. It is assumed that the control input is updated at interval of 0.5s but the plant output is sampled at intervals of 1s.

It is assumed that a multi-rate plant model is obtained using an identification method [18], [19] the ideal sampled response is obtained using the original multi-rate control law Eq. (2) so that the closed-loop characteristic polynomial is set to:

$$
T[z_2^{-1}] = (1 - 0.1z_2^{-1})(1 - 0.2z_2^{-1})(1 - 0.4z_2^{-1})
$$

The reference input is set as the unit step function, and the control system is designed such that the plant output follows the reference input without steady-state error. In this case, the coefficient polynomials in the original multi-rate control law are given as follows:

$$
K[z_2^{-1}] = \begin{bmatrix}
1.5 \\
1.5
\end{bmatrix}
$$

(19)

$$
Y[z_2^{-1}] = \begin{bmatrix}
1.0 - 0.016z_2^{-1} \\
0
\end{bmatrix}
$$

(20)

$$
X[z_2^{-1}] = \begin{bmatrix}
-0.16 + 0.19z_2^{-1} \\
1.0
\end{bmatrix}
$$

(21)

In this section, three control results are shown: the original control law Eq. (2), the conventional extension method [15], and the proposed method.

First, the output and input responses using the original control law are shown in Fig. 2 and Fig. 3, respectively. Fig. 2 shows that the sampled output converges to the reference input without steady-state error, but intersample output oscillates because the control input is changed between sampled outputs. After 18s, the sampled plant output is disturbed by the impulse input, but it rapidly converges to the reference input although intersample response oscillates.

Next, the multi-rate control law is extended using the conventional method [15]. In the conventional method, $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ can be selected independently to the discrete-time reference response. However, the design polynomials are decided not taking into account the disturbance response because the conventional method has been designed in the ideal situation, in which there is no disturbance. Hence, using $U_1[z_2^{-1}] = 0$ and $U_2[z_2^{-1}] = 2.6$, the steady-state intersample ripples are eliminated. The control result using the conventional extension method is shown in Fig. 4 and Fig. 5. It can be seen from Fig. 5 that the control input is not oscillates in the steady state, and Fig. 4 shows that the intersample ripples can be eliminated without changing the samples reference response before 18s. However, after 18s the sampled output response is changed from the original one shown in Fig. 2 because Eq. (16) is not satisfied.

Finally, the simulation result using the proposed extension method is shown. The design polynomials are decided so that Eq. (16) is satisfied, and these are set as $U_1[z_2^{-1}] = -1.3$, and $U_2[z_2^{-1}] = -U_1[z_2^{-1}]$. The output and input results obtained using the proposed are shown in Fig. 6 and Fig. 7, respectively. Fig. 6 shows that the sampled output response using the proposed extended control law is the same as that using the original one, and the intersample ripples can be eliminated in the steady state. Moreover, after 18s, the sampled response using the proposed control law is the same as that using the original control law because Eq. (16) is satisfied.
Fig. 2. Output response using the original multi-rate control law

Fig. 3. Input response using the original multi-rate control law

Fig. 4. Output response using the conventional extended multi-rate control law

Fig. 5. Input response using the conventional extended multi-rate control law

Fig. 6. Output response using the proposed extended multi-rate control law

Fig. 7. Input response using the proposed extended multi-rate control law
VI. CONCLUSION

This paper discussed a design method for a sampled-data multi-rate control system, in which the sampling interval of the plant output is an integer multiple of the hold interval of the control input. In the proposed method, a multi-rate control law is extended without changing the sampled closed-loop system: the reference response from the reference input to the plant output and the disturbance response from the disturbance to the plant output. As a result, the intersample response can be redesigned dependently of the sampled response. The effectiveness of the proposed method was demonstrated through numerical examples. In the future, the effective design method of the design polynomials for the transient response is studied.

REFERENCES