Set-based adaptive estimation for a class of nonlinear systems with time-varying parameters

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Abstract:
In this paper, an adaptive estimation technique is proposed for the estimation of time-varying parameters for a class of continuous-time nonlinear system. A set-based adaptive estimation is used to estimate the time-varying parameters along with an uncertainty set. The proposed method is such that the uncertainty set update is guaranteed to contain the true value of the parameters. Unlike existing techniques that rely on the use of polynomial approximations of the time-varying behaviour of the parameters, the proposed technique does require a functional representation of the time-varying behaviour of the parameter estimates. A simulation example is used to illustrate the developed procedure and ascertain the theoretical results.

Keywords: Nonlinear estimation, adaptive estimation, time-varying parameters

1. INTRODUCTION

The problem of parameter estimation has been of considerable interest during the last two decades. The vast majority of the literature on adaptive estimation and adaptive control relies on the assumption that the parameters of the system are slowly time-varying or constant. In practice however, the time-varying behaviour of the process parameters may be of significant importance. Time-varying parameters can arise from uncertain complex mechanisms not captured by the process model. They can also arise from model-plant mismatch or unmeasured inputs that affect the process dynamics.

Evidently, the ability to estimate such time-varying behaviour can have a significant impact on control system performance. Several researchers have considered the problem of estimation of time-varying parameters. In Zhu and Pagilla [2006], an estimation routine is proposed for a class of nonlinear systems with time-varying parameters. A Taylor series expansion of the time-varying parameters is considered. The time varying behaviour is captured by applying a local polynomial expansion and estimating the (locally) constant Taylor series coefficients using a standard least-squares approach. The authors demonstrate the convergence of the parameter estimates to a neighbourhood of the true parameters within a given resetting period. An related approach is proposed in Chen et al. [2011] where a direct least-squares based method is used to estimate the coefficients of a local polynomial expansion of the time-varying parameters. In Kenné et al. [2008], a sliding mode approach is proposed where convergence of the parameter estimates to the true estimates is guaranteed. The technique relies on an estimate of the sign of the parameter estimation error which may be very difficult to obtain in practice.

Many researchers have considered the problem of unknown input estimation which is very closely related to problem of estimation of time-varying parameters. Combined state and unknown input estimation for a class of nonlinear systems is proposed in Ha and Trinh [2004]. High-gain observer methods are proposed in Triki et al. [2010]. A sliding-mode observer design approach is proposed in Veluvolu and Soh [2011] for the estimation of unknown inputs. In contrast to the estimation of time-varying parameters, the estimation of unknown inputs rely on often restrictive structural assumptions which are related to the observability of the inputs from the available measurement. In parameter estimation, this structural information would in general be limited to an identifiability requirement or a persistency of excitation condition.

The analysis of nonlinear systems with time-varying parameters has been considered in Bastin and Gevers [1988]. In this paper, the authors propose an adaptive observer canonical form and propose an adaptive observer or this class of systems. It is claimed that the estimation of time-varying parameters can be performed subject to a bound on the rate of change of the parameters. A related result is presented in Marino et al. [2011] where an adaptive observer is proposed for a class of nonlinear systems with time-varying parameters and bounded disturbances. As in Bastin and Gevers [1988], estimation of the time-varying parameters is possible subject to a bound on the derivative of the parameters.

The guiding principle in the estimation of time-varying systems is that one can estimate parameters up to a bound on the rate of change of the parameters. As a result, the
proposed technique can be overly conservative since it is almost impossible to estimate this bound a priori. In
this paper, we propose a set-based estimation technique for the estimation of time varying parameters in linearly parameterized nonlinear systems. The approach proposed
is influenced by the identification scheme presented in Adetola and Guay [2010] generalizes the approach from earlier (Adetola and Guay [2009] and Adetola and Guay
[2010]) to a class of nonlinear systems with time-varying parameters. This method ensures convergence of the
parameters to a neighbourhood on their true values and provides an estimate of the size the uncertainty set using a
set-update algorithm that guarantees non-exclusion of the true parameter estimates.

This paper is organized as follows. The problem description and uncertainty set adaptation are presented in
Section 3. This is followed by a simulation example in Section 4 and a brief conclusion in Section 5.

2. PROBLEM DESCRIPTION

Consider a nonlinear system
\[ \dot{x} = f(x) + \sum_{i=1}^{p} g_i(x) \theta_p(t) = f(x) + g(x) \theta(t) \] (1)
where \( f(x) \) and \( g_i(x) \) \((i = 1, \ldots, n)\) are sufficiently smooth vector valued functions, \( g(x) = [g_1(x), \ldots, g_p(x)] \in \mathbb{R}^{n \times p} \), \( x \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R}^p \) is the output, \( \theta(t) \in \mathbb{R}^p \) is an unknown time varying bounded parameter vector assumed to be uniquely identifiable lying within a known compact set \( \Theta^0 = B(0, \Sigma_0) \), where \( \theta_0 = \theta(0) \) is a nominal parameter value, \( \Sigma_0 \) is the radius of the parameter
uncertainty set. The vector-valued functions \( f(x) \), \( g(x) \) is sufficiently smooth. The following assumptions are made about (1).

Assumption 2.1: The state variables \( x(t) \in \mathbb{X} \) a compact subset of \( \mathbb{R}^n \).

Assumption 2.2: The system is observable.

Assumption 2.3: \( \theta(t) \) represents the time varying parameter such that \( ||\theta(t)|| \leq c_1 \).

The aim of this work is to provide the true estimates of plant’s time varying parameters and estimation of the state.

3. PARAMETER AND UNCERTAINTY SET ESTIMATION

3.1 Parameter estimation

Let the estimator model for (1) be chosen as
\[ \dot{\hat{x}} = f(x) + g(x) \hat{\theta}(t) + Ke + w^T \dot{\hat{\theta}}(t), \quad K > 0, \] \( \hat{w}^T = -Kw + g(x), \quad w(0) = 0. \) (2)
resulting in the state prediction error \( e = x - \hat{x} \) and an auxiliary variable \( \eta = e - w^T \theta(t) \) dynamics:
\[ \dot{e} = g(x) \hat{\theta}(t) - Ke - w^T \dot{\hat{\theta}}(t) \] (4)
where \( e(t_0) = x(t_0) - \hat{x}(t_0) \),
\[ \dot{\eta} = -K\eta - w^T \dot{\hat{\theta}}(t), \quad \eta(t_0) = e(t_0) \] (5)
An estimate of \( \eta \) is generated from
\[ \dot{\hat{\eta}} = -K\hat{\eta} \] (6)
with resulting estimation error \( \hat{\eta} = \eta - \hat{\eta} \) dynamics
\[ \dot{\hat{\eta}} = -K\hat{\eta} - w^T \dot{\hat{\theta}}(t), \quad \eta(t_0) = 0. \] (7)
An estimate of \( \eta \) is generated from (6) with resulting estimation error \( \hat{\eta} = \eta - \hat{\eta} \) dynamics given by (7), \( \hat{\eta}(t_0) = \eta(t_0) = 0. \) Therefore \( \hat{\eta}(t_0) = 0. \) Consider a Lyapunov function \( V_{\eta} = \frac{1}{2} \hat{\eta}^T K \hat{\eta} + \frac{1}{2} T^T w^T \dot{\hat{\theta}} \leq -\eta^T K \hat{\eta} + \frac{1}{2} T^T w^T \dot{\hat{\theta}} \) \( (8) \)
where \( k_1 \) is some positive constant. Thus if one assigns \( K \) as
\[ K = \frac{k_2}{2} + \frac{k_1^2}{2} w^T w \]
for some positive constant \( k_2 \) with \( I \), the identity matrix, the inequality above becomes:
\[ V_{\eta}(t) \leq -k_2 V_{\eta} + \frac{1}{2} c_1^2 \]
(9)
Since \( \eta(0) = \hat{\eta}(0) = 0 \), it follows that \( \hat{\eta}(t_0) = 0 \). The inequality (9) yields:
\[ V_{\eta}(t) \leq e^{-k_2 t} V_{\eta}(0) + \int_0^t e^{-k_2 (t-r)} d\tau \frac{c_1^2}{2k_1} \leq \frac{c_1^2}{2k_1 k_2} \]
or
\[ ||\hat{\eta}(t)|| \leq \frac{c_1}{\sqrt{k_1 k_2}} \]
\( \forall t \geq 0 \).
Following Adetola [2008], the parameter estimation scheme has been generated for the above mentioned system.

3.2 Parameter adaptation

Let \( \Sigma \in \mathbb{R}^{n \times n} \) be generated from
\[ \Sigma = \Sigma_0 \] (10)
where \( k_T \) is a positive constant to be assigned. Based on (2),(3) and (6), one considers the parameter update law as proposed in Adetola and Guay [2009] is given by
\[ \dot{\Sigma} = w^T w - k_T \Sigma, \quad \Sigma(t_0) = \alpha I > 0 \] (11)
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where \( k_T \) is a positive constant to be assigned.

Assumption 1. There exists constants \( \alpha_1 > 0 \) and \( T > 0 \) such that
\[ \int_T^{t+T} w(\tau) w^T (\tau) d\tau \geq \alpha_1 I \] (15)
\( \forall t > 0 \).
Theorem 1. Let Assumption 1 be met by the trajectories of (1). The identifier composed of (11) and the parameter update law (12) is such that the estimation error \( \hat{\theta}(t) = \theta(t) - \hat{\theta}(t) \) is bounded and such that
\[
\lim_{t \to \infty} ||\hat{\theta}(t)|| \leq K
\]
where \( K > 0 \) is a strictly positive constant.

Proof:
Consider the Lyapunov function,
\[
V_{\hat{\theta}(t)} = \frac{1}{2} \hat{\theta}(t)^T \Sigma \hat{\theta}(t)
\]
it follows from (11) that
\[
\dot{V}_{\hat{\theta}(t)} = \hat{\theta}(t)^T \Sigma(t) \dot{\hat{\theta}}(t) - k_T \frac{1}{2} \hat{\theta}(t)^T \Sigma \hat{\theta}(t) + \frac{1}{2} \hat{\theta}(t)^T w(t)^T \hat{\theta}(t).
\]
If one substitutes for (12) and \( w^T \hat{\theta}(t) = e - \hat{\eta} - \bar{\eta} \), it follows that, invoking the properties projection algorithm,
\[
\dot{V}_{\hat{\theta}(t)} \leq \hat{\theta}(t)^T \Sigma(t) \dot{\hat{\theta}}(t) - k_T \frac{1}{2} \hat{\theta}(t)^T \Sigma \hat{\theta}(t) - \hat{\theta}(t)^T w(t)(e - \bar{\eta}) + \frac{1}{2} \hat{\theta}(t)^T w(t)^T \hat{\theta}(t)
\]
By Young’s Inequality on the first term, one obtains,
\[
\dot{V}_{\hat{\theta}(t)} \leq \frac{k_3}{2} \hat{\theta}(t)^T \Sigma(t) \hat{\theta}(t) + \frac{1}{2k_3} \hat{\theta}(t)^T \Sigma(t) \hat{\theta}(t)
\]
where \( k_3 \) is some positive constant. It follows that if one sets \( k_T = k_3 + k_4 \) for a positive constant \( k_4 \) then
\[
\dot{V}_{\hat{\theta}(t)} \leq \frac{k_4}{2} \hat{\theta}(t)^T \Sigma(t) \hat{\theta}(t) - \frac{1}{2} (e - \bar{\eta})^T (e - \bar{\eta}) + \frac{1}{2} \eta^T \bar{\eta}
\]
Next, we claim the boundedness of the matrix \( \Sigma(t) \) as follows. By integration, one gets:
\[
\Sigma(t) = e^{-k_T t} \Sigma(0) + \int_0^t e^{-k_T (t - \tau)} w(\tau) w(\tau)^T d\tau
\]
By the boundedness of \( w(\tau) \), one can also write,
\[
\Sigma(t) \leq \Sigma(0) + \beta_2 \int_0^t e^{-k_T (t - \tau)} d\tau I \leq \alpha I + \beta_2 I = \gamma_2 I.
\]
As a result, we get that:
\[
\gamma_2 I \leq \Sigma(t) \leq \gamma_2 I
\]
and
\[
\gamma_2^{-1} I \leq \Sigma(t)^{-1} \leq \gamma_1^{-1} I.
\]
As a result, (17) yields:
\[
\dot{V}_{\hat{\theta}(t)} \leq -k_4 \frac{1}{2} \hat{\theta}(t)^T \Sigma(t) \hat{\theta}(t) - \frac{1}{2} (e - \bar{\eta})^T (e - \bar{\eta}) + \frac{\gamma_2}{k_3} \frac{1}{2} \hat{\theta}(t)^T w(t)^T \hat{\theta}(t)
\]
Taking the limit as \( t \to \infty \), one gets
\[
\lim_{t \to \infty} V_{\hat{\theta}(t)} \leq \left( \frac{\gamma_2}{k_3} + \frac{1}{2k_1} \right) c_1^2
\]
As a result, the parameter estimation error enters a small neighbourhood of the origin parameterized by \( k_1 \) and \( k_3 \), two adjustable positive gains.

3.3 Parameter set adaptation

The analysis from the last section indicates that it will not be possible in general to guarantee that \( \lim_{t \to \infty} \hat{\theta}(t) = 0 \). However, one can always guarantee that the parameter estimation error belongs to a neighbourhood of the origin. In this section, we propose an algorithm that yields an estimate of this neighbourhood which we call the uncertainty set. The uncertainty set is taken to be a ball of radius \( z_0 \) centred at the origin.

Using the analysis from the previous section, an update law that measures the worst-case progress of the parameter identifier in the presence of a disturbance is given by
\[
z_0(t) = \sqrt{\frac{V_z(t)}{4\lambda_{\min}(\Sigma(t))}}
\]
\[
V_z(t) = 4\lambda_{\max}(\Sigma(t)) z_0^2(t)
\]
\[
V_z(t) = -k_4 V_z(t) - \frac{1}{2} (e - \bar{\eta})^T (e - \bar{\eta}) + V_z(t) + \lambda_{\max}(\Sigma(t)) c_1^2
\]
where \( V_z(t) \) represent the solutions of the ordinary differential equation (21) with the initial condition (20), and \( V_z(t) \) is the solution of the initial value problem (22).

The parameter uncertainty set, defined by the ball \( \mathcal{B}(\hat{\theta}_e, z_0) \) is updated using the parameter update law (12) and the error bound (19) according to the following algorithm:

Algorithm 1. (1) Initialize \( z_0(t_i - 1) = z_0 \), \( \hat{\theta}(t_i - 1) = \hat{\theta}^0 \)
(2) At time \( t_i \), update
\[
\begin{cases}
\hat{\theta}(t_i), \Theta(t_i), \quad &\text{if } \left| z_0(t_i) \right| \leq z_0(t_i - 1)

\left| \gamma_1 (t_i - t_i - 1) - (\hat{\theta}(t_i) - \hat{\theta}(t_i - 1)) \right|

\left( \hat{\theta}(t_i - 1), \Theta(t_i - 1) \right), \quad &\text{otherwise}
\end{cases}
\]
(3) Iterate back to step 2, incrementing \( i = i + 1 \).
Algorithm 1 ensures that Θ is only updated when the value of zθ has decreased by an amount which guarantees a contraction of the set. Moreover zθ evolution as given in (19) ensures non-exclusion of θ(t) as given below.

**Lemma 1.** The evolution of Θ = B(0, zθ) under (11), (19) and Algorithm 1 is such that

1. θ ∈ Θ(t0) =⇒ θ ∈ Θ(t) ∀t ≥ t0
2. Θ(t2) ⊆ Θ(t1), t2 ≤ t1 ≤ t2

**Proof:**

(1) By definition, we can establish that V̇θ(t0) ≤ Vθ(t0).

From(9), it follows that

\[
V̇θ(t) ≤ -k_2 V_θ(t) + \frac{c^2}{2k_3}.
\]

By the comparison lemma, the solution of the differential equation \( V̇θ(t) \) guarantees that \( V_θ(t) \) \( \forall t \geq t_0 \).

Similarly, we know that \( V̇θ(t_0) \leq Vθ(t_0) \) (by definition). It follows from (17) that

\[
V̇θ(t) ≤ -k_2 V_θ(t) - \frac{1}{2}(e - ̄e)^T(e - ̄e) + Vθ(t) + λ_{max}(Σ(t)) \frac{c^2}{2k_3}.
\]

Hence, by the comparison lemma, the solution of the initial value problem (21),(20) is such that

\[
V_θ(t) ≤ V_θ(t_0) \forall t ≥ t_0
\]

and since \( V_θ(t) = \frac{1}{2} θ(t)^TΣθ(t) \), it follows that

\[
||θ̂(t)||^2 ≤ \frac{Vθ(t)}{4λ_{min}(Σ(t))} = z_θ^2(t) \forall t ≥ t_0.
\]

Hence, if \( θ̂(t) \in Θ(t_0) \), then \( θ̂(t) \in B(0, z_θ(t)), \forall t ≥ t_0 \).

(2) If \( Θ(t_{i+1}) \not\subseteq Θ(t_i) \), then

\[
\sup_{θ(t_i) \in Θ(t_{i+1})} ||θ̂(t_i)|| ≥ z_θ(t_i)
\]

Now, one can write:

\[
\sup_{θ(t_i) \in Θ(t_{i+1})} ||θ̂(t_i)|| = z_θ(t_i) + ||θ̂(t_i) - θ̂(t_{i+1})||
\]

\[
≤ z_θ(t_{i+1}) + ||θ̂(t_{i+1}) - θ̂(t_i)|| \leq z_θ(t_{i+1}) + ||θ̂(t_{i+1}) - θ̂(t_i)|| + |θ̂(t_{i+1}) - θ̂(t_i)|
\]

\[
≤ z_θ(t_i) + ||θ̂(t_{i+1}) - θ̂(t_i)|| + |c_1(t_{i+1} - t_i)|
\]

which contradicts (25). Hence, Θ update guarantees Θ(t_{i+1}) ⊆ Θ(t_i). And Θ is held constant over update intervals τ ∈ (t_i, t_{i+1}).

**4. SIMULATION EXAMPLE**

**4.1 Example 1**

To illustrate the effectiveness of the proposed method, we consider the following system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -θ_1(t)x_2 - 10θ_2(t)\arctan(900x_2) + 10u
\end{align*}
\]

where \( θ_2(t) = 3\cos(0.5πt) \) and

\[
\begin{align*}
θ_1(t) &= \begin{cases} 
2 + \sin(t) & 0 ≤ t < 40 \\
2 + \sin(40) + \sin(2(t - 40)) & 40 ≤ t < 80 \\
2 + \sin(40) + \sin(80) & t ≥ 80
\end{cases}
\end{align*}
\]

For simulation purposes, it is assumed that \( x_1(0) = 1 \) and \( x_2(0) = -8 \) and \( u = 100\sin(20\pi t) \). The parameter estimates are plotted along with their true value in Figure 1. The parameter estimates follow the true unknown parameters effectively. The uncertainty radius is compared to the norm of the parameter estimation error in Figure 2. As expected, the norm of the parameter estimation error is always less than the uncertainty radius. This confirms that the true unknown uncertain parameters are always contained in the uncertainty set.

![Fig. 1. Plot of the parameter estimates along with the true parameter values.](image1)

![Fig. 2. Upper bound of the estimation error norm ̂θ and the uncertainty set radius ẑθ as a function of time.](image2)

**5. CONCLUSION**

An adaptive estimation technique is proposed for the estimation of time-varying parameters for a class of
continuous-time nonlinear system. A set-based adaptive estimation is used to estimate the time-varying parameters along with an uncertainty set. The proposed method is such that the uncertainty set update is guaranteed to contain the true value of the parameters. Unlike existing techniques that rely on the use of polynomial approximations of the time-varying behaviour of the parameters, the proposed technique does require a functional representation of the time-varying behaviour of the parameter estimates. The techniques yields guaranteed bounds for the uncertainty on the parameter estimates that can be utilized for the monitoring and robust control of uncertain systems.

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