Robust Design of Closed-Loop Nonlinear Systems with Input and State Constraints

Diego A. Muñoz; Wolfgang Marquardt

Abstract: This work focuses on the control design for feedback linearizable nonlinear systems with unknown time-varying disturbances and uncertainty such that bounds on inputs and state constraints are not violated. Exploiting the special structure of the systems considered, controllers based on Lyapunov’s direct methods are easily synthesized, which enforce asymptotic stability in the presence of bounded inputs. However, these controllers do not account explicitly for state constraints in the presence of disturbances and uncertainty. The solution of this task is addressed in this work by an optimization-based method, the so-called normal vector approach.

Keywords: Feedback linearization, Saturation, Lyapunov methods, MIMO, Robustness, Bounded Disturbances, Uncertainty, Transient stability, State constraints.

1. INTRODUCTION

The nonlinear control design for constrained systems is commonly carried out using a two-step approach. After a closed-loop system design has been determined, often without explicitly considering input saturation, the dynamic behavior is investigated and the undesired effects due to actuator constraints are accounted for. The performance degradation caused by input constraints has motivated many studies, where simultaneous approaches are developed to improve the performance of closed-loop systems. For a specific class of chemical processes [Alvarez et al., 1991, 1993, Barron et al., 1997, Barrón and Aguilar, 1998], the bounded control problem has been studied using nonlinear state-feedback, where the fundamental idea is that the controller makes the desired steady state (stable or unstable) a globally and asymptotically stable attractor without saturation. Although the conditions to exclude equilibrium points induced by saturation are established for a class of continuous reactors, it is not straightforward to propose a general extension. Some general results for input-output linearizing SISO systems has been proposed by Kapoor and Daoutidis [1997, 2000] under the assumption that the origin of the so-called “zero dynamics” is locally exponentially stable. The authors defined an appropriate region in the state space, where a control law is locally stabilizing in the absence of bounds but has the ability to guarantee closed-loop stability even when the control law is saturated. The problem can also be solved constructing level sets where the control input stays necessarily unconstrained [Pappas et al., 1995]. In case of MIMO systems, these authors also study feedback linearization under input saturation and identify conditions for which is possible to transform the MIMO problem into $m$ decoupled SISO problems with state-dependent input constraints [Pappas et al., 1995]. However, input-output linearizing feedback control is criticized for applying unnecessary large control action to cancel out nonlinearities that can actually help to stabilize the nonlinear system [Freeman and Kokotovic, 1996].

On the other hand, the concept of a Control Lyapunov Function (CLF) introduced by Artstein [1983] for stability analysis has been converted into a tool for solving the stabilization task [Kokotovic and Arcak, 2001]. In this context, Lin and Sontag [1991] propose a stabilizing feedback law using a bounded control, under the assumption that an appropriate CLF is known. The construction of a CLF is still a hard problem, which can be solved only for strict feedback and feedback linearizable systems [Kokotovic and Arcak, 2001]. Inspired by the results of Lin and Sontag [1991], El-Farra and Christofides [2001, 2003] derive explicit analytical formulas for continuous state feedback bounded robust optimal controllers with well-characterized stability and performance properties. Although this kind of Lyapunov-based control approach accounts explicitly for nonlinearities, uncertainty and input bounds, it does not account explicitly for state constraints.

For constrained systems, our group has developed a new approach based on Constructive Nonlinear Dynamics (CNLD), the so-called normal vector approach [Mönigmann and Marquardt, 2002, Gerhard et al., 2008]. CNLD guarantees desired dynamic properties in the presence of uncertainty and disturbances. Using this method, state and input constraints have been studied in the framework of grazing bifurcations [Gerhard et al., 2006, 2008], i.e., both input and state trajectories are assumed to only hit the constraints in the time-domain tangentially at a single (grazing) point. However, the treatment of input constraints using grazing bifurcations results in suboptimality as it cannot cover the case where an input rests at its bound for a longer time. In order to overcome this shortcoming, we have recently extended our method to allow saturation...
on the manipulated variables under the assumption that it does not create a bifurcation [Muñoz et al., 2010]. However, as mentioned before, input saturation may lead to undesirable effects like a large "overshoot" inducing a limit cycle, an unstable output response or other kinds of performance degradation [Kapoor and Daoutidis, 2000, El-Farra and Christofides, 2003].

In this work we want to combine the advantages of Lyapunov-based control and the normal vector approach to guarantee closed-loop stability with bounded inputs and state constraints under uncertainty and disturbances.

This paper is organized as follows. Section 2 states the system description with necessary assumptions to be fulfilled in order to apply Lyapunov-based control for constrained systems, which is introduced in Section 3. Then, a new method for the design of a bounded robust controller with state constraints is established in Section 4 which guarantees dynamic properties for the transient behavior in the presence of model uncertainty and unknown disturbances. The feasibility of the suggested method is illustrated by means of a simulated chemical process example and compared with other control strategies in Section 5. Finally, conclusions are given in Section 6.

2. SYSTEM DESCRIPTION

We consider input-constrained closed-loop systems represented by a class of continuous systems of ordinary differential equations (ODEs) with the following state-space description

\[
\dot{x} = f(x,d(\alpha,t),t), \quad x(0) = x_0
\]

\[
y_i = h_i(x), \quad i = 1, \ldots, m, \quad \forall \alpha \in \mathcal{A},
\]

where \(x \in \mathcal{X} \subset \mathbb{R}^n_x\) denotes the state variables and \(y_i \in \mathbb{R}\) denotes the \(i\)th output to be controlled. The vector of constrained manipulated inputs \(u = [u_1, \ldots, u_m]^T\) takes values in a nonempty subset \(\mathcal{U} = \{u \in \mathbb{R}^m : \|u\| < u_{\text{max}}\}\), where \(\|\cdot\|\) is the Euclidean norm and \(u_{\text{max}}\) is the positive real number that captures the maximum magnitude of \(u\).

The vector functions \(f \) and \(g_i\) and the scalar functions \(h_i\) are assumed to be sufficiently smooth on their domains of definition. \(d(\alpha,t) \in \mathbb{R}^n_d\) may represent disturbances parameterized by a set of uncertain parameters \(\alpha \in \mathcal{A} \subset \mathbb{R}^n_\alpha\), and time \(t \in \mathbb{R}\), or time-invariant parametric uncertainty in process parameters (e.g., \(\exists j, i, d_j(\alpha,t) = \alpha_i\)). For the uncertain parameters, the following assumption holds:

**Assumption 1.** The set of uncertain parameters \(\alpha\) has the form

\[
\mathcal{A} = \{ \alpha \in \mathbb{R}^n_\alpha | 0 \leq \beta(\alpha) \},
\]

with the differentiable function \(\beta\) mapping from \(\mathbb{R}^n_\alpha\) into \(\mathbb{R}\). For box-type uncertainty regions used in this work, i.e., \(\alpha_i \in [-\Delta \alpha_i, \Delta \alpha_i]\), the smooth approximation

\[
\beta(\alpha) := n_\alpha - \sum_{i=1}^{n_\alpha} \left( \frac{\alpha_i}{\Delta \alpha_i} \right)^2, \quad j \in \mathbb{N},
\]

is used which is obviously an instance of (2).

Note that the uncertain parameters \(\alpha\) are introduced not only to represent a family of bounded disturbances which are interpreted as the most plausible disturbance scenarios against which robustness is required, but also to consider uncertainty in process parameters like kinetic constants or initial conditions.

For state constraints, some time-varying bound on the trajectories of the process model are imposed, i.e.,

\[
0 \leq \tilde{\nu}_j(x,d(\alpha,t),t), \quad j = 1, \ldots, n_\nu, \quad t > 0,
\]

with sufficiently smooth functions \(\tilde{\nu}_j\) mapping from \(\Omega \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_d} \times \mathbb{R}\) into \(\mathbb{R}\). Constraints (4) may represent safety constraints, such as an upper temperature limit in a chemical reactor or a quality constraint such as a bound on a product property. For safe and economical operation it is therefore important to ensure that the constraints (4) hold despite unknown disturbances \(d(\alpha, t)\), \(\forall \alpha \in \mathcal{A}\). Additionally, it is assumed that the origin is the equilibrium point of the nominal system, i.e., for \(u = 0\) and \(d(\alpha, 0) = 0 \Rightarrow f(0,d(\alpha,0)) = 0\).

In order to proceed with the bounded control design, some structural assumptions must be fulfilled:

**Assumption 2.** The process model with \(d(\alpha, 0) = 0\) is input-output linearizable, i.e., there exists a set of integers \((r_1, \ldots, r_m)\) and a coordinate transformation \((\zeta, \eta) = T(x)\) such that the system (1) takes the normal form in the \((\zeta, \eta)\) coordinates [Isidori, 1989].

Defining the tracking error variables \(e^k = \zeta^k - \nu^{(k-1)}_i\), where \(i = 1, \ldots, m\) and \(k = 1, \ldots, r\), the \(\zeta\)-subsystem can be re-written in the following compact form

\[
\dot{e}_i = A e + \vartheta(e, \eta, \nu) + B \cdot C (T^{-1} e, \eta, \nu),
\]

(5)

where \(A, B\) are constant matrices of dimension \((\sum_{i=1}^{m} r_i) \times (\sum_{i=1}^{m} r_i)\) and \((\sum_{i=1}^{m} r_i) \times m\), respectively, \(\vartheta(e, \eta, \nu)\) is a \((\sum_{i=1}^{m} r_i) \times 1\) continuous vector function, \(\nu = (\nu^1, \ldots, \nu^m)^T\), \(\vartheta_i = (v_1, v_2, \ldots, v^r_m)^T\), \(v_i^k\) is a smooth vector function, and \(v_i^k\) is the \(k\)th derivative of the external reference input \(\nu_i\). The \(m \times m\) matrix \(C(T^{-1} e, \eta, \nu)\) is the decoupling matrix of system (1) [Henson and Seborg, 1997, Chapter 4] (see also El-Farra and Christofides [2003] for details).

An additional assumption is imposed on the \(\eta\)-subsystem to guarantee bounded stability of the internal dynamics:

**Assumption 3.** The so-called zero dynamics [Isidori, 1989]

\[
\dot{\eta} = \Psi(e, \eta, \nu)
\]

(6)

is input-to-state stabilizable with respect to \(e\) uniformly in \(\nu\).

We want to stress that feedback linearization will not be used in this work for controller design but only for the construction of control Lyapunov functions [Freeman and Kokotovic, 1995].

3. BOUNDED LYAPUNOV-BASED CONTROL

In order to derive the bounded control law, let us suppose that an appropriate control-Lyapunov function (CLF) is known. A candidate CLF for the \(e\)-subsystem (5) is

\[
V(e) := e^T Pe
\]

where \(P\) is a positive definite matrix that satisfies the Riccati matrix inequality \(A^T P + PA -

$PBB^T P < 0$. We recall from Kokotovic and Arcak [2001] that $V$ is a CLF provided
\[ e \neq 0, L_g V(e) = (0, \ldots, 0) \Rightarrow L_f V(e) < 0, \] where $L_V V$ denotes the Lie derivative of the scalar function $V$ with respect to the function $(\cdot)$. Note that this condition can be easily proven [Freeman and Kokotovic, 1995]. When a CLF candidate is known, Lin and Sontag [1991] provide a formula for a stabilizing feedback law where controls take values in the open unit ball, under the additional assumption that $V$ satisfies the small control property (scp): For each $\delta > 0$ there is an $\epsilon > 0$ such that, if $e \neq 0$ satisfies $\|e\| < \epsilon$, then there is some $u$ with $\|u\| < \delta$ such that $L_f V(e) + L_g V(e) u < 0$. If the controls take values in $U$ as in our setting, $u$ needs to be scaled and $\hat{g}$ needs to be multiplied by $u_{\text{max}}$.

With all these elements, the explicit synthesis formula for the desired bounded multivariable state feedback control law proposed by Lin and Sontag [1991] is given as
\[ u(e) = \kappa(L_f V(e), \|L_g V(e)\|^2(L_g V(e))^T, \] where $\kappa$ is the scalar value function defined by
\[ \kappa = \begin{cases} L_f V + \sqrt{(L_f V)^2 + \|L_g V\|^4}, & \|L_g V\|^2 > 0, \\ 0, & \|L_g V\|^2 = 0, \end{cases} \] which is almost smooth on $\mathbb{R}^n$, takes values in the open unit ball, and globally stabilizes the system [Lin and Sontag, 1991, Theorem 1].

In order to satisfy the scp, Lin and Sontag [1991] showed that this property is equivalent to the inequality
\[ \tilde{\psi}_{\text{CLF}} := -L_f V(e) + \delta \|L_g V(e)\| > 0. \] Therefore, if the evolution of the closed-loop trajectory satisfies this inequality, we can guarantee the desired closed-loop stability. Note that although input bounds are handled with this results, it does not account explicitly for state constraints (4).

4. BOUNDED ROBUST CONTROL DESIGN WITH STATE CONSTRAINT

Under unknown time-varying disturbances, time-invariant uncertainties and constrained inputs, the synthesis of the closed-loop system must guarantee not only stability of the perturbed closed-loop system, but also prevent that trajectories violate state constraints (4). The control design problem we are interested to solve can be represented by the following nonlinear program (NLP):
\[
\begin{align*}
\min_{P} & \quad \phi \\
\text{s.t.} & \quad \dot{e} = f(e, \eta, \nu, d(\alpha, t)) + \hat{g}(e, \eta, \nu, d(\alpha, t)) u \\
& \quad \dot{\eta} = \Psi(e, \eta, \nu, d(\alpha, t)) \\
& \quad T(x_0) = (\kappa^2(0), \eta^T(0))^T \\
& \quad u \text{ defined by Eqs. (8)-(9)} \\
& \quad 0 < \tilde{\psi}_{\text{CLF}}, \\
& \quad 0 < \tilde{\psi}_{\psi}(T^{-1}(e, \eta, \nu), d(\alpha, t), t), \quad j = 1, \ldots, n_{\psi} \quad (11c) \forall \alpha \in A
\end{align*}
\] where $\phi$ is the merit function and equations (11c) constitute semi-infinite constraints that guarantee state constraints and closed-loop stability under bounded inputs and unknown disturbances. $P$ denote the positive definite matrix that should satisfy the Riccati equation (cf. Section 3), which is a degree of freedom in the optimization. The NLP is called semi-infinite as the constraints (11c) have to be fulfilled for all parameter variation $\alpha \in A$ and for all times $t > 0$, which results in an infinite number of constraints.

To solve the semi-infinite problem (SIP) (11), our group developed an algorithm based on constraints on the distance in parameter space between a critical manifold and the robustness region as a criterion for robustness [Mönnigmann and Marquardt, 2005]. Critical manifolds represent those boundaries in the parameter space where feasibility or stability constraints violation occurs. With this family of methods, the SIP is approximated by a NLP with a finite number of constraints. In this paper we use the same method, which is briefly introduced below.

4.1 Critical manifolds of grazing bifurcation

As an example, consider a single state constraint (4) corresponding to (11c) in transformed coordinates, which is a constant upper bound for the state $x_1 \in \mathbb{R}$, i.e., $0 \leq x_{1,\text{max}} - x_1$. Figures 1(a)-(b) show an illustrative example of the transient behavior of state $x_1$ and a single input $u$ after a step disturbance triggered at $t = 0$. The magnitude of the disturbance is parameterized by the parameter $\alpha_1$. The simple upper bound for $x_1$, $\bar{\psi} := x_{1,\text{max}} - x_1$, defines a plane in the $(\alpha_1, x_1)$-space that must not be crossed from below by the time response of $x_1$ after the disturbance. Depending on the response of $x_1$, Figure 1(b) sketches that the trajectory of the input can saturate. As shown in Figure 1(a), there is a critical value of $\alpha_1 = \alpha_1^c$ for which the bounding plane is tangentially touched but not yet crossed. The points where the time response of $x_1$ touches the bounding plane tangentially is a so-called grazing bifurcation [Nordmark, 1991]. The set of points where the bound is crossed, is a parabola-like curve with its extremum located at the grazing bifurcation $t^*, \alpha_1^c$.

In the presence of a second disturbance parameter $\alpha_2$ the parabola-like curve of the crossing points in Fig. 1(a)
unfolds into a fold-like surface and the grazing point into a curve in the \((α_1, α_2, t)\)-space as depicted in Figure 1(c). The projection \(M^{(c)}\) of this curve onto the parameter plane spanned by \(α_1\) and \(α_2\) shown in Figure 1(d) separates the region where the constraint is not violated from the region where disturbances lead to time responses that cross the bound. Figure 1(d) stresses the importance of the grazing point for the design of a dynamical system that has to be restricted to the region, where the constraint is not violated. The design parameters are then optimized under the constraint that the distance between the robustness region and the critical manifolds stays greater than or equal to zero. Since the closest distance occurs along the direction that is normal to \(M^{(c)}\) as well to the boundary of the robustness manifold, this condition is expressed as

\[
α^{rob} = α^c + l \cdot λ, \quad l \geq 0, \quad (12)
\]
with \(α^c\) and \(α^{rob}\) being the closest points on the critical and robustness manifolds, \(l\) is a measure of parametric distance and \(λ\) represents the common normal direction of the critical manifold and the robust manifold. Then the infinite number of constraints (11c) are converted to a finite number of constraints using (12) and the set of algebraic equations which defines the critical manifolds \(M^{(rob)}\) and \(M^{(c)}\) and their normal vector formulations. Due to space limitation, we refrain here from giving the technical details, but refer the interested reader to Gerhard et al. [2008].

4.2 Critical manifolds for transient stability loss

Since (10) guarantees stability, the closed-loop trajectory satisfies this inequality. A situation where disturbances lead to trajectories violating the inequality condition (10) is denoted here as transient stability loss. Hence, the following definition is formulated:

**Definition 4.** A trajectory of a perturbed dynamical system is called critical if inequality (10) becomes active but is not violated.

A practical definition of a critical manifold based on this condition refers to the existence of grazing points to Eq. (10). The application of this condition may obviously lead to conservative results because this condition is too strict in many cases [Wirth et al., 2011]. However, if it is satisfied, we can guarantee that the time-derivative of the CLF is negative-definite along the trajectories of the closed-loop system under uncertainty, disturbances and constraints such that closed-loop robust stability is guaranteed.

4.3 Normal vector systems

The task of finding a control design minimizing an objective \(φ\) and guaranteeing that specified constraints (11c) are never violated despite disturbances is addressed by solving the constrained nonlinear program (11), where the Eqs. (11c) are replaced by

\[
0 = φ^{(j;k)}(P, d(α^{(j;k)}, φ^{(j;k)}, λ^{(j;k)})
\]

\[
0 = φ^{rob(j;k)}(P, α^{rob(j;k)}, λ^{(j;k)})
\]

\[
0 = α^{(j;k)} - α^{rob(j;k)} + l^{(j;k)} \cdot λ^{(j;k)}
\]

\[
0 \leq l^{(j;k)}\]

The upper index \(j \in \{1, \ldots, n_ρ, CLF\}\) enumerates the time-domain constraints. Eqs. (13a) define the so-called augmented normal vector system of grazing points and the boundary of the robust manifold, respectively (see Gerhard et al. [2008] for details). The index \(k = 1, \ldots, K\) enumerates the closest critical points that are taken into account. Eqs. (13b) enforce the minimal distance between a point located on the robust manifold \(α^{rob(j;k)}\) and the closest critical point \(α^{(j;k)}\). Each detected critical point adds a normal vector constraint and introduce new variables. Thus, the degrees of freedom of the NLP are augmented by \(α^{(j;k)}, φ^{(j;k)}, α^{rob(j;k)}, λ^{(j;k)}\) and \(l^{(j;k)}\). The strategy for the solution of the NLP involves four steps as shown in Fig. 2 (for a detailed description we refer the reader to Gerhard et al. [2008]). NPSOL [Gill et al., 1986] is used to compute local solutions of the nonlinear programs. The integrator NIXE (NIXE Is eXtrapolated Euler) [Hannemann et al., 2010] has been included in our implementation to compute the derivatives required in a gradient-driven optimization algorithm.

![Fig. 2. Four-steps algorithm using the normal vector approach.](image)

5. ILLUSTRATIVE EXAMPLE

The case study is taken from El-Farra and Christofides [2001], where a cooled continuous stirred tank reactor (CSTR) is considered with an exothermic first order reaction \(A \rightarrow B\). The CSTR model consists of nonlinear state equations resulting from material and energy balances including reaction kinetics:

\[
\dot{c}_A = \frac{q}{V}(c_{Af} - c_A) - k_0 e^{(-E/RT)} c_A,
\]

\[
\dot{T} = \frac{q}{V}(T_f - T) - \frac{\Delta H k_0}{\rho C_p e^{(-E/RT)}} c_A + \frac{Q}{V \rho C_p}.
\]

The cooling rate \(Q\) [\(J/min\)] is the manipulated variable bounded by \(\|Q\| < Q_{max}\), the reactor temperature \(T\) [\(K\)] is the output and controlled variable, and \(c_A\) [\(mol/l\)] is the concentration of species \(A\) in the reactor. The process parameters are given in Table 1. Defining \(x_1 = c_A, x_2 = T,\)
u = Q and y = x_2, it is easy to show that this system has a relative degree r = 1. If the model (14) is formulated in the form (5) with \( e_1 = c_1 - v_1 = T - T_{sp} \), the CLF is proposed as \( V = \frac{1}{2}p_1(e_1)^2 \) and the control is realized by means of the state feedback control law defined by Eqs. (8)-(9), where

\[
L_f V = p_1 e_1 \left[ \frac{q}{V}(T_f - e_1 - \nu_1) - \frac{\Delta H_0}{\rho C_p} e^{(-E/[R(\nu_1 + e_1)]]}_1 \right]
\]

\[
L_y V = p_1 e_1 \left[ \frac{u_{\text{max}}}{\sqrt{V}} \right].
\]

(15)

\( \eta = c_A - c_{A_{sp}} \) is the state of the zero-dynamics which satisfies Assumption 3 and \( u_{\text{max}} = Q_{\text{max}} = 80 [kJ/s] \). The control law comprises one tuning parameter \( p_1 > 0 \) and the control objective is to drive and maintain the temperature of the reactor at the (unstable) steady state given in Table 1 in the presence of a disturbance in the feed temperature and uncertainty in the reaction enthalpy. The time-varying disturbance and uncertainty are considered in the same form as suggested by El-Farra and Christofides [2001]:

\[
T_f = \begin{cases} 
T_f^{(0)}, & t \leq 0, \\
T_f^{(0)} + dT_f \sin(\omega t), & t > 0,
\end{cases}
\]

(16a)

\[
\Delta H = \begin{cases} 
\Delta H^{(0)}, & t \leq 0, \\
\Delta H^{(0)} + d\Delta H \sin(\omega t), & t > 0,
\end{cases}
\]

(16b)

with \( dT_f \in [-0.1T_f^{(0)}, 0.1T_f^{(0)}] \), \( d\Delta H \in [-\frac{1}{2}\Delta H^{(0)}, \frac{1}{2}\Delta H^{(0)}] \) and the frequency set to \( \omega = 3 \text{min}^{-1} \). Further, the initial condition of the reactor temperature \( T(0) \), which corresponds to \( e_1(0) = T(0) - T_{sp} \) in the new coordinates, is not known exactly and may vary in \([e_1_{sp}, -\Delta^L e_1, e_1_{sp} + \Delta^U e_1]\), where \( \Delta^L e_1 > 0 \) and \( \Delta^U e_1 > 0 \) are considered degrees of freedom in (11). This box-type uncertainty is smoothed by Eq. (3) with \( j = 2 \).

The goal is to minimize the objective

\[
\phi = \int_0^{T_f} (y(t) - y_{sp})^2 dt
\]

for which we can guarantee that the bounded controller drives and maintains the reactor temperature at the desired (unstable) steady state and the following constraints hold at any time in the presence of the disturbance and uncertainty in the initial condition:

\[
\begin{align*}
0 < c_A^{\text{max}} - c_A := \psi_1, & \quad c_A^{\text{max}} = c_{A_{sp}} + 0.1 \\
0 < c_A - c_A^{\text{min}} := \psi_2, & \quad c_A^{\text{min}} = c_{A_{sp}} - 0.1 \\
0 < -L_f V(e) + u_{\text{max}} \|L_y V(e)\| := \psi_{CLF}.
\end{align*}
\]

(17)

The first two constraints guarantee an upper and lower bound on the concentration of A, while the third one makes the bounded control (8)-(9) satisfying the scp property to guarantee closed-loop stability. The control design and initial conditions obtained by solving the NLP guarantees that all possible trajectories do not cross the specific constraints (17). As no critical points are known in advance, the optimization is solved without any normal vector constraints. Unknown critical manifolds are detected by numerical integration and the algorithm sketched in Fig. 2 is used. In this example, two critical points are detected and there is one active after the solution algorithm terminates. The robust solution is summarized in Table 2 and illustrated by Figure 3 (solid line). The normal vector constraint is active at the optimum ensuring the minimal distance between the robustness manifold and the manifold of grazing points. The normal vector constraints related to the stability condition have not been considered because the critical manifold is located far off the uncertainty region \( A \) and hence not controlling the design.

### Table 2. Robust solution of the CSTR.

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
<th>parameter</th>
<th>value</th>
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<td>( t_f )</td>
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<td>59.35</td>
<td>( \Delta^U e_1 )</td>
<td>19.21</td>
</tr>
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<td>( dT_{f}^{(2,1)} )</td>
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</tr>
<tr>
<td>( d\Delta H^{(1,1)} )</td>
<td>-22963.6</td>
<td>( d\Delta H^{(2,1)} )</td>
<td>32885.27</td>
</tr>
<tr>
<td>( t^{(1,1)} )</td>
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<td>( t^{(2,1)} )</td>
<td>0.214</td>
</tr>
<tr>
<td>( t^{(1,1)} )</td>
<td>17.62</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Fig. 3.** \( e_1, \eta_1 \) and input \( u \) profiles at the critical points. Solid line is for the robust solution summarized in Table 2. (●) depicts the grazing point reached at \( t^{(1,1)} = 0.369 \text{min} \) (left column).

Figure 3 also depicts the profiles with a typical input-output linearizing controller (IOL) [Hahn et al., 2004] (dashed-dotted line), the bounded controller proposed by El-Farra and Christofides [2001] (dashed line) and a well-tuned PI controller (dotted line). State constraints were not accounted for design of the three compared controllers. For the IOL controller, both state constraints are violated for the transient behavior, an oscillatory behavior around the set-point is obtained for the first critical point (Fig. 3, left column), while for the second one an unstable behavior is present (Fig. 3, right column). In contrast to the controller designed in this work, the bounded controller proposed by El-Farra and Christofides [2001] and the PI-controller successfully stabilize the process at the desired
steady state but both violate the lower state constraint for the second critical point (Fig. 3, right column).

6. CONCLUSIONS

In this work a Lyapunov-based control for feedback linearizable nonlinear systems has been extended to not only handle input saturation but to also consider constraints on the states. These bounded nonlinear controllers enforce stability and asymptotic reference-input tracking. However, as pointed out by El-Farra and Christofides [2003], the synthesis of these kind of controllers does not account explicitly for state constraints. To overcome this shortcoming, the robust design framework of the normal vector method recently introduced by Marquardt and co-workers, has been adapted to also guarantee that state constraints hold at any time in the presence of model uncertainties and external disturbances. The proposed methodology results in a robustly closed-loop stable control law which systematically addresses not only input and state constraints, but also provides the set of admissible initial conditions from where the steady state can be reached without violating constraints, thus avoiding simulation-based common tuning [El-Farra and Christofides, 2003]. The case study presented in this paper shows the feasibility of the suggested approach. In future work, more general and complex systems will be considered to study industrially relevant case studies.

7. ACKNOWLEDGEMENT

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