Improved LQG benchmark for control performance assessment on ARMAX model process

Zhe Liu, Hong-ye Su, Lei Xie, Yong Gu

State Key Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou 310027, Zhejiang, China
(e-mail: longyanlzh88@163.com)

Abstract: Control performance assessment for MPC systems has attracted a lot of interest in recent years. Compared with the typical MVC benchmark, the LQG benchmark gives a pragmatic assessment result. Based on the traditional LQG benchmark, the equigrid approach can improve the regression performance. In addition to the numerical algorithm, the recursive algorithm and the analytical algorithm were introduced to analyze the essential relationship in the LQG benchmark, which can save the computing cost and improve the regression effect. Based on the ARMAX model, the detailed procedures for these algorithms were discussed. In the simulation to a SISO process and a MIMO process, the different types of LQG benchmarks were calculated.

Keywords: control performance assessment; MPC; LQG benchmark; equigrid.

1. INTRODUCTION

In process industry, the Model Predictive Control (MPC) technology is widely used in recent years. The control performance assessment (CPA) for MPC has attracted a lot of interest because of its significant effectiveness on controller optimization, operation maintenance and economic benefit improvement. This research topic was widely developed since the contribution of Harris in 1989 for the Minimum Variance Control (MVC) performance assessment benchmark, which has been a well-known CPA method (Harris, 1989). An improved approach for CPA was the LQG benchmark introduced by Huang etc. (Huang, 1999a, 2003; Kadali, 2002; Patwardhan, 2002a, 2002b), which added the manipulated variables into dynamic performance index to complete the minimum output variance of MVC. Since the convenience to deal with the manipulation constraints, LQG is more pragmatic for industrial implementation. The traditional LQG tradeoff curve was regressed from an asymmetrical distributed discrete points set, leading to unnecessary computing and unsatisfied regression effect. In recent research a numerical equigrid algorithm was introduced to improve the traditional LQG benchmark (Liu, 2011), which was effective for improving regression performance and saving computing cost. The essential of the numerical algorithm is selecting symmetrically distributed points from existed asymmetrical points set, but not generates equigrid points directly, therefore the total computing cost is increased. In this paper the recursive algorithm and the analytical algorithm were introduced to obtain the equigrid LQG benchmark for control performance assessment, which can calculate the equigrid discrete LQG points directly, thus both of the regression effect and the computing cost can be improved. The analysis bases on ARMAX model, for both single-input single-output (SISO) process and multi-input multi-output (MIMO) process.

2. CONTROL PERFORMANCE ASSESSMENT

The most typical strategy for control performance assessment is the MVC benchmark. The control objective of MVC is to minimize the process variance. Consider a transfer function described process with time delay $r$

$$y(k) = z^{-r}G_p u(k) + G_d \alpha(k)$$

(1)

where $G_p$ and $G_d$ are the process model and the disturbance model respectively, $\alpha(k)$ is a white noise sequence. The process output based on MVC is decided by the disturbance model $G_d$, time delay $r$ and white noise $\alpha(k)$ as

$$y(k) = F(z^{-r}, k) \alpha(k)$$

(2)

where $F(z^{-r}, k)$ is the sum of the first $r$ terms of impulse response sequence inspired to disturbance model $G_d$ (Åström, 1970; Harris, 1989; Huang, 1999b, 2002). The MVC benchmark represents the ideal minimum variance of process output. Then comparing this minimum variance with the current process output variance, the Harris Index can be described as

$$\eta_{Harris} = \frac{\sigma^2_{mv}}{\sigma^2_y}$$

(3)

Consequently the performance margin can be indicated. However, in MVC benchmark the manipulation constraints are not considered, so this benchmark can just provide an instruction for controller improvement.

In practical process industry the constraints for manipulated variables and controlled variables are strictly limited to ensure operation safety and product quality,
therefore taking the manipulation constraints into control performance assessment is more pragmatic in industrial implementation. The LQG benchmark takes the quadratic dynamic index as the performance index, which is

\[ J(\lambda) = E[Y^T W Y] + \lambda E[U^T R U] \]  

(4)

where \( W \) and \( R \) are the output weighting matrix and the manipulation weighting matrix respectively. By varying the manipulation weighting \( \lambda \) and solving the LQG problem respectively, a series of LQG optimal discrete points can be calculated. Then by regressing these points, the obtained LQG tradeoff curve as shown in Fig. 1 represents the relationship between the manipulation variance and the output variance (Huang, 2003). The MVC benchmark is the special minimum output variance point on the LQG tradeoff curve. Ideally, the operation situation will just locate on the curve. However, in industry the disturbance is always inevitable, the practical operation can only appear above the curve.

Fig. 1 Description of LQG tradeoff curve

Xu etc. analyzed the control loop economic performance by constructing a nested steady state optimization problem (Xu, 2007). The outer optimization aimed to solve the steady state performance index, while the corresponding variability and constraints were tuned by the inner optimization. The MPC economic performance was evaluated by solving the benefit potentials through either variability reduction of crucial output variables or constraints optimization and tuning. This method should be realized recursively, however, the recursive procedure cannot describe the essential relationship between the manipulation variance and the output variance explicitly. Integrating the steady state optimization and the LQG benchmark, the above nested optimization in Xu’s work can be reconstructed by the LQG optimization problem (Zhao, 2009b). The traditional LQG benchmark is calculated by varying the manipulation weighting \( \lambda \) with equal interval, consequently the obtained LQG discrete points are distributed asymmetrically, leading to unnecessary computing cost in densely distributed part but unsatisfied regression effect in sparsely distributed part. Recently a numerical algorithm to calculate the equigrid LQG benchmark for MPC economic performance assessment was introduced (Liu, 2011). The essential of numerical algorithm is selecting the symmetrically distributed points from the existed asymmetric points set, but not generating the equigrid points directly, therefore the regression effect was improved but the total computing cost was increased. Comparing with this numerical algorithm, in this section we will introduce other two approaches of recursive algorithm and analytical algorithm to calculate the equigrid LQG benchmark based on ARMAX model process.

3.1 ARMAX model

Consider a SISO process, when it operates around a particular setpoint and after linearization, it can always be described as

\[ y(k) = z^{-1} B(z^{-1}) u(k) + \frac{C(z^{-1})}{A(z^{-1})} \alpha(k) \]  

(6)

where

\[ A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_m z^{-ma} \]

\[ B(z^{-1}) = b_1 z^{-1} + \cdots + b_nh z^{-nh} \]

\[ C(z^{-1}) = 1 + c_1 z^{-1} + \cdots + c_n e^{-nc} \]

Equation (6) is the transfer function form as equation (1) (Camacho, 2003; Huang, 2008). The difference in \( B(z^{-1}) \) is it has no constant term because of the zero-order-hold in sampling, resulting in one sample delay. One characteristic of ARMAX model is the process model and the disturbance model have common denominators. The ARMAX model is useful for designing Kalman filter and Generalized Predictive Control (GPC). When transferred to the state space model form,
it can be described as
\begin{align}
x(k+1) &= Ax(k) + Bu(k) + Ha(k) \\
y(k) &= Cx(k) + a(k)
\end{align}
(7)
The \( A \) matrix is consisted of the poles of the transfer functions, \( H \) is the Kalman gain matrix. \( D \) matrix in output equation is commonly omitted, accordant with \( inG_p \) the numerator order is lower than the denominator order, which owes to zero-order-hold in the sampling. In this paper the LQG benchmark is calculated using the state space described ARMAX model, for both SISO process and MIMO process.

3.2 Three types of equigrid LQG algorithms

A. Numerical algorithm

The method to calculate the traditional LQG benchmark based on the ARMAX model can be found in Zhao’s work (Zhao, 2009b). When the asymmetric discrete points are obtained, the equigrid LQG benchmark can be calculated by the numerical algorithm, following the procedures as (Liu, 2011):

a. Set the normal distance \( d \) according to the practical process requirement;
b. Select the initial LQG benchmark discrete point \((\sigma_{1_s}, \sigma_{2_s})\);
c. Vary \( \lambda \), solve the LQG problem respectively, select the manipulation variance and output variance (standard deviation) relationship points following the normal distance orderly:
\begin{align}
(\sigma_{1_{s+1}} - \sigma_{1_s})^2 + (\sigma_{2_{s+1}} - \sigma_{2_s})^2 &= d^2 \\
\sigma_{1_s} |_{k=0} &= \sigma_{1_s} \quad \sigma_{2_s} |_{k=0} = \sigma_{2_s}
\end{align}
(8)
\[\sigma_{U_s} = \sqrt{Var(U_s)}\]
\[\sigma_{Y_s} = \sqrt{Var(Y_s)}\]

The corresponding \( \lambda \) and optimal control law can be obtained;
d. Regress the equigrid LQG benchmark from the discrete points calculated by the optimized \( \lambda \), the relationship equation can be obtained as
\[\sigma_{Y} = f_{eq}(\sigma_{U})\]
(9)

This numerical algorithm is generated from the traditional LQG benchmark which is available in MIMO process, therefore the numerical algorithm is suitable for both SISO process and MIMO process.

B. Recursive algorithm

Consider a state space described ARMAX model
\begin{align}
X(k+1) &= AX(k) + BU(k) + HA(k) \\
Y(k) &= CX(k) + a(k)
\end{align}
(10)
The quadratic optimal state feedback control law is (Hendricks, 2008)
\[U(k) = -L\hat{X}(k)\]
(11)
where
\[L = (B^T SB + \lambda R)^{-1}(B^T SA + N^T)\]
(12)
is the state feedback gain matrix, \( \hat{X}(k) \) is the estimated state, \( S \) can be obtained by solving the Riccati equation \(A^T SA - (A^T SB + N)(B^T SB + \lambda R)^{-1}(B^T SA + N^T) + Q = 0\)
(13)
where \( Q = C^T WC \) is the state weighting matrix in the quadratic dynamic index, \( N \) is the state and manipulation weighting matrix, generally \( N = 0 \). In the recursive algorithm, the manipulation weighting \( \lambda R \) is unknown, which should be calculated according to the specified equal distance \( d \). In LQG benchmark with two dimensional coordinates \((\sigma_{1_s}, \sigma_{2_s})\), the distance between two adjacent points is
\[d = \sqrt{[\sigma^2(Y_{s+1}) - \sigma^2(Y_s)]^2 + [\sigma^2(U_{s+1}) - \sigma^2(U_s)]^2}\]
(14)
where
\[\sigma^2(Y_s) = \sigma^2(CX_s) = Var(Y_s)\]
\[\sigma^2(U_s) = \sigma^2(-L\hat{X}_s) = Var(U_s)\]
The \( \sigma^2(X_s) \) and \( \sigma^2(\hat{X}_s) \) can be calculated by solving the Lyapunov equation as in the traditional LQG benchmark (Zhao, 2009b, Liu, 2011). Then the relationship between \( L_s \) and \( L_{s+1} \) is constructed. Substituting the calculated \( L_{s+1} \) into equation (12), and integrating equation (13), the expected unknown parameter \( \lambda R \) of the next step can be calculated. Following the above steps recursively, the equigrid LQG optimal discrete points set can be obtained by the recursive algorithm. Then regressing these points, the relationship between the manipulation variance and the output variance can be described as the optimal LQG tradeoff curve equation.

From the above discussion, the recursive algorithm is available for both SISO process and MIMO process. The difference between the numerical algorithm and the recursive algorithm is the former one operates forward, from the manipulation weighting \( \lambda R \) to LQG points, while the latter one operates backward, through the specified equigrid points to calculate the manipulation weighting.

The procedures of the recursive algorithm can be described as:

a. Give the initial parameters of \( S_0 \) and \( \lambda R_0 \);
b. Solve the Riccati matrix equation and the state feedback control law. The unknown parameters are \( S_i \) and \( \lambda R_i \), substitute them with the initial values, then the parameter \( L_i \) connected with the following steps can be calculated;
c. Solve the manipulation variance \( Var(U_i) \) and the output variance \( Var(Y_i) \). Their connections are the state variance \( Var(X_i) \) and the state estimation variance \( Var(\hat{X}_i) \) with the common parameter \( L_i \). \( L_i \) can be obtained by solving the Lyapunov matrix equation. The inner relationship between \( Var(U_i) \) and \( Var(Y_i) \) can be indicated;
d. According to the normal distance between adjacent two points as equation (14), which is described by the manipulation variance and the output variance, and following step b, the relationship between two....
steps of $L$ and $L_{i+1}$ is indicated. $\text{Var}(X_i)$, $\text{Var}(\hat{X}_i)$, $\text{Var}(U_i)$, $\text{Var}(Y_i)$ can be calculated. Then the next step of $S_{i+1}$ and $(\lambda R)_{i+1}$ can be obtained; e. Go back to step b to repeat the procedures recursively.

### C. Analytical algorithm

The above two algorithms more or less depend on the Matlab optimization toolbox, to solve the key roles of Riccati equation and Lyapunov equation for LQG benchmark, therefore the essential relationships between the manipulation weighting and the LQG tradeoff discrete points are not clear. The analytical algorithm can identify their relationships explicitly. However, this algorithm is just available for first-order SISO process described by ARMAX model. This restriction is caused by the complexity of solving the matrix equations. Although the Riccati equation and the Lyapunov equation can be solved by the Linear Matrix Inequality (LMI) method and so on (Lancaster, 1985), however, these methods commonly give the numerical solutions, which are not suitable for the analytical algorithm to obtain the equigrid LQG benchmark.

Consider a SISO process described by the transfer function as equation (6). For it is an ARMAX model, the denominators of the process and disturbance models are the same. In addition, for the process is first-order, the denominators have only one pole, when transformed to state space model as equation (7), the system matrix $A$ has only one element, therefore the process can be described as

\[ x(k+1) = Ax(k) + Bu(k) + Ha(k) \]  
\[ y(k) = Cx(k) + a(k) \]

where the parameters are all scalars, the to be solved Riccati matrix equation for LQG benchmark is simplified to an algebraic equation. Then the state estimation and the quadratic optimal state feedback control law is

\[ \hat{x}(k+1) = (a - hc - bL)\hat{x}(k) + hy(k) \]  
\[ u(k) = -L\hat{x}(k) \]

where

\[ L = \left( b^2 s + r \right)^{-1} bsa \]  
\[ \hat{x}(k) \text{ is the state estimation}, \quad s \text{ can be obtained by solving the algebraic Riccati equation} \]

\[ b^2 s^2 - (a^2 r + b^2 q - r)s - qr = 0 \]

where $q$ and $r$ are the state weighting and the manipulation weighting respectively. Integrating the system equation and the Kalman estimator as

\[ \begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} a & -bL \\ hc & a-hc-bL \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} h \\ h \end{bmatrix} \alpha(k) \]

which can be described as

\[ \hat{X}(k+1) = A_{ij} \hat{X}(k) + B_{ij} \alpha(k) \]

Solve the covariance for equation (20), we can obtain

\[ \begin{bmatrix} \text{Var}(x) & \text{Cov}(x\hat{x}) \\ \text{Cov}(x\hat{x}) & \text{Var}(\hat{x}) \end{bmatrix} \]

simplified as

\[ \text{Var}(\hat{x}) = A_{ij} \text{Var}(\hat{x}) A_{ij}^T + B_{ij} \text{Var}(\alpha) B_{ij}^T \]  
(23)

where $\text{Cov}$ and $\text{Var}$ represents the covariance and the variance respectively. This is the famous Lyapunov matrix equation. In recursive algorithm, this matrix equation is solved by Matlab optimization toolbox, but here we will get the explicit solution, so $\text{Var}(x)$ and $\text{Var}(\hat{x})$ should be solved analytically. To realize that, the notion of Kronecker product and vec-function, the proposition of their relationship will be adopted (Lancaster, 1985). Equation (23) can be reconstructed as

\[ (A_{ij} \otimes A_{ij} - I) \text{vec} \text{Var}(\hat{x}) = -(B_{ij} \otimes B_{ij}) \text{vec} \text{Var}(\alpha) \]

which is

\[ \begin{bmatrix} a^2 - 1 & -2abL & b^2 L^2 \\ ahc & a^2 - \alpha c - bL - b^2 c L & -1 - bL(a - \alpha c - bL) \\ h^2 c^2 & 2hc(a - \alpha c - bL) & (a - \alpha c - bL)^2 - 1 \end{bmatrix} \]

Then $\text{Var}(x)$ and $\text{Var}(\hat{x})$ can be solved. Thus the process manipulation variance and output variance can be obtained respectively:

\[ \text{Var}(u) = L^2 \text{Var}(\hat{x}) \]
\[ \text{Var}(y) = c^2 \text{Var}(x) + \text{Var}(\alpha) \]

The same as the recursive algorithm, when the specified normal distance $d$ and the initial point $(\sigma_{u0}^2, \sigma_{y0}^2)$ are given, the adjacent points can be calculated one by one, by solving the quadratic optimal control backwards, and the corresponding manipulation weighting $r$ can be obtained.

Regressing the LQG optimal discrete points calculated by the analytical algorithm, the LQG tradeoff curve equation can be obtained as the important condition of steady state optimization. The meaning of this analytical algorithm is not only for equalizing the LQG benchmark, but also in analyzing the explicit relationship between the manipulation weighting and the LQG benchmark. In this algorithm the inner principles of LQG benchmark are decomposed to basic algebraic operation and do not depend on the Matlab toolbox at all, which can be transferred to other computer languages conveniently.

The procedures of analytical algorithm are mostly the same as the recursive algorithm, the difference is in solving the matrix equations.

### 3.3 Analysis of the different algorithms

**Remark 1** In the above three equigrid algorithms, the numerical algorithm and the recursive algorithm are available for both SISO process and MIMO process.
When dealing with MIMO process, the manipulation variance and the output variance should be evaluated by the trace of their covariance matrix respectively, the manipulation weighting to be adjusted aims for coordinating the sum of the controlled variables variances and the sum of the manipulated variables variances as

\[ J = \text{trace}(E(Y^TWY)) + \lambda \text{trace}(U^T RU) \]  
(28)

which can also be described as

\[ J = \sum_{j=1}^{p} w_j \text{Var}_j + \lambda \sum_{i=1}^{m} r_i \text{Var}_i \]  
(29)

where \( w_j \) and \( r_i \) are the diagonal elements of \( W \) and \( R \) respectively. To obtain the two dimensional equigrid LQG benchmark, we always fix \( w_j \) and \( r_i \), only the weighting \( \lambda \) is adjusted. This is a simplified strategy to deal with MIMO dynamic performance index, since the free degrees among the manipulated variables and among the controlled variables are artificially restricted. In this situation, we suppose the importance matching among the manipulation variables and among the controlled variables have been preset. Actually, for a MIMO process with \( m \) inputs and \( p \) outputs the free degree is \( m + p - 1 \). When adjusting the \( m + p - 1 \) weightings, we can get a \( m + p \) dimensional Pareto optimal surface. Also using the regression method, a corresponding nonlinear equation with \( m + p \) variables can be obtained as the LQG constraint condition for steady state optimization. However, for the Pareto optimal surface the equigrid is hardly to be realized.

**Remark 2** The analytical algorithm is restricted in first-order ARMAX SISO process. Although solving the LQG problem is available in MIMO process as the other two algorithms, the matrix equations need not be solved explicitly since the Matlab solver can realize that. The only reason for this restriction is, for a general Riccati matrix equation, there is no effective analytical method to solve that except the numerical method such as LMI. So that the parameters to be dealt with must be scalars, the Riccati matrix equation can be simplified to an algebraic equation which can be solved analytically. When solving the covariance to equation (20), the variables \( x \) and \( z \) are correlated thus the term \( Cov(xz) \) exists, so that the Lyapunov matrix equation (23) can not be simplified to an algebraic equation. However, this special matrix equation can be solved by the Kroncker product and the vec-function analytically. For the elements are scalars, the computing complexity can be reduced to some extent.

**Remark 3** In the analytical algorithm, actually the analytical relationships among \( \text{Var}(u) \), \( \text{Var}(y) \) and \( L \) have been solved, logically, the intermediate variable \( L \) could be eliminated so that the direct function \( \text{Var}(y) = f(\text{Var}(u)) \) should be built as the benchmark for steady state optimization, and the LQG tradeoff curve need not be obtained by the regression method. However, since the complexity in solving the Riccati equation and the Lyapunov equation, even the Riccati equation has been simplified to an algebraic equation, it is difficult to be realized.

### 4. SIMULATION ANALYSIS

In this section we will calculate the equigrid LQG benchmark using the above three types of algorithms. Consider a SISO process with ARMAX model

\[ y(k) = G_p(z^{-1})u(k) + G_z(z^{-1})\alpha(k) \]

\[ = \frac{0.2155z^{-1}}{1-0.9918z^{-1}}u(k) + \frac{1-0.13z^{-1}}{1-0.9918z^{-1}}\alpha(k) \]  
(30)

where \( \alpha(t) \) is the Gaussian white noise, its standard deviation is 0.1. By solving the LQG problem according to the above procedures, the four types of LQG benchmarks, which are calculated by the traditional algorithm, the numerical equigrid algorithm, the recursive equigrid algorithm, and the analytical equigrid algorithm are shown in Fig.2 respectively.

From the figure we can see the LQG benchmarks calculated by the recursive algorithm and the analytical algorithm are overlapped. The reason is the procedures of these two types of algorithms are almost the same, the difference is in solving the matrix equations. In the former one, the matrix equations are solved by the Matlab optimization toolbox, its essential is numerical optimization. While in the latter one the Riccati equation is simplified to an algebraic equation, and the Lyapunov equation is solved analytically, therefore the essential relationship between the manipulated variance and the output variance is indicated.

Also consider a MIMO process with two manipulated variables and two controlled variables, the transfer function with ARMAX model is

\[ Y(k) = G_{pu}(z^{-1})U(k) + G_{pu}(z^{-1})\alpha(k) \]  
(31)

where

\[ G_{pu}(z^{-1}) = \begin{bmatrix} 0.9653z^{-1} - 0.1035z^{-2} & 1.072z^{-1} - 0.13z^{-2} \\ 1-0.1975z^{-1} + 0.0195z^{-2} & 1-0.1975z^{-1} + 0.0195z^{-2} \\ 0.0941z^{-1} - 0.0007z^{-2} & 0.1193z^{-1} - 0.0036z^{-2} \\ 1-0.1975z^{-1} + 0.0195z^{-2} & 1-0.1975z^{-1} + 0.0195z^{-2} \end{bmatrix} \]

\[ G_{pu}(z^{-1}) = \begin{bmatrix} 1+1.727z^{-1} - 0.025z^{-2} & 2.473z^{-1} - 0.054z^{-2} \\ 1-0.1975z^{-1} + 0.0195z^{-2} & 1-0.1975z^{-1} + 0.0195z^{-2} \\ 1-0.1975z^{-1} + 0.0195z^{-2} & 1-0.1975z^{-1} + 0.0195z^{-2} \end{bmatrix} \]

For this process we use the recursive algorithm, the MIMO equigrid LQG benchmark is calculated as shown in Fig. 3.
5. CONCLUSIONS

Performance assessment for MPC control systems is significant because it can indicate whether the controller is operating in a healthy situation. As the typical assessment strategy, MVC benchmark can give the idealistic optimal output but does not consider the manipulation constraints. The LQG benchmark takes the manipulation effect into the quadratic dynamic index, which is a pragmatic assessment strategy. However, the traditional LQG benchmark is obtained from a set of asymmetric distributed points set, leading to unnecessary computing and unsatisfied regression result. In previous work we have showed a numerical equi-grid LQG benchmark to tackle this problem. This algorithm can not indicate the inner relationship between the manipulation variance and the output variance. In this paper we introduced the recursive algorithm and the analytical algorithm to calculate the equi-grid LQG benchmark, and analyzed the essential characteristic of the LQG performance assessment based on the ARMAX model process. For these three types of equi-grid LQG benchmarks, the former two are available for both SISO process and MIMO process. The third one is only suitable for the first-order SISO process since the complexity in solving the matrix equations. However, this analytical algorithm explicitly shows the essential relationships among the LQG parameters and the analytical LQG benchmark. In simulation analysis we calculated the LQG benchmarks using the above algorithms for a SISO process and a MIMO process.

REFERENCES