Extremum seeking control of the CANON process - existence of sub-optimal stationary solutions

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Abstract: The paper considers extremum seeking control (ESC) for on-line optimization of the CANON process, a new and potentially highly effective process for ammonium removal from wastewater. For gradient estimation we employ the classical method based on periodic excitation. From simulations we find that the ESC scheme can lock onto sub-optimal stationary solutions, far removed from the optimal solution, and that the ESC may have multiple stationary solutions for given controller parameters. The cause of this is investigated through analysis of a general dynamic model. Based on the analysis, it is shown that for systems for which the optimum corresponds to the input-output transfer-function having a transmission zero at the origin, there will in general exist a number of stationary solutions to the ESC with periodic excitation. The solutions are characterized by the phase lag of the system, rather than a zero gradient of the objective function, and are hence in general not related to the optimality conditions. For systems that can be described by Hammerstein or Wiener models, as typically considered in ESC, the solution will in general correspond to the zero gradient condition fulfilled at the optimum. As shown, the CANON process can not be described by Hammerstein or Wiener models, and this then explains the observed existence of sub-optimal stationary solutions.

Keywords: Real time optimization, extremum seeking, periodic excitation, adaptive control, wastewater treatment plant

1. INTRODUCTION

Extremum seeking control (ESC) is a real-time optimization method used to achieve and maintain optimal operating conditions even for complex processes during run-time. The main advantages of ESC is that it is based on a strict feedback control scheme, thereby suppressing the effects of uncertainty and disturbances, and that it can be used without any explicit process model, e.g., Sternby [1979], Dochain et al. [2011]. Both these properties make ESC particularly suitable for the Completely Autotrophic Nitrogen removal Over Nitrite (CANON) process considered in this paper.

The CANON process is essentially a biochemical reactor used to remove ammonium from concentrated wastewater streams. The process has the potential to significantly reduce the costs for wastewater treatment plants (WWTP) since it reduces the need for aeration and chemical additives as compared to more conventional processes such as the nitrification-denitrification process [Khin and Annachhatre, 2004]. However, the CANON process is challenging from a control perspective. First, it has a sharp optimum with respect to dissolved oxygen (DO), the main control variable, with large losses for relatively small deviations from the optimal DO. Furthermore, the complexity of the process makes it challenging to derive reasonably accurate predictive models, and off-line predictions of the optimal conditions are therefore of little use. Finally, the fact that the feed to WWTPs typically is highly variable implies that the optimum will vary significantly over time.

The ESC scheme we consider in this paper employs sinusoidal perturbations for gradient estimation. ESC with periodic excitation is a classic control method that dates back to the beginning of the 20th century. The main developments took place in the 50’s and 60’s, but then with a focus on systems that could be described by static or Hammerstein/Wiener models [Sternby, 1979]. More recently, Krstić and Wang [2000] consider general dynamic models and derive local stability results based on averaging and singular perturbations. Their results show that the ESC with periodic excitation under certain conditions will converge towards a region around the optimum, with an error proportional to the square of the amplitude of the excitation signal. A key assumption in their proof is that the excitation signal is sufficiently slow compared to the process dynamics. This is not practical for slow processes, such as the CANON process, as it would imply that it would take months or years to arrive at the optimum and then only in the absence of disturbances. The impact of using higher excitation frequencies is considered in Chiova et al. [2007] and it is shown that the local error also will be proportional to the square of the excitation frequency. Similar to Krstić and Wang [2000], they base their results on local approximations around the optimal solution.
When employing ESC with periodic excitation for the CANON process in this paper, simulations reveal that the system can lock on to operating points far removed from the optimum. Furthermore, depending on the initial conditions, the algorithm will converge to different stationary solutions. Thus, a local analysis around the optimal point is less relevant and we therefore aim at deriving the exact stationary solutions of the ESC applied to a general dynamic system. We show that the stationary solutions can be characterized by simple conditions on the gain and/or phase of the locally linearized system. We relate these results to the dynamic properties of systems that can be described by Hammerstein/Wiener models and those that require more general dynamic models, respectively, and show that stationary solutions far from the optimum as well as multiple stationary solutions will only exist for processes that cannot be described by Hammerstein/Wiener models, such as the CANON process.

We start the paper by briefly describing the ESC algorithm with periodic excitation. We then present simulation results for the CANON process with ESC, revealing the existence of sub-optimal solutions as well as multiple stationary solutions. To explain these results, we proceed to derive expressions for the stationary solutions of the algorithm in terms of local frequency responses. We then relate the characteristics of the stationary solutions to previous results on the dynamics of systems with steady-state input multiplicity. To show that the ESC algorithm may converge to different stationary solutions depending on the initial conditions, we derive simple results on stability to show that several solutions may be locally stable for given controller parameters. Finally, we present a simple example for which analytical results may be derived and for which we show that also when the stationary solution is unique may it in principle be arbitrarily far removed from the optimum. Before concluding the paper, we return to the CANON process to confirm that the observed results can be explained by the results derived in this paper. We also briefly discuss possible remedies to the ESC with periodic excitation that will ensure convergence to a solution in the close vicinity of the optimal solution.

2. EXTREMUM SEEKING CONTROL WITH PERIODIC EXCITATION

The principle idea behind extremum seeking control is to use gradient feedback to bring a process to the maximum or minimum, corresponding to the zero gradient point, of the input-output map in which the output represents the objective function and the input is the main control variable. There exists several approaches to ESC, differing mainly in the way the local gradient is obtained, e.g., based on sliding mode [Yu and Ozguner, 2002] or numerical optimization methods [Zhang and Ordonez, 2011]. In this paper we consider the classical and much studied variant based on sinusoidal perturbations [Sternby, 1979, Ariyur and Krstic, 2003, Dochain et al., 2011]. The corresponding ESC loop is outlined in Fig. 1, which also defines the various signals of the scheme. The main motivation for choosing this particular scheme is that it is model independent and also relatively simple to implement.

We consider the process to be described by a general set of nonlinear differential equations combined with a nonlinear state-to-output map. It is assumed that the system is asymptotically stable for all inputs \( \theta \) and can be described by a state space model of the form

\[
\dot{x} = f(x, \theta) \\
y = h(x)
\]

(1)

The assumption of asymptotic stability can easily be relaxed by introducing a stabilizing feedback control law. The functions \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R} \) are assumed to be sufficiently smooth such that all necessary derivatives exist. Furthermore, we assume that there exist a sufficiently smooth function \( l : \mathbb{R} \to \mathbb{R}^n \) such that

\[ 0 = f(x, \theta) \]

(2)

if and only if

\[ x = l(\theta) \]

The assumptions above implies that the stationary solutions of (1) are parametrized by \( \theta \) and that the composite function

\[ h \circ l : \mathbb{R} \to \mathbb{R} \]

(3)

exists and is sufficiently smooth. The function (3) is the steady-state map between \( \theta \) and \( y \), and as such, it is the function we want to optimize by employing ESC. We will assume that (3) has an extremum which is either a maximum or a minimum.

The addition of the sinusoid in Fig. 1 is motivated by the fact that the product of the sinusoid itself and the system response to the sinusoid will have a DC component which is proportional to the local gradient of the input-output map \( h \circ l \), provided the system acts as a static map. The purpose of the high-pass filter \( F_H \) is to remove the DC component from the process response, while the low-pass filter \( F_L \) serves to retain only the DC component of the predicted gradient.

We next apply the outlined scheme to the CANON process.

3. ESC OF THE CANON PROCESS

The CANON process is an efficient biological process used to remove ammonium from ammonium rich wastewater streams. If utilized correctly, it has the potential to substantially lower the cost of nitrogen removal in wastewater treatment plants [Khin and Annachhatre, 2004, Zhu et al., 2008]. The cost reduction is mainly due to lower oxygen

![Fig. 1. Structure of the ESC system.](image-url)
demands, which reduce the aeration costs, and the fact that there is no need for an external carbon source. However, a sensitivity study of the CANON process indicates that the process exhibit a sharp optimum in concentration of dissolved dinitrogen gas $N_2$, the desired end product, with respect to the DO concentration [Hao et al., 2002]. With only small deviations in the DO concentration from the optimum, the process can become highly ineffective instead. Thus, to realize the benefits of the process, it is of vital importance to keep the DO concentration close to the optimal value.

We develop a simplified model of the CANON process in order to test the viability of employing the ESC to locate the optimum on-line. The biofilm of the reactor is modelled as a series of discrete layers with fixed dimensions. The transport between the layers is driven by diffusive flow and the reaction kinetics within each layer are those given in Hao et al. [2002].

To obtain a model of the complete process, the biofilm model is finally combined with a model of a mixed tank with constant inflow. We here employ 10 layers in the biofilm, yielding a model with a total of 90 ordinary differential equations.

Note that the model we use is not intended to yield precise quantitative predictions, but rather describe the qualitative aspects of the relationship between DO input and $N_2$ in the outflow. The input-output map, as predicted by the model, is shown in Fig. 2. As can be seen, the optimal conversion of $NH_3$ into $N_2$ is achieved for a DO concentration around 0.38 [g/l].

We next repeat the simulation, now with an initial DO content of 0.1 [g/l] closer to the optimum. Indeed, one could in this case directly conclude that the scheme does not work since the stationary solution has a lower $N_2$ concentration than the starting point. However, if one instead had started at a lower DO concentration then it would be hard to say if the ESC solution was close to optimum or not.

Fig. 3. Simulation of ESC control of CANON process with initial DO content of 0.1 [g/l].

Fig. 4. Steady state map between DO [g/l] and $N_2$ [g/l] for the simplified CANON-model. Stable stationary solutions of the ESC-loop are marked with a $\ast$, unstable stationary solutions are marked with a $\circ$.

There are several interesting observations to make from the simulations above. The first is that there exists stationary solutions to the ESC scheme which are far removed from the optimum, so far that results based on local analysis around the optimum does not apply. Second, the ESC may have multiple attracting stationary solutions. To better understand these results, we next consider determination of the stationary solutions of the ESC scheme applied to a general nonlinear dynamical system.

4. STATIONARY SOLUTIONS OF THE ESC SCHEME

Consider the system given in (1) controlled by an ESC-loop as shown in Fig. 1. We are interested in determining the stationary solutions for which the signal $\xi(t) = 0$, corresponding to $\dot{\theta} = \theta$ being constant. With $\xi = 0$, the input to the process becomes

$$\theta(t) = \dot{\theta} + a \sin(\omega t)$$

This input will in turn yield a stationary response in the process output $y(t)$ which is composed of a DC component, resulting from $\dot{\theta}$, combined with the frequency response for $a \sin(\omega t)$. If we assume that the amplitude of the sinusoid
a is small, then the frequency response can be described by the transfer-function \( G(s) \) obtained by linearizing the process around the steady-state corresponding to \( \theta = \theta^* \), i.e.,

\[
y(t) = h \circ (l(\theta) + G(\omega) \alpha \sin(\omega t + \varphi_G))
\]

The presence of the high-pass filter \( F_H \) will effectively remove the DC component of \( y \), resulting in the response

\[
y(t) - \eta(t) = [G(\omega)] [F_H(\omega) \alpha \sin(\omega t + \varphi)]
\]

where \( \varphi = \arg(G(\omega)) + \arg(F_H(\omega)) \) is the combined phase lag of the system and the high-pass filter. The signal \( y - \eta \) is “demodulated” by multiplication with \( \alpha \sin(\omega t) \) which yields

\[
(y(t) - \eta(t)) \alpha \sin(\omega t) = [G(\omega)] [F_H(\omega)] [a^2 \sin(\omega t + \varphi) \sin(\omega t)].
\]

The trigonometric identity

\[
\sin(a) \sin(b) = \frac{1}{2} (\cos(a - b) - \cos(a + b))
\]

yields

\[
(y(t) - \eta(t)) \alpha \sin(\omega t) = \frac{a^2}{2} [G(\omega)] [F_H(\omega)] (\cos(\omega t + \varphi) - \cos(2\omega t + \varphi)).
\]

Note that the demodulated signal consists of a DC component and a sinusoidal component with twice the excitation frequency. Low-pass filtering the demodulated signal yields

\[
\xi = \frac{a^2}{2} |F_L(0)| [G(\omega)] |F_H(\omega)| \cos(\varphi) - \frac{a^2}{2} |F_L(\omega)| [G(\omega)] |F_H(\omega)| \cos(2\omega t + \varphi + \arg(F_L(\omega)))
\]

The low pass filter is assumed to effectively filter out the high frequency component, i.e.,

\[
|F_L(\omega)| = 0,
\]

which yields

\[
\xi = \frac{a^2}{2} |F_L(0)| [G(\omega)] |F_H(\omega)| \cos(\varphi).
\]

Since \( \xi = 0 \) is required to yield a constant \( \dot{\theta} \), it follows that we for stationarity require

\[
\frac{a^2}{2} |F_L(0)| [G(\omega)] |F_H(\omega)| \cos(\varphi) = 0
\]

Clearly, \( \frac{a^2}{2} |F_L(0)| |F_H(\omega)| > 0 \), so the only possibility for (6) to be true is if either

\[
|G(\omega)| = 0
\]

or

\[
\cos(\varphi) = 0 \Rightarrow \varphi = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \ldots
\]

From the analysis above we draw the conclusion that the stationary solutions are characterized either by the system being output invariant with \( |G(\omega)| = 0 \) or the phase lag fulfilling (8). Since these criteria can be fulfilled irrespectively of the optimality, conditions there can exist stationary solutions unrelated to the optimum. Furthermore, the phase lag \( \varphi \) can in principle vary with the input \( \theta \) in such a way that (8) can be fulfilled for several different stationary points. Some systems could therefore have multiple stationary solutions for a single excitation frequency.

The derivation above is not strict since it depends on (4), a condition that few filters actually fulfill. However, if we consider the full ESC-loop, it is clear that high-frequency components will be attenuated not only by the low-pass filter but also by the integrator and typically by the process itself as well. Furthermore, if we consider the average of \( \xi \) over one period of time, \( T = \pi/\omega \), we get

\[
\frac{1}{T} \int_0^T \xi dt = \frac{1}{T} \frac{a^2}{2} |G(\omega)| |F_H(\omega)| \int_0^T (|F_L(0)| \cos(\varphi) - |F_L(\omega)| \cos(2\omega t + \varphi + \arg(F_L(\omega)))) dt = \frac{1}{T} \frac{a^2}{2} |G(\omega)| |F_H(\omega)| \left( \int_0^T |F_L(0)| \cos(\varphi) dt - \int_0^T |F_L(\omega)| \cos(2\omega t + \varphi + \arg(F_L(\omega))) dt \right)
\]

\[
= \frac{a^2}{2} |F_L(0)| [G(\omega)] |F_H(\omega)| \cos(\varphi).
\]

This is exactly what we get from assuming (4), i.e., by making the assumption we study the average behaviour of the system which makes sense since we are interested in stationary solutions. If the assumption (4) is not made, then \( \dot{\theta} \) would not be constant for stationary solutions and we would instead have to consider limit-cycles of small amplitude.

### 4.1 Relation of stationary solutions to optimality

From the above we find that stationary solutions of the ESC scheme either have \( |G(\omega)| = 0 \) or a phase lag \( \varphi = \frac{\pi}{2} + n\pi \). To better understand how this relates to properties of a general dynamic system at the optimum, it is interesting to relate it to properties of systems with steady-state input multiplicity. Steady state input multiplicity is a property that all systems viable for ESC exhibits, i.e., there are multiple inputs yielding the same output at steady-state due to the existence of a maximum or minimum. Such systems have been shown to possess certain dynamical properties [Jacobsen, 1994] that are relevant for the stationary solutions of the ESC as derived above.

Let

\[
G(s) = K \frac{b_0 + b_1 s + \cdots + b_m s^n}{a_0 + a_1 s + \cdots + a_n s^n}, \quad n \geq m
\]

be the transfer-function of (1) linearized about a steady-state solution \( x = l(\dot{\theta}) \). Then the stationary gain is given by

\[
G(0) = K \frac{b_0}{a_0} = (h \circ l)'(\dot{\theta})
\]

Let \( \theta^* \) be the value for which \( h \circ l \) achieves its extremum. Then it follows that \( (h \circ l)' \) switch sign through zero at \( \theta = \theta^* \), i.e. the sign of \( (h \circ l)'(\theta^* + \varepsilon) \) is opposite of \( (h \circ l)'(\theta^* - \varepsilon) \) for small values of \( \varepsilon \). This implies that the stationary gain \( G(0) \) also will switch sign through zero when linearized about \( x^* = l(\theta^*) \). This can only happen if either of

\[
K = \pm 0, \quad b_0 = \pm 0
\]

is true at the optimum (\( a_0 = \pm \infty \) is not possible for proper systems). If \( K = 0 \), then all dynamics disappear at the extremum since \( G(s) = 0/s \), i.e., the output is invariant at the optimum. If instead \( b_0 = 0 \) and some \( b_k \neq 0 \), then a real zero will cross between the LHP and the RHP and
there exist such that . In this case the system will have a dynamic response even at the extremum and, furthermore, the linearized system will be non-minimum phase, at least locally, on one side of the extremum.

Consider now the case of a Hammerstein/Wiener model, as considered in most previous studies on ESC. For such models it is clear that the optimum corresponds to , and hence the optimum is a stationary solution of ESC for sufficiently small . Furthermore, the phase lag of such models does not vary with and hence there will only be singular frequencies for which the ESC can lock on to a solution with . Thus, for essentially any choice of the excitation frequency there will be a unique stationary solution of the ESC which is the optimal solution for small . Furthermore, the deviation from optimality for larger will only depend on the degree of non-symmetry of the mapping about the optimum. As shown in Krstić and Wang [2000], the solution will also be stable for an appropriate choice of controller parameters, including a sufficiently small excitation frequency .

Consider next the case in which the optimum corresponds to , i.e. a transmission zero crosses the imaginary axis through zero as passes the extremum. In this case, the system has a zero at and hence a phase lag of at . Thus, the stationary solution will asymptotically approach the optimum as . For solutions close to the extremum there will be a zero close to , either in the LHP or RHP, and hence a small non-zero frequency for which the phase lag . Thus, for small non-zero excitation frequencies the ESC will converge to a solution in the vicinity of the optimum. Since the zero moves away from the origin as passes away from the optimum, it implies that the distance to the optimum will increase with increasing frequency. This also corresponds well with the results based on local approximations around the optimum in Chiova et al. [2007]. Note that the slower the zero moves with changes in , the larger the distance to the optimum will be for a given excitation frequency.

Also, note that as the zero has moved some distance from the imaginary axis the impact of other poles and zeros are likely to interfere with its phase contribution and we may not get a phase lag for any solution in the vicinity of the optimum. Thus, for sufficiently large frequencies there will probably not be any stationary solutions related to the existence of a process optimum.

Finally, it is clear that a process may have frequencies where the phase lag without any relation whatsoever to the optimality conditions discussed above. If this frequency varies with , then we will have a continuous range of excitation frequencies for which a sub-optimal stationary solution will exist. In such cases it also likely that multiple solutions will exist, of which one is related to the optimality of the process while the others are not. However, there may also exist situations where all stationary solutions are sub-optimal in the sense that they are not related to the optimality conditions of the process. This is shown for a simple example process below. First, we derive a simple stability condition for the stationary solutions of the ESC as derived above.

4.2 Stability of the stationary solutions

As shown above, depending on the dynamic properties of the process and the excitation frequency , the ESC may lock on to different types of stationary solutions. In practice, one will of course only observe stable stationary solutions and hence it is of interest to determine if all the various types of stationary solutions can be stable, at least for some choices of controller parameters.

To simplify the stability analysis we will assume that the control is so slow that the process in combination with the low-pass and high-pass filters acts as a static map from to . Note that the control can be slow even if the excitation frequency is relatively high since the response time also depends on other parameters such as excitation amplitude and integrator gain . Also note that the purpose of the stability analysis presented here simply is to show that in principle all stationary solutions discussed above can be asymptotically stable.

Consider the ESC closed loop in Fig. 1. This can be simplified into Fig. 4 in which all blocks but the integrator block have been included in the block labelled .

To investigate the stability of the simplified loop we seek to find an algebraic expression for the relation .

If is varying slowly, then the local response of the system (1) to small and relatively fast perturbations can be approximated by the system linearized about the current . Thus, we approximate the system by a linear parameter varying (LPV) system with as a parameter. We form

\[
\begin{align*}
\dot{x} &= A(\hat{\theta})x + B(\hat{\theta})\theta \\
\hat{y} &= C(\hat{\theta})x
\end{align*}
\]

which yields the parametrized transfer-function

\[
G(s, \hat{\theta}) = C(\hat{\theta})(sI - A(\hat{\theta}))^{-1}B(\hat{\theta}).
\]

If is fast compared to the variations in , we can consider the problem using separate time scales. For the fast time scale, we approximate as a constant and follow the same steps as when we derived (5) to characterize the stationary solutions. We get

\[
\xi = \frac{\omega^2}{2} |F_L(0)||G(i\omega, \hat{\theta})||F_H(i\omega)| \cos(\phi(\hat{\theta})) = L(\hat{\theta}).
\]

This static map is the relation between and in the slow time scale. Again, we are interested in stationary solutions corresponding to . To determine
the stability of such solutions we consider a linearization of $L$ around the stationary solutions for $L(\bar{\theta}) = 0$

$$L(\hat{\theta}) \approx \frac{dL(\bar{\theta})}{d\theta} \hat{\theta}, \quad \hat{\theta} = \theta - \bar{\theta}.$$  

Now if we replace $L$ by its linear approximation in the closed loop in Fig. 4, it should be clear the closed loop will have a single pole at

$$k \frac{dL(\bar{\theta})}{d\theta}.$$  

The stability of the loop is determined by the sign of the pole and the stability criterion thus becomes

$$k \frac{dL(\bar{\theta})}{d\theta} < 0 \quad (10)$$

Condition (10) can in principle be satisfied for any type of stationary solution discussed above, and hence all types of solutions can in principle be stable. Also, note that any stationary solution can be made stable by simply choosing the appropriate sign of the controller gain $k$.

5. EXAMPLES

In order to illustrate the results derived above in a transparent fashion which furthermore can be reproduced by the reader, we here first consider a simple low order system before returning to the more complex CANON process.

5.1 A simple example

Consider the 2nd order system

$$\begin{align*}
\dot{x}_1 &= -x_1 + \theta^2 \\
\dot{x}_2 &= -x_1 - x_2 + 10\theta \\
y &= x_2
\end{align*}$$

If we take the cost function to be the equilibrium map from $\theta$ to $y$ given by

$$h \circ l(\theta) = \theta(10 - \theta), \quad (11)$$

then we see that the optimal steady-state output is $y^* = 25$ for $\theta^* = 5$. Now suppose that we do not know the model and want to determine the optimum using ESC with periodic excitation.

Linearizing the system about some constant $\bar{\theta}$ we get

$$G(s) = \frac{10s + 10 - 2\bar{\theta}}{(s + 1)^2}.$$  

We immediately see that the zero gain condition $|G(i\omega)| = 0 \forall \omega$ in (7) is not fulfilled for any $\hat{\theta}$. Furthermore, the transfer-function of the linearized system has a zero at $s = 0$ for the optimal input $\bar{\theta} = 5$. Thus, any stationary solution must be characterized by the phase lag condition (8), i.e.,

$$\varphi = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \ldots$$

Now, by determining the phase lag of the system, substituting this into (8) and then solving for $\hat{\theta}$ we find

$$\hat{\theta} = 5 - \frac{5\omega}{\tan\left(\frac{\pi}{2} + 2\arctan(\omega) - \arg(F_H(i\omega))\right)} \quad (12)$$

which describe how the stationary solution depend on the frequency.

In order to test the theory, we implemented the model and the ESC-scheme in Matlab using

$$k = 0.25, \quad a = 0.2,$$

$$F_H(s) = \frac{s}{s + \omega_h}, \quad \omega_h = 0.01,$$

$$F_L(s) = \frac{\omega_l}{s + \omega_l}, \quad \omega_l = 0.01.$$  

The results from these simulations using different values of the excitation frequency $\omega$ are given in Fig. 5 and Table 1.

![Fig. 5. Stationary solutions of the simple example process for various excitation frequencies $\omega$.](image)

Table 1. Results of simulations of ESC for simple example process with $k > 0.$

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\theta$</th>
<th>$\frac{x(\theta)}{y}$</th>
<th>Stable</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>5.05</td>
<td>0.98</td>
<td>Yes</td>
</tr>
<tr>
<td>0.5</td>
<td>6.81</td>
<td>0.64</td>
<td>Yes</td>
</tr>
<tr>
<td>0.55</td>
<td>9.17</td>
<td>0.22</td>
<td>Yes</td>
</tr>
<tr>
<td>0.7</td>
<td>14.20</td>
<td>0.24</td>
<td>Yes</td>
</tr>
<tr>
<td>2</td>
<td>-8.47</td>
<td>0.12</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>-6.23</td>
<td>0.080</td>
<td>No</td>
</tr>
</tbody>
</table>

As can be seen from the Figure and Table, the deviation from the optimum increases with increasing excitation frequency $\omega$. This is as expected from the discussion above, and also from the local analysis in Chioua et al. [2007]. However, we also see that for some $\omega$ there will not exist any stable stationary solution to the ESC whatsoever, assuming $k > 0$. Finally, we also note from Table 1 that the stability of the various stationary solutions with finite control bandwidth is well predicted by the stability analysis above.

5.2 Extended example

The change in $\varphi$ with $\theta$ in the above example is too small for the system to exhibit multiple stationary solutions. Consider instead the system described by

$$G_2(s, \theta) = G(s, \theta)G_T(s, \hat{\theta})$$

where $G(s)$ is given by the simple example above and

$$G_T(s, \hat{\theta}) = \frac{1}{(\tau(\hat{\theta})s + 1)^m}, \quad \tau(\hat{\theta}) = \frac{1}{1 - 0.095\hat{\theta}}$$

represents a transport process which retention time depends on the input $\theta$. We limit $\theta$ to be less than 10 so
that the system remains open-loop stable. The steady-state mapping \( h \circ l(\theta) \) is identical to that in the simple example above.

Analysis of \( \varphi(\hat{\theta}) \) show that if we try to optimize this system using ESC with an excitation frequency \( \omega = 0.03 \) and the same high-pass filter as above, then there will exist three stationary solutions at \( \theta = 5.05, \theta = 8.8, \) and \( \theta = 9.9 \). See also Fig. 6. Simulations show that for positive \( k \), the solution converges to either \( \theta = 5.05 \) or \( \theta = 9.9 \), depending on the initial conditions. For negative \( k \), the solution converge to \( \theta = 8.8 \). The stability results are in all cases confirmed by the stability analysis presented above.

![Stationary solutions of extended example for excitation frequency \( \omega = 0.03 \). Solutions (1) and (2) are stable while solution (3) is unstable for positive \( k \).](image)

**Fig. 6.** Stationary solutions of extended example for excitation frequency \( \omega = 0.03 \). Solutions (1) and (2) are stable while solution (3) is unstable for positive \( k \).

### 5.3 The CANON process revisited

For the CANON process, we found that, with an excitation frequency \( \omega = 4 \text{ rad/h} \), the ESC algorithm could converge to two different stationary solutions of which one was close to the optimum and one was far removed from the optimum. See also Fig. 2. By considering linearizations of the model for various DO levels we find that the two operating points correspond to a phase lag of \( -\pi/2 \) and \( \pi/2 \), respectively, and that both are stable according to the stability condition (10). Furthermore, we also find a third stationary solution for the phase lag is \( \pi/2 \), but this solution is found to be unstable for positive integral gain \( k \). The results are summarized in Table 2. Linearization also reveals that the CANON process has a transmission zero at \( s = 0 \) at the optimum, and hence the ESC will only converge to the optimum as the excitation frequency goes to zero.

**Table 2. Results of simulations of CANON with ESC**

<table>
<thead>
<tr>
<th>( \theta ) (DO [g/l])</th>
<th>( y ) (N(_2) [g/l])</th>
<th>( \varphi ) [rad]</th>
<th>( \text{sgn} \left( \frac{dL(\theta)}{d\theta} \right) )</th>
<th>Stable</th>
</tr>
</thead>
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<tr>
<td>0.083</td>
<td>0.52</td>
<td>-1.57</td>
<td>-</td>
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</tr>
<tr>
<td>0.46</td>
<td>10.14</td>
<td>1.57</td>
<td>+</td>
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<tr>
<td>0.44</td>
<td>10.3</td>
<td>1.57</td>
<td>-</td>
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**REFERENCES**


6. SUMMARY AND CONCLUSIONS

We have in this paper considered extremum seeking control of a complex and relatively slow process, the CANON process for ammonium removal from wastewater. Direct application of the ESC scheme with periodic excitation revealed that the scheme could converge to stationary solutions far removed from the optimum, and furthermore that multiple stationary solutions could exist. An analysis of a general dynamic model revealed that the stationary solutions of the ESC are characterized either by the linearized dynamics having zero gain for all frequencies \( |G(i\omega)| = 0 \) \( \forall \omega \), or the phase lag at the excitation frequency being \( \pi + n\pi \). This result was related to previous results on properties of dynamic systems close to extremum points in the input-output map [Jacobsen, 1994], and based on this we could conclude that both conditions for stationarity are likely to be fulfilled in some vicinity of the optimum for sufficiently low excitation frequencies. However, there may also exist stationary solutions fulfilling the phase lag condition at operating points with no relation to the optimality condition whatsoever. Furthermore, for higher excitation frequencies, there may exist no stationary solutions related to the optimality of the process. The use of relatively low excitation frequencies would make the ESC impractical for slow processes like the CANON process as it would take months or years to arrive at the optimum and then only in the absence of disturbances. Thus, future research should consider whether it is possible to modify the simple ESC scheme with periodic excitation to ensure convergence to the close vicinity of the optimum also for higher excitation frequencies. We believe that utilization of the knowledge of the general behavior of dynamic processes around extremum points, as discussed above, will prove crucial to solve this problem.

