Abstract: A procedure for identifying a linear MIMO model with LFT-type uncertainty is presented. The uncertainty is minimized subject to the requirement that known input-output data do not invalidate the model. The optimization problem is solved stepwise. First, the uncertainty is minimized frequency-by-frequency with respect to the sampled frequency response of a nominal model subject to data-matching constraints, which can be expressed as matrix inequalities. Depending on the uncertainty structure, the matrix inequalities may be linear or bilinear with respect to frequency samples of the nominal model. We show that the bilinear matrix inequalities (BMIs) can be transformed into linear matrix inequalities (LMIs) for a certain type of uncertainty model. The resulting optimization problem is then convex. Next, a state-space model is fitted to the frequency responses subject to the same data-matching constraints. Prior to this fitting, possible time delays in the data are removed by a special technique. In this way, the order or the state-space model can be minimized. Naturally, the time delays are re-inserted in the transfer function calculated from the state-space model. An application to distillation modelling is included.

Keywords: Uncertain systems, MIMO systems, time-delay systems, frequency-domain system identification, linear matrix inequalities (LMIs), bilinear matrix inequalities (BMIs).

1. INTRODUCTION

For robust control design, a model with information about the model uncertainty, i.e., the possible discrepancy between a nominal model and the true system, is very useful. A standard model type for this purpose is a model consisting of a linear nominal model augmented by an uncertainty description in the form of a norm-bounded uncertainty (Zhou et al., 1996).

A central issue in identification for control is to find the best nominal model for controller design (Van den Hof and Schrama, 1995; Hjalmarssson, 2005; Gevers, 2005). Most studies have been restricted to single-input single-output (SISO) systems, but in recent times also multiple-input multiple-output (MIMO) systems have been considered (Barenthin et al., 2008; Bazanella et al., 2010; Agüero et al., 2012).

Since the purpose of the model is controller design, an iterative approach with alternating closed-loop identification and controller design is suggested. However, Bölöng and Mäkilä (1998) showed that the same (or better) performance can be achieved with a single open-loop identification experiment and an iterative $\mathcal{H}_\infty$ modelling/controller design procedure. Robust optimal experimental design for open-loop identification has been considered by Rojas et al. (2007).

The best nominal model for controller design is not necessarily a model that minimizes the open-loop prediction error of the experiment data. For identification using an $\mathcal{H}_\infty$ criterion, the size of the uncertainty is more relevant. However, up till recently there was no clear consensus as to what constitutes an operational control-oriented uncertainty set (Gevers, 2005). According to Douma and Van den Hof (2005), the various uncertainty types are all equivalent provided that the identification methods deliver optimal uncertainty sets rather than an uncertainty bound around a prefixed nominal model. Thus, the uncertainty set should be minimized over both the nominal model and the uncertainty bound, as done by Häggblom (2006). For an LFT type of uncertainty, it has recently been shown that the $\mathcal{H}_\infty$ norm of the uncertainty is a rigorous measure of the worst-case degradation of the stability margin for a system under feedback control irrespective of the particular type of uncertainty structure (Vinnicombe, 2001; Lanzon and Papageorgiou, 2009). An important task is then to choose a structure, which allows a comparatively small uncertainty bound.

In this paper we present a procedure for identifying a linear multiple input, multiple output (MIMO) model with uncertainty bounds such that the model can explain all known input-output data with a minimal $\mathcal{H}_\infty$ norm uncertainty. The optimization problem is solved in steps.

The maximum singular value of the uncertainty is first minimized frequency-by-frequency subject to data-matching constraints, which can be expressed as matrix inequalities. Depending on the uncertainty structure, the matrix inequalities may be linear or bilinear with respect to frequency samples of the nominal model. We show in this paper that the bilinear matrix inequalities (BMIs) can be transformed into linear matrix inequalities (LMIs) for a certain type of uncertainty models. The resulting optimization problem is then convex. For a review of LMIs and BMIs, see VanAntwerp and Braatz (2000).
Next we fit a state-space model to the calculated frequency responses subject to the data-matching LMI constraints. Because time delays pose difficulties in state-space modelling, we use a special technique to remove the effect of possible time delays from the calculated frequency-response data prior to fitting the state-space model. After this, the time delays can be reintroduced in the corresponding transfer function model. In this way, a good fit can be obtained with a reduced model order.

2. PROBLEM FORMULATION

It is assumed that information about the system to be modelled is available in the form of input-output data. These data could be obtained through identification experiments designed to reveal the relevant properties of the system. In the proposed method, sampled frequency responses of the data are needed, i.e.,

\[ \{ u_k(j\omega), y_k(j\omega) : \omega \in \Omega \}, \quad k = 1, \ldots, N, \]

where \( N \) is the number of data sets, \( \Omega \) is a chosen set of frequency points, \( u_k \) is the input signal and \( y_k \) is the output signal in the \( k \)-th data set. The signal dimensions are arbitrary.

Noisy data is a nuisance. However, if identification experiments have been conducted to produce the data, the input signal is usually well defined and its frequency response can readily be calculated. One way of solving the problem with noise in the output signal is to fit a filter, \( G_k(s) \), to the data and to calculate a noise-free output by (Böling et al., 2004)

\[ y_k(j\omega) = G_k(j\omega)u_k(j\omega). \]

(2)

Note that \( G_k(s) \) is not considered to be a true model; \( u_k \) may, e.g., be the input in a short interval of a longer experiment, where it is not properly exciting the full system (Häggbloom and Böling, 1998). Hence, \( G_k \) is assumed to apply only to the input \( u_k \), not necessarily to any other input. This is a convenient way of separating noise from relevant dynamics.

In this paper, we consider output-multiplicative uncertainty, which can be expressed as

\[ G(s) = G_0(s) + \alpha(s)\Delta(s)G_0(s), \quad \|\Delta\|_\infty \leq 1, \]

(3)

where \( G \) represents the true system, \( G_0 \) is a nominal transfer function model, \( \Delta \) is MIMO uncertainty bounded by \( \|\Delta\|_\infty \leq 1 \), and \( \alpha \) is a scalar uncertainty weight, which is considered to belong to the uncertainty. Here we only consider unstructured \( \Delta \) uncertainties. Other types of LFT uncertainty models are considered in Häggbloom (2010).

We want an uncertainty model that, for all inputs \( u_k \), \( k = 1, \ldots, N \), can produce the corresponding output \( y_k \) with some allowed \( \Delta = \Delta_k \). In addition, we want the norm \( \|\Delta\|_\infty \) to be as small as possible. Without loss of generality, \( \alpha \) can be assumed to be real valued and positive, but \( \alpha \) may depend on the frequency. Formulated in terms of sampled frequency responses, we thus want to minimize \( \max_{\omega} \alpha(\omega) \) subject to the data-matching constraints

\[ y_k(j\omega) = \left[ I + \alpha(\omega)\Delta_k(j\omega) \right] G_0(j\omega)u_k(j\omega), \]

\[ \bar{\sigma}(\Delta_k(j\omega)) \leq 1, \quad \forall \omega \in \Omega, \forall k, \]

where \( \bar{\sigma}(\cdot) \) denotes the maximum singular value.

It is obvious from (4) that the minimum of \( \max_{\omega} \alpha(\omega) \) depends on \( G_0 \). Because the data-matching constraints at different frequencies are independent of each other when sampled frequency functions and responses are considered, a first part of the minimization problem can be solved by minimizing \( \alpha(\omega) \) with respect to \( G_0(j\omega) \), subject to (4), for \( \forall \omega \in \Omega \). After this, a model can be fitted to the calculated frequency responses \( G_0(j\omega) \).

3. MINIMIZING THE UNCERTAINTY WEIGHT

For each frequency \( \omega \in \Omega \), the data-matching constraint (4) can be written as

\[ y_k = G_0u_k + \alpha\Delta_k G_0u_k, \quad \bar{\sigma}(\Delta_k) \leq 1, \quad \forall k, \]

(5)

where the arguments “\( \omega \)” and “\( j\omega \)” have been suppressed for convenience. The constraint can be written as a matrix inequality by means of the following lemma adapted from Chen and Gu (2000).

**Lemma 1.** Consider the matrix equation

\[ C = AB. \]

There is a solution \( \Delta, \bar{\sigma}(\Delta) \leq 1 \), if and only if

\[ \begin{bmatrix} AA^* & C \\ C^* & B^*B \end{bmatrix} \succ 0, \]

(7)

where \( X^* \) is the complex-conjugate transpose of a matrix \( X \) and \( X \succ 0 \) denotes that \( X \) is positive semidefinite.

According to Lemma 1, a necessary and sufficient requirement for a \( \Delta_k \), \( \bar{\sigma}(\Delta_k) \leq 1 \), to exist such that (5) is satisfied exactly, is

\[ \begin{bmatrix} 1\alpha^2 & (y_k - G_0u_k) \\ (y_k - G_0u_k)^* & u_k^*G_0G_0u_k \end{bmatrix} \succ 0, \quad \forall k. \]

(8)

Unfortunately, (8) is a bilinear matrix inequality (BMI) when \( G_0 \) is a decision variable. This means that the problem of minimizing \( \alpha^2 \) with respect to \( G_0 \) is nonconvex and thus difficult to solve.

As shown below, it is possible to write the uncertainty model (5) in a form, which allows the data-matching constraints to be expressed as LMIs. The vector \( G_0u_k \) can be solved from (5) and used to eliminate \( G_0u_k \) from the uncertainty term of (5). This results in

\[ y_k = G_0u_k + \alpha\Delta_k(I - \alpha\Delta_k)^{-1}y_k \]

(9)

with the additional requirement \( \alpha \leq 1 \). Because \( \Delta_k \) is assumed to be unstructured with \( \bar{\sigma}(\Delta_k) \leq 1 \), Lemma 10.2.4 in Chen and Gu (2000) can be used to transform (9) into...
(1 - α^2)^{-1} y_k = G_0 u_k - α(1 - α^2)^{-1/2} \Delta_k \Delta_k^{-1/2} y_k, \quad (10)

where \(\sigma(\Delta_k) \leq 1\). In general, \(\Delta_k \neq \Delta_k^{-1}\), but the bound on \(\Delta_k\) is nonconservative (i.e., necessary and sufficient).

Because \(\alpha\) is a scalar, (10) can easily be simplified, but it is in a suitable form for application of Lemma 1, which yields

\[
\begin{bmatrix}
\beta I & (1 + \beta y_k - G_0 u_k) \\
((1 + \beta) y_k - G_0 u_k)^* & (1 + \beta) y_k^* y_k
\end{bmatrix} \succeq 0, \quad \forall k,
\]

(11)

where \(\beta = \alpha^2 (1 - \alpha^2)^{-1}\).

Equation (11) is an LMI in terms of the decision variables \(\beta\) and \(G_0\). Since \(\beta\) decreases with decreasing \(\alpha\) when \(0 < \alpha < 1\), minimization of \(\alpha\) subject to (8) is equivalent to minimization of \(\beta\) subject to (11).

4. DEALING WITH TIME DELAYS

We intend to fit a state-space model to the calculated frequency responses \(G_0(j \omega)\) because we are then able to enforce the data-matching conditions also for the model. Unfortunately, a state-space model does not handle time delays well. They can be modelled approximately by introducing extra states into the model, thus increasing the model order. Here we present a solution to avoid increasing the model order.

Consider a transfer function matrix

\[
G(s) = \begin{bmatrix}
G_{11}(s)e^{-\theta_{11}s} & G_{12}(s)e^{-\theta_{12}s} \\
G_{21}(s)e^{-\theta_{21}s} & G_{22}(s)e^{-\theta_{22}s}
\end{bmatrix},
\]

(13)

where the time delays \(\theta_{ij}\) are separated from the rest of the transfer functions \(G_{ij}\). For simplicity, we here consider a 2 \times 2 system, but the principle can easily be extended to larger systems. This matrix can be factorized as

\[
G(s) = L_y(s)G^\odot(s)L_u(s),
\]

(14)

where \(L_y\) and \(L_u\) are diagonal matrices of time delays,

\[
L_y(s) = \begin{bmatrix}
e^{-\theta_{11}s} & 0 \\
0 & e^{-\theta_{22}s}
\end{bmatrix}, \quad L_u(s) = \begin{bmatrix}
e^{-\theta_{11}s} & 0 \\
0 & e^{-\theta_{22}s}
\end{bmatrix}.
\]

(15)

In general, all time delays cannot be transferred from \(G(s)\) to \(L_y(s)\) and \(L_u(s)\), but we can determine a maximum of transferrable time delays by solving the simple optimization problem

\[
\max(\theta_{11} + \theta_{12} + \theta_{21} + \theta_{22}) \text{ s.t. } \\
\theta_{11} + \theta_{21} \leq \theta_{12}, \quad \theta_{21} + \theta_{22} \leq \theta_{12}, \quad \theta_{12} + \theta_{22} \leq \theta_{11}.
\]

(16)

Note that we do not need to know the transfer functions \(G_{ij}\), it is sufficient to know the time delays \(\theta_{ij}, \ i, j \in \{1, 2\}\), in order to solve (16).

If we want to extract the same maximum time delays from several models \(G_k\), with time delays \(\theta_{kij}\), we include \(\theta_{ij} + \theta_{ij} \leq \theta_{kij}, \ k = 1, \ldots, N, \ i, j \in \{1, 2\}\), as constraints in (16) for all such models.

When \(L_y\) and \(L_u\) have been determined, we can calculate frequency response samples of new inputs and outputs according to

\[
u_k^\odot(j \omega) = L_u(j \omega)u_k(j \omega) \quad \text{and} \quad y_k^\odot(j \omega) = L_y(j \omega)y_k(j \omega), \quad \forall \omega \in \Omega, \forall k.
\]

(17)

Here, \(u_k^\odot\) and \(y_k^\odot\) are inputs and outputs for the (almost) delay-free transfer matrices \(G_k^\odot\) (which may be unknown).

5. FINDING A NOMINAL MODEL

We want to find a model \(\tilde{G}_0(s)\) that closely approximates the calculated frequency responses \(G_0(j \omega), \ \forall \omega \in \Omega\). There are many methods for doing this. We use a degree-constrained Nevanlinna-Pick interpolation method (Meché and Bosgra, 2004), where the model is parameterized as a state-space model. Because of this, we first determine an essentially delay-free model \(G^\odot_0(s)\), parameterized as

\[
\tilde{G}_0^\odot(s) = C(sl - A)^{-1}B + D.
\]

(18)

Details of the procedure are given in Jafarian and Hägglomb (2010). The interpolation method is used without data-matching constraints. Hence, the constraints are almost certainly violated by the initial model. However, this can easily be remedied. In accordance with (14),

\[
\tilde{G}_0(s) = L_y(s)\tilde{G}_0^\odot(s)L_u(s).
\]

(19)

Replacing \(G_0\) in (11) by \(\tilde{G}_0\) and combination with (17)-(19), yields

\[
\begin{bmatrix}
\beta I & (1 + \beta) y_k^\odot - C(j \omega I - A)^{-1} B + D \mu_k^\odot \\
((1 + \beta) y_k^\odot)^* & (1 + \beta) y_k^\odot y_k^\odot
\end{bmatrix} \succeq 0,
\]

(20)

where (*) indicates the complex-conjugate transpose of the elements in the symmetrical position of the full matrix. This matrix inequality is linear in the variables \(\beta\), \(C\) (or \(B\)) and \(D\). Thus, \(\alpha\) can be minimized subject to the data-matching constraints (20) by minimizing \(\beta\) with respect to \(C\) and \(D\) for \(\forall \omega \in \Omega\) and \(\forall k\). After this, the nominal model \(\tilde{G}_0(s)\) is obtained by (18) and (19).

6. APPLICATION TO DISTILLATION MODELLING

In this section the presented uncertainty modelling method is applied to a two-product distillation column. A distillation column is a multivariable system usually characterized by a strong directionality, which means that the transfer matrix is ill-conditioned and nearly singular. Such a system tends to be difficult to identify and to control (Hägglom and Böling, 1998).
6.1 Identification Data

Identification data and models determined by Häggblom and Böling (1998) are here used for uncertainty modelling. The distillation column was excited by a series of consecutive step changes in the so-called high- and low-gain directions, see Fig. 1. Because of nonlinearity and difficult dynamics, a single linear model fitted to the experimental data does not describe the behaviour well, as illustrated by Fig. 2. However, linear models fitted to each step response, resulting in six \( 2 \times 2 \) models \( G_k(s), \ k = 1, \ldots, 6 \), fit the experimental data essentially perfectly. Here, these models are used as filters to obtain noiseless output data. They also give information about time delays.

6.2 Calculation of Optimal Frequency Responses

Our previous studies have indicated that the input-output data can reasonably well be captured by an output-multiplicative uncertainty model when the linear model fitted to all experimental data is used as the nominal model in the uncertainty description (Nyström et al., 2003). However, that model is not optimized to minimize the uncertainty, which is the objective here.

The input data from the identification experiment are converted to frequency response data and the models \( G_k(s), \ k = 1, \ldots, 6 \), are used as filters to calculate noise-free frequency responses of the output according to (2). Thus, data as specified in (1) are available for six data sets. The set of frequency points, \( \Omega \), are chosen as 99 logarithmically equally-spaced points in the range 0.001…2\( \pi \) rad/min.

Optimal frequency response samples of \( G_0(j\omega) \) are calculated by minimizing \( \alpha \), which is done by minimizing \( \beta \) with respect to \( G_0 \) subject to (11). The Matlab toolbox YALMIP [15] together with the SeDuMi solver are used for the optimization. Fig. 3 shows the minimum uncertainty weight \( \alpha \) as a function of frequency for the frequency response of the optimal nominal model as well as the model fitted in the time domain to the entire experiment. As can be seen, the improvement is considerable.

6.3 State-Space Modelling

In order to demonstrate the advantage of time-delay removal as described in Section 4, a state-space model is first fitted to the calculated frequency response samples without time-delay removal. This is done by fitting the model directly to the input-output data defined in (1). In the data-matching constraint (20), \( u_k^o \) and \( y_k^o \) are replaced by \( u_k \) and \( y_k \), respectively.

Due to the complexity of the optimization problem, only a subset of the full range of frequency points are used when fitting the model. Three different model orders are used: 10, 14 and 20. As can be seen from Fig. 4, only the model with 20 states allows an uncertainty that is close to the optimal one over most of the frequency range.

6.4 Time-Delay Compensation

To reduce the order of the nominal model, time delays are determined as explained in Section 4 and extracted from the input and output signals according to (17). The same state-space modelling procedure as in the previous section is applied to the new frequency response samples. A shown by Fig. 4, a 10th order model now gives an uncertainty description that is closer to the optimal solution than the 20th order model without time-delay compensation.

![Fig. 1. Identification of a distillation column with excitations in low- and high-gain directions (\( y = \) distillate comp., \( x = \) bottoms comp., \( D = \) distillate flow rate, \( L = \) reflux flow rate, \( V = \) flow rate of steam to reboiler).](image1)

![Fig. 2. Fit of a linear MIMO model (full line = exp. data, dotted lined = model fit, \( y = \) dist. comp., \( x = \) bot. comp.).](image2)
7. CONCLUSIONS

A procedure for identifying a linear MIMO model with LFT-type uncertainty has been presented. The uncertainty is minimized subject to the requirement that known input-output data do not invalidate the model. The optimization problem is solved stepwise. First, the uncertainty is minimized frequency-by-frequency with respect to the sampled frequency response of a nominal model subject to data-matching constraints, which can be expressed as matrix inequalities. Next, a state-space model is fitted to the frequency responses subject to the same data-matching constraints. Prior to this fitting, possible time delays in the data are removed by a special technique and later re-inserted in the transfer function calculated from the state-space model. As shown in an application to distillation modelling, it was possible by this technique to reduce the order of the nominal uncertainty model from 20 to 10 without increasing the model uncertainty.

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