An extended AUDI algorithm for simultaneous identification of forward and backward paths in closed-loop systems

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Abstract: In closed-loop system identification, most of the existing methods only focus on the forward path, yet few on simultaneous identification of the forward and backward paths. Meanwhile, an augmented UD identification (AUDI) algorithm has been proved effective in open-loop system identification, but it only extracts the forward path information, while not including the backward path information provided in an augmented information matrix, which is helpful for the closed-loop system identification. In this paper, an extended AUDI (EAUDI) is proposed to simultaneously identify the model orders and parameters of both forward and backward paths of a closed-loop system. The conditions of identifiability and uniform convergence for closed-loop systems using the EAUDI algorithm are also given. The effectiveness of this algorithm is demonstrated by a numerical example.

1. INTRODUCTION

With the increasing demand of industrial production and the improvement of environmental requirements, more and more advanced control methods are used in industry, which usually require a mathematical model (Tufa and Ramasamy, 2011). System identification is an art of modelling; thus, the study of system identification is of great practical significance (MacArthur and Zhan, 2007).

System identification includes two parts: identification of model structure and identification of model parameters. For a linear system, model structure identification means to determine the order of a model. Model order identification and parameters identification are often mutually dependent. However, most of the existing methods identify the model order and model parameters separately (Fang and Xiao, 1988). One common method is separately identifying the model parameters and loss functions for all possible orders from 0 to $n$, where $n$ is the maximum possible order of the model. An appropriate model order is then determined by examining the loss functions (Niu \textit{et al.}, 1990). However, this method has a disadvantage that its computational burden would be very high if $n$ is large.

To the best of our knowledge, there are few methods which can simultaneously identify the model order and parameters. An augmented UD identification (AUDI) algorithm is one of such methods (Niu \textit{et al.}, 1990). The AUDI algorithm is based on an augmented information matrix (AIM), and decomposes the AIM using $\mathbf{U}\mathbf{D}\mathbf{U}^T$ factorization. The AUDI algorithm was used to simultaneously identify the order and parameters of SISO open-loop systems (Niu \textit{et al.}, 1990), and the orders and parameters of MIMO open-loop systems are simultaneously identified (Niu \textit{et al.}, 1991) using the AUDI algorithm.

In the field of system identification, closed-loop identification has been paid special attention (Jeng and Lin, 2011; Welsh and Goodwin, 2002). Safety and production restrictions are often strong reasons for not allowing identification experiments in open-loop for many industrial production processes, and thus experimental data can only be obtained under closed-loop conditions (Bendtsen \textit{et al.}, 2008; Welsh and Goodwin, 2002; Gilson and Hof, 2005). The main difficulty in closed-loop identification results from the correlation between disturbances and control signals, induced by the loop (Jeng and Lin, 2011; Badwe, Patwardhan and Gudi, 2011; Betts \textit{et al.}, 2009; Wang and Qin, 2006). The closed-loop identification approaches can be broadly classified into three categories: direct, indirect and joint input-output identification (Betts \textit{et al.}, 2009; MacArthur and Zhan, 2007; Ljung, 1999). Most of these methods only focus on the forward path of a closed-loop system, but few on simultaneous identification of the forward and backward paths.

Meanwhile, an augmented UD identification (AUDI) algorithm has been proved effective in open-loop system identification, but it only extracts the forward path information, while not including the backward path information provided in an augmented information matrix, which is helpful for the closed-loop system identification. To the best of our knowledge, most of the existing literature about the AUDI algorithm focuses on the identification of an open-loop system, and yet little on a closed-loop system. Hence, in this paper, the information of the data matrix in the AUDI algorithm will be further extracted and an extended
augmented UD identification (EAUDI) is proposed to simultaneously identify the model orders and parameters of both the forward and backward paths of a closed-loop system. The conditions of identifiability and uniform convergence for closed-loop systems using the EAUDI algorithm are also given.

The rest of this paper is organized as follows. The EAUDI algorithm for a closed-loop system is presented in detail in Section 2, where the conditions of identifiability and uniform convergence are also given. In Section 3, the effectiveness of the EAUDI algorithm is demonstrated by a numerical example, followed by conclusions in Section 4.

**Remark:**

The controller model of some systems is unknown in advance, such as biological systems and economic systems. In fact, EAUDI algorithm can not only be used to identify controlled systems, where the backward path usually refers to a controller, but also to any system that can be represented as a feedback connection structure, which is more important. For EAUDI algorithm, the system can include two or more paths as long as these paths can be represented as a feedback connection structure.

The main purpose of this paper is to elaborate the idea of EAUDI algorithm. In order to make it easier to understand, we elaborate the idea in a control system, which is more familiar to us.

### 2. EAUDI for closed-loop systems

Consider the following linear closed-loop model

\[
A(z^{-1})z(k) = B(z^{-1})u(k) + v(k)
\]

\[
P(z^{-1})u(k) = Q(z^{-1})z(k) + w(k)
\]

where

\[
A(z^{-1}) = 1 + a(1)z^{-1} + \cdots + a(n_a)z^{-n_a}
\]

\[
B(z^{-1}) = b[d]z^{-d} + \cdots + b(n_b)z^{-n_b}
\]

\[
P(z^{-1}) = p[1]z^{-r} + \cdots + p(n_p)z^{-n_p}
\]

\[
Q(z^{-1}) = q(c)z^{-c} + \cdots + q(n_q)z^{-n_q}
\]

where \(z(k)\) and \(u(k)\) are the output and input variables respectively, and \(v(k)\) and \(w(k)\) are white noises; \(a(i)\) and \(b(i)\) are parameters of the forward path model; \(p(i)\) and \(q(i)\) are parameters of the backward path model; \(n_a, n_b, n_p, n_q\) and \(n_c\) are the corresponding orders; \(c\) and \(d\) are the delay of backward path and forward path respectively.

Define data vectors as

\[
\begin{bmatrix}
\hat{\theta}_1^{(1)} \\
\hat{\theta}_2^{(1)} \\
\vdots \\
\hat{\theta}_n^{(1)} \\
\hat{\theta}_1^{(2)} \\
\hat{\theta}_2^{(2)} \\
\vdots \\
\hat{\theta}_n^{(2)} \\
\vdots \\
\hat{\theta}_1^{(n)} \\
\hat{\theta}_2^{(n)} \\
\vdots \\
\hat{\theta}_n^{(n)}
\end{bmatrix}
\]

and the loss function matrix \(D(k)\) has the form

\[
D(k) = \text{diag}[J_{f}^{(0)}(k) \ J_{b_{1}}^{(1)}(k) \ J_{b_{1}}^{(1)}(k) \ J_{b_{1}}^{(2)}(k) \ J_{b_{1}}^{(2)}(k) \ \cdots \ J_{b_{1}}^{(n)}(k) \ J_{b_{1}}^{(n)}(k)]
\]

where superscripts “(i)” \((i=0,1,2,\ldots,n)\) stand for the model orders. For instance, \(\hat{\theta}_1^{(1)}\) is the first parameter of the \(n\)th
order model parameters. The subscripts “f” and “b” represent forward and backward paths, respectively.

Define

\[ \Phi^T(k)U(k) = e^T(k) \]  

(14)

where

\[ e(k) = [e_1(k), e_2(k), \cdots, e_{2n}(k), e_{2n+1}(k)]^T \]  

(15)

Equation (14) is equivalent to the following \((2n+1)\) equations

\[
\begin{align*}
0 & \Rightarrow z(k-n) \\
z(k-n) & \Rightarrow u(k-n+1) \\
z(k-n), a(k-n) & \Rightarrow z(k-n+1) \\
z(k-n), a(k-n), z(k-n+1) & \Rightarrow u(k-n+1) \\
& \vdots \\
z(k-n), a(k-n), z(k-n+1), \cdots z(k-1) & \Rightarrow u(k-1) \\
z(k-n), a(k-n), z(k-n+1), \cdots z(k-1), u(k-1) & \Rightarrow z(k)
\end{align*}
\]

where “\(\Rightarrow\)” means to use linear combination of the left hand side to fit the right hand side and \(e(k)\) is the loss function of the \(i\)th equation (Niu et al., 1995).

The AUDI algorithm used in open-loop systems does not efficiently use the information contained in the AIM, and it extracts only the forward path information \((\hat{\Theta}, J_f(k))\), while the information of backward path \((\hat{\alpha}, J_b(k))\) was not used. In the AUDI algorithm, we further extract the information of backward path contained in AIM.

Define the forward path order

\[ n_f = \text{max}\{n_p, n_b\} \]  

(17)

and the backward path order

\[ n_b = \text{max}\{n_p, n_q\} \]  

(18)

Equation (14) describes a set of equalities. The forward path \((5)\) is the \(n\)th odd column of \((14)\). The order of \((5)\) can be determined by the forward path loss functions \((J_f(k))\), and its parameters, e.g., \((7)\) are the corresponding odd column of \(U(k)\). Meanwhile, the backward path \((6)\) is the \(n\)th even column of \((14)\). The order of \((6)\) can be determined by the backward path loss functions \((J_b(k))\), and its parameters, e.g., \((8)\) are also the corresponding even column of \(U(k)\).

In the identification of the forward path, we maximize correlation between the output and input via the forward path and minimize correlation of the input and the output via the backward path. However, the worst case for identification of the forward path is exactly the best condition for identifying the backward path (Niu et al., 1995).

Hence, we should study the conditions of identifiability and uniform convergence for both the forward path and backward path of \((1)\) using the EAUDI algorithm. The EAUDI is a least squares (LS) type algorithm. Firstly, the conditions of identifiability and uniform convergence for the LS identification algorithm are discussed. Considering of the characteristics of the EAUDI algorithm – UD decomposition, the conditions of identifiability and uniform convergence for EAUDI algorithm are given following.

**Lemma 1** (Fang and Xiao, 1988):

The condition of identifiability for the forward path model of \((1)\) using the LS identification algorithm is

\[
w(k) \neq 0 \text{ or } \begin{cases} w(k)=0 \\
 0 \leq n_p + d \leq n_p \\
n_q + d \geq n_q
\end{cases}
\]  

(19)

The condition of uniform convergence for the forward path model of \((1)\) using the LS identification algorithm is

\[ c \neq 0 \text{ or } d \neq 0 \]  

(20)

**Lemma 2:**

The condition of identifiability for both the forward path and backward path of \((1)\) using the LS identification algorithm is

\[
\{w(k) \neq 0\} \cap \{v(k) \neq 0\} \\
\{w(k)=0\} \cap \{n_p + d \leq n_p \text{ or } n_q + d \geq n_q\} \\
\{v(k)=0\} \cap \{n_p \leq n_p + c \text{ or } n_q \leq n_q + c\}
\]

(21)

The condition of uniform convergence for both the forward path and backward path of \((1)\) using the LS identification algorithm is

\[ c \neq 0 \text{ or } d \neq 0 \]  

(22)

**Proof of Lemma 2**

(1) Proof of identifiability

From \((6)\), the result of the backward path model of \((1)\) using the LS identification algorithm is

\[ \hat{\alpha} = \left( \sum_{k=1}^{n} g(k)g^T(k) \right)^{-1} \sum_{k=1}^{n} g(k)u(k) \]

Thus the condition of identifiability for the backward path means to prove...
\[ R_z = P \lim_{L \to \infty} \left[ \frac{1}{L} \sum_{k=1}^{L} g(k) g'(k) \right] \]

is a non-singular matrix. Using the same method by Fang and Xiao (Fang and Xiao, 1988), the condition of identifiability for the backward path model is

\[ \nu(k) \neq 0 \quad \text{or} \quad \begin{cases} \nu(k) = 0 \\ n_p \leq n_a + c \quad \text{or} \quad n_q \leq n_a + c \end{cases} \quad (23) \]

Combining (19) and (23), the condition of identifiability for both the forward path and backward path (21) is proved.

(2) Proof of uniform convergence

Using the same method for the proof of the condition of uniform convergence for the forward path by Fang and Xiao (Fang and Xiao, 2002), the condition of uniform convergence for the backward path model is

\[ d \neq 0 \quad \text{or} \quad c \neq 0 \quad (24) \]

Combining (20) and (24), the condition of uniform convergence for both the forward path and backward path (22) is proved.

Based on Lemma 2, the condition of identifiability and uniform convergence for both the forward path and backward path model of (1) using the EAUDI algorithm is given as described in Theorem 3.

**Theorem 3:**

The condition of identifiability for both the forward path and backward path model of (1) using the EAUDI algorithm is

\[ \{w(k) \neq 0\} \cap \{\nu(k) \neq 0\} \]

\[ \{c \neq 0\} \cap \{d \neq 0\} \quad (25) \]

The condition of uniform convergence for both the forward path and backward path model of (1) using the EAUDI algorithm is

\[ c \neq 0 \quad \text{or} \quad d \neq 0 \quad (26) \]

**Proof of Theorem 3**

(1) Proof of identifiability

**Necessity**

The EAUDI is a LS type algorithm, so the condition of identifiability for EAUDI algorithm must satisfy (21). Equation (21) guarantees the elements of \{z(k-n), u(k-n),...,-z(k-1), u(k-1)\} are linearly independent. And considering of the characteristics of EAUDI algorithm–UD decomposition, data product matrix \( S(k) \) should be non-singular, so the elements of \( \varphi(k) = [-z(k-n), u(k-n),...,-z(k-1), u(k-1)]^T \) should be linearly independent. Due to (21), we shall just prove \( z(k) \) is linearly independent to the elements of \( \{z(k-n), u(k-n),...,z(k-1), u(k-1)\} \). From (1), if \( \nu(k) = 0 \), \( z(k) \) is linearly dependent to \( \{z(k-n), u(k-n),...,z(k-1), u(k-1)\} \). Thus, \( \nu(k) \neq 0 \). For the backward path, the result of \( w(k) \neq 0 \) can be obtained using the same method.

**Sufficiency**

From (11), we know that (1) can be identified by the EAUDI algorithm, as long as data product matrix \( S(k) \) is non-singular. Thus the elements of \( \varphi(k) = [-z(k-n), u(k-n),...,-z(k-1), u(k-1),-z(k)]^T \) should be linearly independent. And the condition \( \nu(k) \neq 0 \), \( w(k) \neq 0 \) can guarantee that the elements of \( \varphi(k) = [-z(k-n), u(k-n),...,-z(k-1), u(k-1),-z(k)]^T \) are linearly independent. Thus the sufficiency is proved.

(2) Proof of uniform convergence

The EAUDI is a LS type algorithm, so the condition of uniform convergence using EAUDI algorithm is the same as that of using the LS identification algorithm. Thus, (26) is proved.

\[ \square \]

3. Numerical Example

Consider the following closed-loop model

\[ z(k) - 1.5z(k - 1) + 0.7z(k - 2) = u(k - 1) + 0.5u(k - 2) + \nu(k) \]

\[ u(k - 1) = -0.5z(k - 1) + w(k - 1) \]

where \( z(k) \) and \( u(k) \) are the output and input variables and \( \nu(k) \) and \( w(k) \) are white noise. The maximum possible order is taken as 4 and SNR(\( z \)) = 2.1. The definition of signal-to-noise ratio (SNR) is

\[ SNR = \sqrt{\frac{\text{var}(z(k)) - \text{var}(e(k))}{\text{var}(e(k))}} \quad (28) \]

Using the EAUDI algorithm, the results are as follows.

\[ \begin{align*}
D(k) &= \text{diag}[J_f^{(0)}(k), J_f^{(1)}(k), J_f^{(2)}(k), J_f^{(3)}(k), J_f^{(4)}(k)] \\
J_f^{(0)}(k), J_f^{(1)}(k), J_f^{(2)}(k), J_f^{(3)}(k), J_f^{(4)}(k)] &= \text{diag}[10566 500 7270 498 125] \\
&= \text{diag}[497 124 496 124] 
\end{align*} \]
U(k) =
\[
\begin{bmatrix}
1 & -0.49 & 0.96 & -0.02 & 0.69 & -0.06 & -0.02 & 0.08 & -0.00 \\
1 & 0.86 & 0.05 & 0.51 & -0.07 & 0.01 & 0.07 & 0.01 \\
1 & -0.50 & -1.50 & 0.14 & 0.72 & -0.24 & -0.03 \\
1 & 1.00 & -0.05 & 0.49 & 0.06 & 0.06 & 0.00 \\
1 & -0.60 & -1.52 & 0.25 & 0.73 \\
1 & 1.01 & -0.06 & 0.49 \\
1 & -0.60 & -1.53 \\
1 & 1.00 \\
\end{bmatrix}
\] (30)

From the loss function of forward path $J_f(\theta)(k)$ and backward path $J_b(\theta)(k)$, we obtain that the forward path model order is 2 and the backward path model order is 1. The parameters of forward path and backward path from the corresponding columns of $U(k)$ are shown in Table 1. The efficacy of the EAUDI algorithm for (27) is shown in Figure 1.

Table 1. Comparison of parameters estimated for (27) by the EAUDI algorithm and true values

<table>
<thead>
<tr>
<th>$\hat{a}$</th>
<th>$\hat{b}$</th>
<th>$a_1$</th>
<th>$b_1$</th>
<th>$p_1$</th>
<th>$q_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>0.69</td>
<td>0.51</td>
<td>-1.50</td>
<td>1.00</td>
<td>1</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.7</td>
<td>0.5</td>
<td>-1.5</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 1. The result of the EAUDI algorithm for (27)

It can be seen in Table 1 and Figure 1, the EAUDI algorithm is effective in the identification of a closed-loop system.

4. CONCLUSIONS

In this paper, the AUDI algorithm has been reviewed and the fact has been found that the AUDI algorithm for open-loop systems extracts only the forward path information, but the information of backward path contained in an AIM was not used, which is helpful for the closed-loop system identification. Meanwhile, most of the existing closed-loop identification methods only focus on the forward path, but few on simultaneous identification of the forward and backward paths. Hence, the EAUDI algorithm is proposed to simultaneously identify the forward and backward paths of closed-loop systems. Through a numerical example, the effectiveness of the EAUDI algorithm used in the identification of a closed-loop system is demonstrated. We only consider the identification of SISO closed-loop systems using the EAUDI algorithm in this paper, and this algorithm can be extended to multi-input-multi-output (MIMO) and ARMAX (coloured noises) processes, which will be given in a separate paper.

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