Fixed Interval Smoothing of Nonlinear/Non-Gaussian Dynamic Systems in Cell Space

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Abstract: State estimation problems such as optimal filtering and smoothing do not lend themselves to analytical treatment in general nonlinear/non-Gaussian dynamic systems. The fixed interval smoothing problem aims to construct the marginal conditional probability density function of the state given past and future measurements relative to the state. For linear Gaussian systems it is derived as the Rauch-Tung-Striebel smoother. Theoretical solutions are available for nonlinear systems in the form of forward-filter-backward-smoother strategy and the two-filter-smoother formula that combines the results of two independent filters. Recently, these methods have been implemented in the context of sequential Monte Carlo filters with importance sampling. In this paper the fixed interval smoothing is numerically approximated by recursive computation of the conditional density as a piecewise constant function. The key for this algorithm is a coarse-grained representation of the system dynamics as an approximate aggregate Markov chain in discretized state space or cell space. The proposed approach is demonstrated with a simulation example involving a nonlinear CSTR model.

Keywords: Estimation theory; Kalman filters; Nonlinear systems; Monte Carlo method; Optimal estimation.

1. INTRODUCTION

State estimation is the task of determining the trajectory of the unknown state vector of a dynamic system, given its observable noisy measurements and uncertain system models. The problem is commonly solved in a probabilistic sense using Bayesian inference [Lee, 1964, Ho and Lee, 1964]. The complete solution is found in the probability density function (pdf) of the state conditioned on measurements. The filtering problem is defined as the conditional density of the current state given all measurements until now. It is of particular interest to online operations such as process monitoring, control and optimization. The current state estimate is indispensable for designing the appropriate process inputs to satisfy the operations criteria.

A related problem to filtering is fixed interval smoothing over a fixed set of sequential measurements. It is defined as the estimation of past states with a knowledge of the history of all measurements collected at both past and future sampling instances relative to the smoothed state. Smoothing by nature is an offline task that operates on batches of previously stored measurements. For instance, high uncertainty in the knowledge of the system’s initial condition may cause slow convergence of the filtered estimates to the true state. Smoothing can potentially help in correcting the poor convergence as well as establishing the initial condition to a higher degree of accuracy. Postprocessing streams of audio and video data are representative applications of smoothing algorithms [Prado et al., 1999, Vermaak et al., 2002, Okuma et al., 2004]. Smoothing is applied for computing the arrival cost parameters in moving horizon estimation [Rao and Rawlings, 2002].

The conditional pdfs of the filter and the smoother provide complete statistical information from which representative state estimate are drawn. Unfortunately, for general nonlinear systems the conditional pdfs are non-Gaussian and they are continually translated, distorted and spread over the state space. It is impossible to determine conditional pdfs that cannot be parameterized in finite forms [Kushner, 1967, Tanazaki, 1996]. Closed-form recursive solutions exist only for linear Gaussian systems in the form of the Kalman filter and the companion Rauch-Tung-Striebel (RTS) smoother [Rauch et al., 1965]. Suboptimal estimation using Taylor series approximations of the nonlinear functions is employed in the extended Kalman filter and smoother [Jazwinski, 1970].

The theoretical solutions to filtering and smoothing problems have been known for decades [Ho and Lee, 1964]. In recent years several numerical implementations are reported that circumvent the need for functional approximations and rely instead on various approximations of non-Gaussian densities. The sequential Monte Carlo strategies provide several particle filters [Gordon et al., 1993, Arulampalam et al., 2002] and particle smoothers [Godsill et al., 2004, Briers et al., 2010]. Methods based on numerical integration on state space grids are known as grid based filters [Kitagawa, 1987, Kramer and Sorenson, 1988, Bucy and Senne, 1971, Terwiesch and Agarwal, 1995]. A related smoother uses Gaussian-sum approximations of the conditional densities [Kitagawa, 1994]. These methods are classified into two types: (1) the forward-filter-backward-smoother (FFBS) strategy and (2) two-filter-smoother (TFS) formula. Recently a generalized two-filter recursive smoothing formula is proposed [Briers et al., 2010]. The method of unscented RTS smoother employs the unscented transform to...
realize the evolution of the moments of pdfs while retaining certain Gaussian assumptions of the RTS smoother [Särkkä, 2008]. All these probability density approximation methods entail computational cost in the form of function calls to make large number of system predictions or the transformation of the state space grids at each sampling instance, which may be sometimes repetitive.

Estimation in cell space is a combination of grid based approaches and Monte Carlo sampling based approaches [Ungarala et al., 2006, 2008]. The state space is approximated as a finite collection of discrete hypervolumes known as cells. The system dynamics in state space are approximated by coarse grained cell to cell stochastic transitions that are governed by an aggregate Markov chain. A modeling task converts the state transition equation into a discretized cell probability transition operator using Monte Carlo simulations known as cell-to-cell mapping [Hsu, 1987]. The one-time modeling task relieves the grid or cell space based methods from much of the computational load during filtering. The filtering task consists of recursively propagating the filtering density as a piecewise constant function, without numerous calls to the state transition function in state space.

In this paper the cell filter is extended to the cell smoother approach. Both the FFBS recursions and the TFS recursions are shown to be achieved in cell space on piecewise constant densities, which is expected to reach the accuracy of sequential Monte Carlo methods at a reduced computational demand. Applications of estimation in cell space is unfortunately limited to low dimensional problems because the demands on storage quickly outweigh the computational efficiency due to the “curse of dimensionality” encountered in dynamic programming problems [Bellman and Kalaba, 1965]. However limited, the grid based methods still serve a useful purpose by benchmarking the solutions even for low-dimensional systems for comparing various estimation algorithms.

The rest of the paper is organized as follows. The decision theoretic solutions to nonlinear/non-Gaussian filter and smoother are outlined. The numerical implementation of the solutions in cell space on piecewise constant density is known as the filtering probability density from which an aggregate Markov chain. A modeling task converts the state in state space are approximated by coarse grained cell to cell stochastic transitions that are governed by a linear integral operator known as the Chapman-Kolmogorov equation with the help of the state transition probability density $p(x_{k}\mid x_{k-1})$.

More generally, this temporal evolution of the state probability density is caused by a linear integral operator known as the Fokas operator $P_f$ [Lasota and Mackey, 1994]

$$p(x_{k}\mid y_{1:k-1}) = P_f[p(x_{k-1}\mid y_{1:k-1})],$$

which operates on general Lebesgue measures subject to arbitrary nonsingular transition functions $f$ in state space. The implicit Fokas operator is defined as

$$\int P_f[p(x_{k-1}\mid y_{1:k-1})] = \int \int p_w(w_{k-1}) \, dw_{k-1} \times p(x_{k-1}\mid y_{1:k-1}) \, dx_{k-1}. $$

In some nonlinear cases, it is possible to implement the Fokas operator with simplifying assumptions of the state transition function $f$ being monotonic, nonsingular and continuously differentiable. However, closed-form computation of $P_f$ can be generally cumbersome or impossible to perform when the inverse map $f^{-1}$ possesses complicated and discontinuous geometry.

2. STATE ESTIMATION

A general dynamic system is represented by its state vector $x_k \in \mathbb{R}^n$ and the associated vector of measurements $y_k \in \mathbb{R}^m$ at the current time instance $k$. The dynamics of the state are modeled by a discrete-time stochastic forward mapping equation and the measurements are given as functions of the states corrupted with noise,

$$x_k = f(x_{k-1}, w_{k-1}), \quad y_k = h(x_k, v_k),$$

where $f: (\mathbb{R}^n \times \mathbb{R}^m) \to \mathbb{R}^n$ and $h: (\mathbb{R}^n \times \mathbb{R}^m) \to \mathbb{R}^m$ are vector valued nonlinear functions. The system noise $w_k \in \mathbb{R}^m$ and the measurement noise $v_k \in \mathbb{R}^m$ are uncorrelated white noise sequences with known probability density functions (pdfs) $p_w(w_k)$ and $p_v(v_k)$ respectively. The initial condition of the state $x_0$, uncorrelated with the noise sequences, is characterized by a known pdf $p(x_0)$. The noise sequences are not necessarily additive and all pdfs are generally considered to be non-Gaussian.

The state of the system is typically not directly accessible. As a consequence, the states must be estimated from available measurements. Given a sequence of measurements $y_{1:k} = \{y_1, \ldots, y_k\}$ the state estimation problem involves the determination of the state sequence $x_{1:k} = \{x_1, \ldots, x_k\}$ by computing the joint conditional pdf of the states $p(x_{1:k}\mid y_{1:k})$. This problem is typically posed in a Bayesian framework as [Jazwinski, 1970]

$$p(x_{1:k}\mid y_{1:k}) = \frac{p(x_{1:k}, y_{1:k})}{p(y_{1:k})}. \quad (3)$$

The joint density of the states and measurements $p(x_{1:k}, y_{1:k})$ is factored into a product of the likelihood function $p(y_{1:k}\mid x_{1:k})$, which determines the likelihood of the measurements being related to the states, and the a priori density of the states $p(x_{1:k})$.

2.1 Filtering

Often it is of practical interest in engineering dynamic systems to determine the marginal density of the current state conditioned on all available measurements $p(x_k\mid y_{1:k})$. The marginal density is known as the filtering probability density from which a point estimate is made as representative of the current state. For example, in applications of process monitoring (tracking) and control (guidance) a knowledge of the deviation of the state from its setpoint is useful to design an appropriate control input with the objective of driving the deviation to zero.
The likelihood function in the Bayes equation (4) is determined in closed-form under the restrictive assumption that the measurement function is nonsingular and continuously differentiable,

\[ p(y_k|x_k) = p(y_k|h^{-1}(x_k,y_k)) \frac{\partial h^{-1}}{\partial y_k} = L_h(x_k,y_k). \]  

(8)

Hence, the filtering density is recursively computed by

\[ p(x_k|y_{1:k}) \propto L_h(x_k,y_k)P_f[p(x_{k-1}|y_{1:k-1})], \]  

(9)

where it is noted that the normalizing proportionality constant on the lefthand side is the inverse of the joint density of the measurements \( p(y_{1:k}) \), which is independent of the states.

The conditional mean \( \hat{x}_k \) is a frequently employed point estimate of the current state vector under a minimum variance criterion

\[ \hat{x}_k = \int x_k p(x_k|y_{1:k}) \, dx_k. \]  

(10)

Unfortunately, the conceptually appealing Bayesian decision theoretic approach to filtering is difficult to implement for systems involving arbitrary nonlinear state space maps \( f \) and \( h \) and non-Gaussian pdfs. In special cases, closed-form solutions for the recursion of a finite number of conditional moments are possible, viz., the Kalman filter for the first two moments of Gaussian random variables subject to linear dynamics.

2.2 Smoothing

The smoothing problem is formulated as the estimation of the marginal density of the state at any time instance \( l \) conditioned on all available measurements both before and after that particular time instance, i.e., \( p(x_l|y_{1:k}) \) for \( l = 1, \ldots, k \) and a fixed \( k \). Beginning from the initial condition, each smoothed estimate is conditioned on all future measurement information relative to its time index, except for the last smoothed estimate, which is also the filtered estimate. The conditioning on future information can be expected to bring the smoothed estimates closer to the true states than filtering alone.

Consider the joint conditional density \( p(x_k|y_{1:k}) \) in Eq. (3). The smoothing marginal density is interpreted as a \( k-1 \) integral as follows

\[ p(x_l|y_{1:k}) = \int \cdots \int p(x_k|y_{1:k}) \, dx_k \cdots dx_{k-l} \, dx_{k-l+1} \cdots dx_k. \]  

(11)

However, it is desirable to pose the smoothing density in a recursive formulation for efficient computation similar to the recursion of the filtering density.

The desired recursion is formulated by the following factorization of the joint conditional density [Godsil et al., 2004, Briers et al., 2010]

\[ p(x_k|y_{1:k}) = p(x_k|y_{1:k}) \prod_{l=1}^{k-1} p(x_l|x_{l+1},y_{1:k}), \]

\[ = p(x_k|y_{1:k}) \prod_{l=1}^{k-1} p(x_l|x_{l+1},y_{1:k}), \]

\[ = p(x_k|y_{1:k}) \prod_{l=1}^{k-1} \frac{p(x_{l+1}|x_l)p(x_l|y_{1:k})}{p(x_{l+1}|y_{1:k})}. \]  

(12)

Subsequent integration of the joint conditional density with respect to \( \{x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_k\} \) results in the marginal smoothing density of a particular \( x_l \) in the desired recursive form

\[ p(x_l|y_{1:k}) = p(x_l|y_{1:l}) \prod_{l=1}^{k-1} \frac{p(x_{l+1}|x_l)p(x_l|y_{1:k})}{p(x_{l+1}|y_{1:k})}. \]  

(13)

The above recursive formulation can be written in terms of the Foias operator operating separately twice in each recursive step as follows

\[ p(x_l|y_{1:k}) = p(x_l|y_{1:l})P_f\left( \frac{p(x_{l+1}|y_{1:k})}{p(x_{l+1}|y_{1:l})} \right). \]  

(14)

Algorithmically speaking, a forward sweeping filter in Eq. (9) is implemented first to compute and store the filtering densities \( p(x_l|y_{1:k}) \) for \( l = 1, \ldots, k \), which is initialized with the prior \( p(x_0) \). Subsequently, a backward sweeping smoother is implemented for the smoothing densities via the recursion in Eq. (14), for \( l = k-1, \ldots, 1 \), which is initialized with the last filtered density \( p(x_k|y_{1:k}) \) for a fixed \( k \). This approach is termed as the forward-filter-backward-smoother (FFBS) algorithm.

The FFBS algorithm has a practical difficulty in the sense that the predictive density \( P_f[p(x_{l|1:y_{1:l-1}})] \) in the denominator of the smoother recursion in Eq. (14) is naturally assumed to be nonzero. In numerical approximations, this term can potentially result in indeterminate forms.

The algorithm of two-filter-smoother (TFS) is an alternative to the FFBS algorithm, which avoids the operation of division by a probability density function [Kitagawa, 1994]. The TFS approach is realized by the following factorization of the marginal smoothing density

\[ p(x_l|y_{1:k}) = \frac{p(x_l|y_{1:l-1})p(x_l|x_l)}{p(y_{1:k}|y_{1:l-1})}, \]  

(15)

which is rewritten up to a constant of proportionality as

\[ p(x_l|y_{1:k}) \propto P_f[p(x_{l-1}|y_{1:l-1})]p(y_{1:k}|x_l). \]  

(16)

On the lefthand side of Eq. (15), the first term in the numerator \( p(x_l|y_{1:l-1}) \) is a predictive density determined by a forward sweeping filter through the Foias operator. The second term in the numerator \( p(y_{1:k}|x_l) \) is known as the backward information filter that is realized via a backward recursive filtering scheme. The forward and backward filters are implemented independent of each other and the results are combined to determine the smoothing densities.

The backward information filter possesses the following recursive formula

\[ p(y_{1:k}|x_l) = p(y_{1:l}|x_l) \int p(x_{l+1}|x_l)p(y_{l+1:k}|x_{l+1}) \, dx_{l+1}, \]  

(17)

or in terms of the Foias operator it is written as

\[ p(y_{1:k}|x_l) = p(y_{1:l}|x_l)P_f[p(y_{1:k}|x_{l+1})]. \]  

(18)

Together with Eq. (16), the above recursion formula completes the TFS algorithm. The backward information filter is initialized with \( p(y_{1:k}|x_l) \), i.e., the likelihood function at \( k \) and run backward for \( l = k-1, \ldots, 1 \). The two-filter-smoothing approach also has a practical difficulty because the backward information filter \( p(y_{1:k}|x_l) \) is not a probability density function and it is not guaranteed to be finite. A generalized two filter smoother is available for cases where \( p(y_{1:k}|x_l) \) cannot be determined exactly by the traditional two filter formula [Briers et al., 2010].
3. SMOOTHING IN CELL SPACE

3.1 Preliminaries

Let a finite region of the state space $R \subset \mathbb{R}^n$, be partitioned into a finite number of connected sets called cells, $z = \{z^1, \ldots, z^N\}$. Each cell represents a small hypervolume of a multidimensional state space. All the state space outside the region of interest is denoted by a single infinite sized cell called the sink cell $z^0$. The temporal evolution of the system is approximated by a finite state Markov chain over the cell space in the form of coarse-grained dynamics. Regardless of the dimensionality of the state space, the cell space is considered as one dimensional.

Consider the discretization of a finite region in measurement space $S \subset \mathbb{R}^p$, where measurements of corresponding $x \in \mathbb{R}$ are obtained. Let $S$ be partitioned into a finite set of measurement cells, $\{d^1, \ldots, d^M\}$. A single measurement sink cell $d^0$ represents the infinite measurement space outside $S$.

The marginal filtering density $p(z_k|y_{1:k})$ is approximated as a piecewise constant function or probability mass vector (pmv),

$$p(z_k|d^1_{1:k}) = \begin{bmatrix} m^0_k \\ m^1_k \\ \vdots \\ m^N_k \end{bmatrix} \quad \text{with} \quad \sum_{i=0}^{N} m^i_k = 1. \quad (19)$$

where $m^i_k$ represents the probability mass associated with a cell $z^i$ conditioned on the sequence of observed measurement cells $d^1_{1:k}$ until $k$.

The evolution of the system is a finite state aggregate Markov chain over the cell space. The transition between cells is represented by a dynamic relationship known as cell mapping,

$$z^i_k = F(z^i_{k-1}, w_{k-1}) \quad (20)$$

where $F : (\mathbb{Z}^n \times \mathbb{R}^n) \rightarrow \mathbb{Z}^n$. Transitions from the cells $\{z^1, \ldots, z^N\}$ into the sink cell $z^0$ are considered as terminal. The transitions are governed by the cell transition probability mass $p^{ij}$

$$p^{ij} = \int p(z^i|z^j) dz. \quad (21)$$

It is interpreted as the fraction of the volume in any cell $z^j$ that is mapped into its image cell $z^i$ by the action of the cell map $F$. It is approximately realized by Monte Carlo sampling and simulation as follows,

$$p^{ij} \approx \frac{s^j}{s^i} \quad (22)$$

where $s^j$ is the number of uniformly sampled initial conditions in cell $z^j$ and $s^i$ is the number of mapped images falling in cell $z^i$ due to the action of the point map $f$. The generalized cell mapping method is a systematic means of computing the transition probabilities [Hsu, 1987].

The evolution of the filter pmv can be shown as the following linear transformation,

$$\begin{bmatrix} m^0_k \\ m^1_k \\ \vdots \\ m^N_k \end{bmatrix} = \begin{bmatrix} m^0_{k-1} \\ m^1_{k-1} \\ \vdots \\ m^N_{k-1} \end{bmatrix} \begin{bmatrix} p^{00} & p^{01} & \cdots & p^{0N} \\ p^{10} & p^{11} & \cdots & p^{1N} \\ \vdots & \vdots & \ddots & \vdots \\ p^{N0} & p^{N1} & \cdots & p^{NN} \end{bmatrix} \begin{bmatrix} 0 \\ m^0_{k-1} \\ \vdots \\ m^N_{k-1} \end{bmatrix}. \quad (23)$$

which is a discretized form of the linear integral Foias operator $P_f$, summarized as [Ungarala et al., 2006]

$$p(z_k|d^1_{1:k-1}) = P_F p(z_{k-1}|d^1_{1:k-1}). \quad (24)$$

The cell measurement map corresponding to the state measurement map $h$ is

$$d^i_k = H(z^i_k, v_k), \quad (25)$$

where $H : (\mathbb{Z}^n \times \mathbb{R}^n) \rightarrow \mathbb{Z}^n$. The likelihood of obtaining a measurement cell $d$ when the state cell is $z$ is given by the following cell likelihood mass matrix corresponding to the map $H$,

$$L_H(z, d) = \begin{bmatrix} p^{00} & p^{01} & \cdots & p^{0N} \\ p^{10} & p^{11} & \cdots & p^{1N} \\ \vdots & \vdots & \ddots & \vdots \\ p^{M0} & p^{M1} & \cdots & p^{MN} \end{bmatrix}, \quad (26)$$

where $p^{ij}$ is the likelihood mass with respect to the measurement cell $d^j$ and the state cell $z^i$. It is obtained by integrating over the volume of the data cell using Monte Carlo sampling. Given a measurement cell $d^i_k$ at time $k$, the cell likelihood mass vector (lmv) is presented by the $i$th row in the likelihood matrix $L_H$.

$$p(d^i_k|z) = \begin{bmatrix} p^{i0} \\ p^{i1} \\ \vdots \\ p^{iN} \end{bmatrix}. \quad (27)$$

3.2 Forward Filter

The forward sweeping filter density is numerically approximated by the recursive formulation of the filter pmv as follows

$$p(z_k|d^1_{1:k}) \propto p(d^i_k|z) \odot P_F p(z_{k-1}|d^1_{1:k-1}), \quad (28)$$

where $\odot$ represents the Hadamard product or element by element multiplication of two vectors.

3.3 Backward Smoother

The backward sweeping smoothing density is approximated by the following recursion of the smoothing pmv using the previously computed and stored filter pmv from the forward sweep

$$p(z_k|d^1_{1:k}) = p(z_k|d^1_{1:k}) \odot \left(p(z_{k+1}|d^1_{k}) \odot P_F p(z_{k+1}|d^1_{k+1})\right)^T P_F. \quad (29)$$

where $\odot$ represents element by element division of two vectors. Note that the discretized Foias operator $P_F$ is a numerical approximation which assigns zero transition probability masses to certain cell transitions. This may cause zero probability masses for some cells in the predictive pmv from the forward filter. It requires a small roughening or perturbation of the zero valued elements of the predictive pmv before the element division operation is performed.

3.4 Two-Filter-Smoother

The two filter smoother is numerically implemented by combining the results of a forward sweeping filter and an independent backward information filter

$$p(z_k|d^1_{1:k}) \propto P_F p(z_{k-1}|d^1_{1:k-1}) \odot p(d^i_k|z), \quad (30)$$

where the backward information filter is implemented by the following recursion

$$p(d^i_k|z) = p(d^i_k|z) \odot \left(p(d^i_{k+1}|z_{k+1})\right)^T P_F. \quad (31)$$
Table 1. Parameter set for CSTR model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>100 L/min</td>
</tr>
<tr>
<td>$V$</td>
<td>100 L</td>
</tr>
<tr>
<td>$C_0$</td>
<td>1 gmol/L</td>
</tr>
<tr>
<td>$T_0$</td>
<td>350 K</td>
</tr>
<tr>
<td>$T_r$</td>
<td>305 K</td>
</tr>
<tr>
<td>$k$</td>
<td>$7.2 \times 10^{10}$ L/min</td>
</tr>
<tr>
<td>$E_a/R$</td>
<td>8750 K</td>
</tr>
<tr>
<td>$\Delta H$</td>
<td>$-5 \times 10^3$ J/gmol</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1000 g/L</td>
</tr>
<tr>
<td>$C_p$</td>
<td>0.239 J/kg</td>
</tr>
<tr>
<td>$U$</td>
<td>$5 \times 10^3$ J/cm$^2$ min K</td>
</tr>
<tr>
<td>$A$</td>
<td>10 cm$^2$</td>
</tr>
<tr>
<td>$C_r$</td>
<td>1 gmol/L</td>
</tr>
<tr>
<td>$T_c$</td>
<td>100 K</td>
</tr>
</tbody>
</table>

It should be noted that the smoothing recursions in cell space can be easily implemented with the forward sweeping filter results from other approaches such as the particle filters.

4. SIMULATION EXAMPLE

Consider an adiabatic continuous stirred tank reactor (CSTR) with the reactor concentration and temperature modeled by the following coupled nonlinear ordinary differential equations [Chen et al., 2004].

\[
\frac{dC}{dt} = \frac{q}{V} (C_0 - C) - kCe^{-\frac{T}{T_c}} + \frac{k \Delta H}{\rho C_p} kCe^{-\frac{T}{T_c}} - \frac{UA}{\rho C_p} (T - T_c).
\]

The parameter set is shown in Table 1 and the state variables are normalized with the reference values $C_r$ and $T_r$ respectively.

The system is simulated by numerical integration using the Runge-Kutta 4th-5th order method and sampled at a $0.4$ s interval. The mean squared error (MSE) of estimation for filtering and smoothing is defined as,

\[
MSE = \frac{1}{Kn} \sum_{k=1}^{K} (x_k - \hat{x_k})^T (x_k - \hat{x_k}),
\]

where $K$ is the number of measurements and $n$ is the length of the state vector.

Table 2 lists the computational cost in CPU seconds and the mean squared error averaged over 100 realizations for two different cell sizes. The computation time for the smoother is higher than the filter because it includes the time for the forward filter as well as the time to store and retrieve the results for backward smoothing.

As expected the MSE for the smoother is lower than that of the filter. Fig. 1 shows typical results of the smoother in CSTR state space. The smoother yields estimates closer to the true state and also corrects the initial slow convergence of the filter initialized by poor quality information about the initial condition. The smoothed initial condition is recovered as $[0.52, 3.52]$, close to the true initial condition.

The extended Kalman filter and smoother results are included in the table for comparison. The smoother is the equivalent of the Rauch-Tung-Striebel linear smoother [Rauch et al., 1965]. The large MSE values for both the filter and smoother are a result of the linearization errors in computing the covariance matrices. The smoother did not fare better than the filter in terms of the MSE. The CPU time is considerably high because the Runge-Kutta numerical integration is carried out at each sample time for the filter and the smoother. However, the smoothed initial condition fares better with $[0.52, 3.44]$.

5. SUMMARY

This paper focuses on the problem of state estimation in nonlinear and non-Gaussian systems. Of specific interest here is the fixed interval smoothing of states conditioned on the entire set of available measurements. The problem is important because establishing the initial condition of the state and reassessing the state at any time instance with the hindsight of future information can be useful. For example, belief inference applications in Bayesian learning depend upon smoothing to estimate system parameters. Smoothing is also critical to many applications related to speech and video data processing used in target tracking and localization. The soothing update is a popular method for arrival cost parameters in moving horizon estimation.
The cell filter is a computationally efficient numerical approximation of the theoretical nonlinear/non-Gaussian Bayesian filter, limited to implementation in low dimensional systems due to excessive storage requirements in high dimensions. In this paper, the concept of filtering is extended to smoothing using the forward-filter-backward-smoother recursive method and the two-filter-smoother formula. Both implementations can be used by combining the results of other forward sweeping filters such as the sequential Monte Carlo methods. The ability of the cell smoothing algorithms to correct slow convergence of the cell filter initialized with poor quality prior information is demonstrated with a CSTR simulation case study.

REFERENCES


