Integration of RTO with MPC through the gradient of a convex function

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Abstract: Predictive controllers are usually implemented as part of a hierarchical structure of the process operation where the Real Time Optimization (RTO) and the Model Predictive Control (MPC) are executed in separated layers. Here, it is proposed to use the gradient of a convex function that links the optimizing targets to the MPC controller. The convex function is a $n+1$ dimensional paraboloid of the process inputs, with the minimum corresponding to the input targets resulting from the RTO routine. For this MPC controller, the infinite prediction horizon and terminal state constraint are considered, and it operates according to the zone control strategy. Simulation results of a linear system are provided.

Keywords: Model predictive control, optimizing control, Real Time Optimization.

1. INTRODUCTION

Model Predictive Control (MPC) is an optimization based control algorithm that uses an explicit model to predict the future response of a process system. At each time step, the MPC controller computes an input sequence as a solution to an open-loop optimization problem subject to constraints on inputs and states, but only the first input is injected into the system. In practice, MPC controllers are implemented as part of a multilayer hierarchy of control functions (Ying and Joseph, 1999; Kassman et al., 2000; Tatjewski, 2008; Darby et al., 2011). In this hierarchy, the Real Time Optimization (RTO) lies at a layer above the MPC layer and determines the optimal economic steady-state settings of the process, which can be represented as targets to some or all of the system inputs as well as set-points to some of the system outputs. Then, the MPC controller receives the targets from the RTO layer and must follow them to reach a more profitable operating point. This integration of RTO with MPC is known as two-layer approach. Some studies (Zanin et al., 2002; Biegler and Zavala, 2009; De Souza et al., 2010; Ochoa et al., 2010) have attempted to integrate the RTO layer to the MPC controller, in such a way that both the economic objective and the dynamic regulation are solved in one single layer. In several cases, the zone control strategy is considered when MPC receives optimizing targets from an upper layer (Gonzalez and Odloak, 2009; Ferramosca et al., 2010) or when the economic objective is integrated to the MPC controller (Zanin et al., 2002; De Souza et al., 2010).

In this strategy, the aim of the MPC layer is not to guide the outputs to exact values or set-points, but only to maintain them inside appropriate ranges or zones.

One of the main issues in the study of MPC strategies is the stability of the closed loop system. A popular approach to obtain a stable MPC consists of adopting an infinite prediction horizon (Rawlings and Muske, 1993). For stable systems, the infinite-horizon open-loop objective function is reduced to a finite-horizon objective by defining a terminal state penalty, which is obtained from the solution of a Lyapunov equation. The stability of this controller was demonstrated for the regulator case of systems with constraints on the inputs and states and where the system steady-state lies at the origin. Odloak (2004) developed a version of the stable MPC which is suitable for implementation. Considering a prediction model in the incremental form, this strategy can deal with the output offset. The resulting optimization problem includes a terminal state constraint which is softened by adding slack variables. The stability of the MPC that receives optimal targets from an upper layer has also been studied. Gonzalez and Odloak (2009) presented an MPC controller with nominal stability that receives input targets directly from the RTO layer and operates according to the zone control. Alvarez and Odloak (2010) proposed a stable algorithm for the integration of RTO and MPC which includes an intermediary layer for target recalculation.

The objective of this paper is to develop a stable MPC controller which uses the gradient of a convex function of the
inputs to link the RTO targets to the MPC. An infinite prediction horizon and terminal state constraint are adopted as well as the zone control strategy. The paper is organized as follows. In section 2 we present a previous MPC controller which incorporates the gradient of the economic function to solve the RTO and MPC problem in one-single layer. Then, the proposed MPC is presented in the section 3. Next, the simulation results are discussed for a linear system. Finally, the conclusions are presented.

2. THE MPC WITH ECONOMIC GRADIENT

Aiming to integrate RTO with MPC in one single layer, De Souza et al. (2010) proposed to include the gradient of the economic function as an additional term in the objective function of the MPC. This approach incorporates the economic objective into the MPC controller such that the RTO and MPC are solved in a single optimization routine, under the structure of one single layer. Consider a multivariate system with \( nu \) inputs. The economic function associated with the predicted steady-state can be represented as follows:

\[
F_e = f_{eco}(u) \tag{1}
\]

If the vector that represents the control action is changed to \( u + \Delta u \), the first order approximation to the gradient of the economic function can be represented as follows:

\[
\mathcal{Z}_{u+\Delta u} = \frac{dF}{du} \bigg|_{u+\Delta u} = \frac{dF}{du} + \frac{d^2F}{du^2} \Delta u \tag{2}
\]

The above equation can be represented as follows:

\[
\mathcal{Z}_{u+\Delta u} = d + G\Delta u \tag{3}
\]

where \( \Delta u = u(k + m - 1) - u(k - 1) \) is the total move of the input vector, \( m \) is the control horizon of the MPC controller, \( d \) is the gradient vector at the present time and \( G \) is the Hessian of the economic function with respect to the inputs. The gradient vector \( \mathcal{Z}_{u+\Delta u} \) can be considered as a deviation vector in relation to the reference, which is equivalent to considering that the gradient of the economic function is zero at the optimum. So, if bringing this vector to zero is considered as one of the controller objectives, it will have an economic component. Therefore, the set of error equations represented in equation (3) can be included in the objective function of the MPC.

In order to minimize the gradient of the economic function in the control problem, De Souza et al. (2010) proposed to solve the following optimization problem:

**Problem P1:**

\[
\min_{\Delta u_k} V_k = \sum_{j=1}^{m} \left[ y(k+j) - y_{sp,k} \right]^T Q \left[ y(k+j) - y_{sp,k} \right] + \sum_{j=0}^{m-1} \Delta u(k+j) \Delta u(k+j) + \mathcal{Z}_{u+\Delta u}^T \mathcal{R} \mathcal{Z}_{u+\Delta u} + \mathcal{Z}_{e+\Delta e}^T \mathcal{P} \mathcal{Z}_{e+\Delta e} \tag{4}
\]

subject to:

\[
-\Delta u_{max} \leq \Delta u(k+j) \leq \Delta u_{max}, \quad j = 0, 1, \ldots, m-1 \quad (5)
\]

\[
\Delta u(k+j) = 0, \quad j \geq m \quad (6)
\]

\[
u_{min} \leq u(k+j) \leq u_{max}, \quad j = 0, 1, \ldots, m-1 \quad (7)
\]

using for prediction a state-space model in the incremental form:

\[
x(k+1) = Ax(k) + Bu(k) \quad (8)
\]

\[
y(k) = Cx(k) \quad (9)
\]

The weights \( Q, R \) and \( P \) are positive definite matrices of appropriate dimensions. Notice that this is a conventional MPC that includes a term that integrates the economic objective in the optimization problem, the main idea of this strategy is to minimize the gradient of the economic function \( \mathcal{Z}_{e+\Delta e} \) in order to solve the economic optimization and the control problem in one single layer.

De Souza et al. (2010) showed several advantages of this approach, one of them is the reduction in the computational load as Problem P1 maintains the form of a QP. A successful application of this approach to an industrial distillation system in an oil refinery in Brazil is reported in Porfírio and Odloak (2011).

It can be observed that the method proposed by De Souza et al. (2010) requires the economic function of the process to be convex (in case of minimizing) or concave (in case of maximizing), such that the optimum will be reached where the gradient is equal to zero, or the gradient is minimum. For some processes this condition cannot be assured, for example the economic function of the styrene reactor considered in Alvarez (2012) has the shape of a saddle in the region around the nominal operating point. Then for this process the method described in this section cannot be applied. Even for highly nonlinear processes that present the convexity condition at the operating region, the presence of disturbances can change the shape of this region. In this case, a disturbance can send the operating point to a neighborhood where the economic function has more than one stationary point with minimums and maximums. Notice that the economic function is calculated from the states of the rigorous process model, which is nonlinear in most cases.

A second issue about this method lies on the stability. The on-line calculation of the gradient can turn the closed-loop system unstable. The vector \( d \) and the matrix \( G \), in a dynamic environment, change at each sampling time as defined by the rigorous model, and the controller may not take into account these changes. Thus, for the MPC controller, the term \( \mathcal{Z}_{u+\Delta u}^T \mathcal{P} \mathcal{Z}_{e+\Delta e} \) in the objective function can act as a disturbance and turn it unstable.

In the section 3 we present a two-layer approach to integrate RTO with MPC. This approach is based on the gradient minimization, but instead of the economic function we propose a convex target function, assuming that an upper RTO sends optimal targets for the inputs. It will be shown
that it guarantees both economic optimality and stability of the closed-loop system.

3. STABLE MPC WITH GRADIENT OF A CONVEX FUNCTION

Let us assume that the RTO is executed in an upper layer and the solution of this layer produce \(nu\) optimal targets for the inputs denoted as \(u_{i,RTO}\) where \(i = 1,2,\ldots,nu\). We define the following convex function:

\[
F_j(u(k)) = \sum_{i=1}^{nu} K_{ii}(u_i(k) - u_{i,RTO})^2
\]  \hspace{1cm} (10)

where \(u(k) = [u_1(k) \ldots u_{nu}(k)]\) is the input vector. The gradient of this function can be calculated from (2) and (3) and then the vector \(d\) and the matrix \(G\) are introduced into an infinite horizon MPC controller. In the section 3.1 the structure of the prediction model is described. Section 3.2 describes the development of the stable MPC with gradient of a convex function of the targets.

3.1. Prediction model

Consider a stable system with \(nu\) inputs and \(ny\) outputs, and assume that the poles relating \(u_i\) to any output \(y_j\) are non-repeated. For this kind of system, Odloak (2004) considers the following state-space model that is suitable to the implementation of MPC:

\[
\begin{bmatrix}
\dot{x}'(k+1) \\
\dot{x}^d(k+1)
\end{bmatrix} =
\begin{bmatrix}
I_{ny} & 0 \\
0 & F
\end{bmatrix}
\begin{bmatrix}
x'(k) \\
x^d(k)
\end{bmatrix} +
\begin{bmatrix}
D^b \\
D^d FN
\end{bmatrix} \Delta u(k)
\]  \hspace{1cm} (11)

\[
y(k) = [I_{ny} \ \psi'] \begin{bmatrix}
x'(k) \\
x^d(k)
\end{bmatrix}
\]  \hspace{1cm} (12)

where

\[
\Delta u(k) = u(k) - u(k-1)
\]

\[
x' = [x_1 \ \cdots \ x_{ny}]^T, \ x' \in \mathbb{R}^{ny},
\]

\[
x^d = [x_{ny+1} \ x_{ny+2} \ \cdots \ x_{ny+nd}]^T, \ x^d \in \mathbb{C}^{nd},
\]

\[
F \in \mathbb{C}^{nd \times nd},
\]

\[
\psi = \begin{bmatrix}
\Phi \\
0
\end{bmatrix}, \ \psi \in \mathbb{R}^{ny+nd},
\]

\[
\Phi = [1 \ \cdots \ 1], \ \Phi \in \mathbb{R}^{nu,nu}
\]

\(F, \ D^b\) and \(D^d\) are obtained from the step response model. Observe that the input in the model defined by (11) and (12) is \(\Delta u(k)\), which means that the output integrates the input. Then, in the model defined by (11) and (12), the state vector can be split in two components: \(x'\) that corresponds to the integrating poles produced by the incremental form of the model, and \(x^d\) that corresponds to the system modes. It can be shown that \(x'\) is equal to the predicted output steady-state and component \(x^d\) corresponds to the stable modes of the system. When the system approaches to the steady-state, component \(x'\) tends to zero. \(F\) is a diagonal matrix with components of the form \(e^{rf}\), where \(r_i\) is a pole of the system and \(T\) is the sampling period. It is assumed that the system has \(nd\) stable poles and \(D^d\) is the gain matrix of the system. To build up matrix \(\Phi\), it is also assumed that \(na\) is the number of poles associated to any input \(u_i\) and any output \(y_j\).

3.2. MPC formulation

The model defined by (11) and (12) is considered for prediction in the following MPC optimization problem (Odloak, 2004):

Problem P2:

\[
\min_{\Delta u, \delta, \delta_i} V_i =
\sum_{j=0}^{m-1} \left( y(k+j|k) - \delta_{sp,j} \right)^T \underbrace{Q}_{ny \times ny} \left( y(k+j|k) - \delta_{sp,j} \right)
+ \sum_{j=0}^{m-1} \Delta u(k+j|k)^T R \Delta u(k+j|k) + \delta_i^T S \delta_i
\]  \hspace{1cm} (13)

subject to (5), (6), (7) and

\[
x'(k+m) - \delta_{sp,j} = 0
\]  \hspace{1cm} (14)

where \(Q, R\) and \(S\) are positive weighting matrices of appropriate dimensions. The slack variables \(\delta_{sp,j} \in \mathbb{R}^{ny}\) are intended to improve the feasibility of the equality constraint (14), which forces the output error to be small or null. This MPC controller considers an infinite prediction horizon. In order to avoid the cost \(V_i\) to be unbounded, the first term of the objective function (13) is split in two terms, the first one is a finite sum from \(j=0\) to \(m-1\), and the second one, which is an infinite sum from \(m\) to \(+\infty\), is reduced through the terminal constraint (14), and can be written in terms of the component \(x^d\) of the state at \(k+m\):

\[
V_i = \sum_{j=0}^{m-1} \left( y(k+j|k) - \delta_{sp,j} \right)^T \underbrace{Q}_{ny \times ny} \left( y(k+j|k) - \delta_{sp,j} \right)
+ x^d(k+m)^T \underbrace{Q}_n x^d(k+m)
+ \sum_{j=0}^{m-1} \Delta u(k+j|k)^T R \Delta u(k+j|k) + \delta_i^T S \delta_i
\]  \hspace{1cm} (15)

where \(\underbrace{Q}_n\) is calculated off-line from the Lyapunov equation for the discrete time system defined by the equations (11) and (12):

\[
\underbrace{Q}_n - F^T \underbrace{Q}_n F = F^T \psi^T \underbrace{Q}_n \psi F
\]  \hspace{1cm} (16)

The controller defined by Problem P2 has guarantee of stability and recursive feasibility is also assured. The weight \(S\) should be large enough to assure the convergence of the process to the desired steady-state. In order to integrate the RTO targets to the stable MPC, the gradient of the function defined in (10) is introduced as a
quadratic term into the objective function (13), corresponding to a stable MPC controller. Then (13) can be written as follows:

\[ V_k = \sum_{j=0}^{m} (y(k+j | k) - y_{sp,k} - \delta_{s,k})^T Q (y(k+j | k) - y_{sp,k} - \delta_{s,k}) + \sum_{j=0}^{m-1} \Delta u(k+j | k)^T R \Delta u(k+j | k) + (d_t + G \Delta \pi_k)^T P (d_t + G \Delta \pi_k) + \delta_{s,k}^T S \delta_{s,k} \]

(17)

where \( \Delta \pi_k = (u(k+m-1 | k) - u(k-1 | k)) \)

As the MPC controller is to follow targets for the inputs, the zone control strategy was adopted for the outputs (Gonzalez and Odloak, 2009). It consists on considering the output set-points \( y_{sp} \) as variables of the optimization problem constrained to upper and lower bounds that define the output zone \([y_{min}, y_{max}]\). Then the output set-point becomes a variable of the control problem that can be written as follows:

**Problem P3:**

\[
\min \ V_k = \sum_{j=0}^{m} (y(k+j | k) - y_{sp,k} - \delta_{s,k})^T Q (y(k+j | k) - y_{sp,k} - \delta_{s,k}) + \sum_{j=0}^{m-1} \Delta u(k+j | k)^T R \Delta u(k+j | k) + (d_t + G \Delta \pi_k)^T P (d_t + G \Delta \pi_k) + \delta_{s,k}^T S \delta_{s,k} \\
\text{subject to (5) to (7), (14) and} \\
y_{min} \leq y_{sp,k} \leq y_{max} \\
\]

(18)

(19)

The controller described by Problem P3 has guarantee of stability since it is derived from the Problem P2, which is stable (Odloak, 2004), and the inclusion of the gradient term does not interrupt the convergence of the objective function. As the convex function does not depend on the system states, the computation of \( d_k \) is not affected by disturbances.

**Convergence of problem P3:** Notice that if there is an optimal solution of the problem P3 at time step \( k \):

\[
\Delta \pi_k^* = \begin{bmatrix} \Delta u^* (k | k)^T & \ldots & \Delta u^* (k+m-1 | k)^T \end{bmatrix}, y_{sp,k}^*, \delta_{s,k}^* \}
\]

for the undisturbed system, a feasible solution exists at time step \( k+1 \):

\[
\begin{bmatrix} \Delta \pi_k^* + \Delta \pi_{k+1} \end{bmatrix} = \begin{bmatrix} \Delta u^* (k+1 | k)^T & \ldots & \Delta u^* (k+m-1 | k)^T \end{bmatrix}, y_{sp,k+1}^*, \delta_{s,k+1}^* = \delta_{s,k}^* 
\]

The calculation of the cost corresponding to each solution leads to the following inequality:

\[
V_k^* - V_{k+1}^* \geq 0
\]

The optimal value of the objective function at time \( k+1 \) follows that \( V_k^* \leq V_{k+1}^* \). Then we observe that the successive costs of problem P3 are strictly decreasing \( V_k^* \leq V_{k+1}^* \) and for a large enough time \( k \) follows that \( V_k^* = V_{k+1}^* \). This means that the objective function converges and that at steady-state the conditions \( y(k | k) - y_{sp,k} = \delta_{s,k}^* \) and \( \Delta u(k | k) = 0 \) should hold.

The details of the proof of convergence of problem P3 are presented in Alvarez (2012).

The controller defined by problem P3 does not suffer from the same drawbacks as the one-layer strategy defined by problem P1. On one hand, the convexity of the function is assured by the convex target function and on the other hand the stability is guaranteed. The MPC with gradient of a convex target function was successfully tested in simulation for a nonlinear styrene reactor, see Alvarez (2012). It is also important to remark that the method proposed in this paper can be generalized to convex functions with different structures, since the paraboloid presented in (10) is one possible convex function of the targets. In comparison to the integration proposed in Alvarez and Odloak (2010), this strategy is solved in two layers. Note also that there are less constraints and optimization variables, and for the method proposed here there is no need to force the inputs to follow the targets with a constraint. The structure of the MPC controller is fixed for different optimization scenarios, since the structure of the MPC controller depends only on \( d \) and \( G \).

**4. SIMULATION RESULTS**

The results presented here are not intended to compare the strategy of De Souza et al (2010) to the one presented in section 3, but to show the ability of the MPC integration in two layers to follow the targets, which are assumed to be the solution of an upper RTO layer, even in presence of disturbances. The process system considered here is part of the FCC system presented in Sotomayor and Odloak (2005). The FCC subsystem has two inputs and three outputs, the manipulated input variables are the flow rate of air to the catalyst regenerator \( u_t \) and the opening of the regenerated catalyst \( u_{12} \). The controlled outputs are the riser temperature \( y_1 \), the regenerator dense phase temperature \( y_2 \), and the regenerator dilute phase temperature \( y_3 \).

The transfer function model is represented as follows:

\[
\begin{bmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{bmatrix} = \begin{bmatrix} 0.45 & 0.20 \\ 2.98s+1 & 1.71s+1 \\ 1.5 & 0.19s - 3.81 \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \]

The constraints are \( u_{max} = [250 101], u_{min} = [75 25], \Delta u_{max} = [25 25], y_{max} = [570 800 900], y_{min} = [490 550 550] \). The
tuning parameters considered are $m = 3$, $Q_y = \text{diag}(1 1 1)$, $R = \text{diag}([0.05 0.05])$, $S = 10^4 \cdot \text{diag}(1 1 1)$, $P = \text{diag}(1 1)$. The convex function has the following parameters: $K_{u1} = 5$; $K_{u2} = 5$. The initial conditions for the simulation are: $y_0 = [549.5 704.3 690.6]$ and $u_0 = [230.6 60.2]$; and the RTO targets are initially: $u_{\text{RTO}} = [220 40]$.

At first, the process starts at non-optimal conditions and the inputs have to reach their targets. Figure 1 shows the outputs, which are driven to a new steady-state inside the zone. Figure 2 shows that the inputs tend to reach the targets in a short period of time. In Figure 4 it is observed that the components of the gradient vector tend to zero quite rapidly. Then, at $t=80\text{min}$, an unmeasured disturbance is introduced in the system. The disturbance corresponds to a decrease of 30 ton/h in $u_1$ and an increase of 30% in $u_2$. This disturbance remains until $t=100\text{min}$, when the inputs recover the values calculated by the controller. At this point, the second and third outputs have been pushed to outside the control zones, as can be seen in Figure 1. But as the inputs reach their targets, the outputs are moved back to inside the respective zones. A temporary increase in the objective function can also be seen in Figure 3. When the controller stabilizes the process, changes in the input targets are introduced at time $t=170\text{h}$. Observe that both inputs approach their targets very closely but output $y_3$ reaches its minimum bound and for this reason the two inputs do not reach their targets exactly. Outputs $y_1$ and $y_2$, are kept inside their zones.

Finally, at $t=280\text{h}$, the zone of the third output is narrowed as the lower bound changes from 550K to 600K. It can be observed that the controller is able to adjust the output to this new value while both inputs stabilize at a minimum distance to the target values. This means that for this output zone, the input targets are not reachable, and the controller avoids overpassing the output zone by adjusting the inputs to the closest possible values. Figure 3 shows that the objective function value is different to zero because the gradient
component corresponding to the second input remains different from zero, as shown in Figure 4.

Fig. 4. Gradient vector components of the convex function when linear system is controlled by the stable MPC with gradient.

5. CONCLUSIONS

In this paper an MPC controller with an economic target represented by the gradient of a convex function is presented. The idea of minimizing the gradient of a function as a quadratic term in the MPC objective function comes from the work of De Souza et al. (2010) that proposes to include the gradient of the economic function in the MPC controller resulting in a one-layer structure. This approach has a lot of potential but issues as convexity and stability should be studied, as discussed at the end of section 2. Then, for the integrating strategy presented in section 3, instead of incorporating the economic function, it is considered a convex function that links the MPC to the optimizing targets, resulting in a two-layer integration. The convex function is a $n+1$ dimensional paraboloid of the process inputs, with the minimum corresponding to the input targets resulting from the RTO layer. The controller formulation is also based on the infinite horizon MPC presented by Odloak (2004), which is stable. The outputs are controlled by zones instead of fixed set-points. The resultant MPC controller is stable for the nominal system. The convergence of the MPC controller is remarked in section 3. Simulation results of the application of the two-layer strategy to an FCC subsystem showed good performance of this controller for changes in the target values and the output zones. Future work could be focused on the extension of this controller to systems with uncertainty in the model parameters, to produce a more robust integration.

REFERENCES


