Tube based Quasi-min-max Output Feedback MPC for LPV Systems

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Abstract: Output feedback model predictive control of polytopic linear parameter varying (LPV) systems subject to input constraints is considered in this paper. The proposed control scheme incorporates robust observer and robust state feedback control. To handle disturbances and model uncertainty, disturbance invariant tube and quasi-min-max MPC are combined to achieve recursive feasibility and robust stability. Simulation results are provided to verify the effectiveness of the proposed algorithm.

Keywords: LPV systems, Constrained control, Quasi-min-max MPC, Disturbance invariant tube.

1. INTRODUCTION

Model predictive control (MPC), originated in 1970s, has made great impact in both industries and academia (Mayne [2000]). It explicitly uses a model of the plant to predict future system response, and decides control inputs by online optimization to minimize a certain performance index. As its name suggests, models play an important role in MPC controller design. Various model structures have been adopted for MPC design, such as finite impulse response model, LTI state space model and many others. To handle model uncertainty and/or nonlinearity, linear parameter varying (LPV) systems have been studied since the late 1990s (Besselmann [2010]).

For LPV systems, the time varying parameters are measured online, but their future evolution is uncertain, and contained in a prescribed bounded set. Quasi-min-max MPC (Lu [2000]) was proposed by extending robust MPC approach in Kothare [1996] from polytopic systems to LPV systems. It assumes that system state is exactly measurable. However, only output variables are available in practice generally. Thus, output feedback control is necessary for real applications of such algorithms. The common approach for output feedback MPC implementation adopted in industry is to combine the state observer and state feedback control. But the problem is not trivial since MPC controller plus state estimator which are designed separately cannot guarantee the closed loop stability, which is consistent with the general result for nonlinear systems (Atassi [1999]). In the literature, there are few reported works on output feedback MPC design. An off-line robust output feedback MPC was proposed in Wan [2002], which exploits a cut-and-try procedure. It also assumes that the time varying parameter will converge eventually, which may not be satisfied in practice. A parameter-dependent dynamic output feedback controller was proposed in Ding [2010], which synthesizes the control law via convex optimization online. The closed loop stability was specified by quadratic boundedness. However the resultant optimization is complex, which may limit its application only to low order systems. Recently, an output feedback MPC based on quasi-min-max algorithm has been proposed in Park [2011]. However, the stability is not rigorously proven, since the interaction between observer and controller is not addressed. Therefore, stabilizing output feedback MPC for LPV systems with acceptable computational complexity is needed. Moreover, a nominal LPV model may be not enough to describe the real behavior of industrial systems since model uncertainty and external disturbances are inevitable.

An output feedback MPC combining Tube-based Robust MPC and quasi-min-max algorithm is proposed in this paper. The disturbances and measurement noises are included in the LPV model, which make the model assumption more realistic. The disturbance invariant tube approach for LTI systems is advocated in Mayne [2005] due to its simplicity to deal with external disturbances. Recently Tube MPC approach has been extended to nonlinear systems (Rakovic [2011], Cannon [2011]). For LPV systems discussed here, the disturbance invariant tube is used to bound disturbance-like response, while quasi-min-max approach is adopted to handle model parametric uncertainty and to regulate the center trajectory of the tube. If the initial state estimation error, disturbances and measurement noises are small enough, the proposed approach can robustly stabilize the system if the online optimization is feasible initially. The state will eventually converge to a neighborhood of the origin. The computational complexity is similar to it of the state feedback Quasi-min-max algorithm (Lu [2000]).

The rest of the paper is structured as follows. In section 2, the problem considered is defined; observer design is

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explained in section 3; section 4 shows the use of tube in controller design; the online optimization is formulated in section 5 and theoretical properties are analyzed; numerical examples are provided in section 6; finally, conclusions are drawn in section 7.

Notation: Given two sets $A$ and $B$, then $A \oplus B = \{a + ba \in A, b \in B\}$ (Minkowski sum) and $A \otimes B = \{ba \in A \subset A\}$ (Pontryagin difference). Given set $E \subset R^n$, $M \in R^{m \times n}$, the linear mapping $F$ of $E$ via $M$ is defined as $F = \{y \in R^n | \exists x \in E, y = Mx\}$. The symbol $*$ is used for convenience to denote
\[
\begin{bmatrix}
M \\
N \\
H
\end{bmatrix} = \begin{bmatrix}
M^T \\
N^T
\end{bmatrix}.
\]

2. PROBLEM STATEMENT
The discrete-time system to be controlled is described by
\[
\begin{align*}
x(k+1) &= A(k)x(k) + B(k)u(k) + w(k), \quad (1a) \\
y(k) &= Cx(k) + v(k), \quad (1b)
\end{align*}
\]
where $x \in R^n$ is the state, $u \in R^m$ is the control input, $y \in R^p$ is the measurement; $w \in R^q$ and $v \in R^r$ are persistent disturbances and measurement noises, respectively, and $w$ and $v$ are bounded by polytopic set as $w \in W$ and $v \in V$. $A(k)$ and $B(k)$ are time varying matrices with appropriate dimensions, which are assumed to be exactly known at the current moment, and unknown but vary in the convex hull $\Omega = Col([A_1|B_1|, \ldots, [A_J|B_J|])$ in the future as
\[
[A(k)|B(k)] = \sum_{j=1}^J [A_j|B_j] \lambda_j(k),
\]
where $\sum_{j=1}^J \lambda_j(k) = 1$, and $\lambda_j(k) \ge 0$ for $1 \le j \le J$.

In addition, the input signals are bounded as
\[
|u_j(k)| \le u_j, 1 \le j \le m, k \ge 0,
\]
which are decided by the actuator limits. Output constraints may be considered in the same way as input constraints.

Therefore the uncertainty of the system (1) includes the unknown time varying system dynamic, external disturbances and sensor noise. Due to various non-vanishing uncertainties, it is difficult to achieve asymptotic stability. The objective of the controller is then to stabilize the system and to steer the state to the neighborhood around the origin, while satisfying the input constraints (3). Since only output variables are available, the state observer + state feedback controller scheme, commonly used, is adopted.

3. OBSERVER DESIGN
The following Luenberger state observer is adopted to estimate the state of system:
\[
\dot{x}(k+1) = A(k)x(k) + B(k)u(k) + L_e(y(k) - C\tilde{x}(k)), (4)
\]
where $\tilde{x}(k)$ is the estimation of $x(k)$, and $L_e$ is the observer gain. The estimation error is defined as $e(k) = x(k) - \tilde{x}(k)$, and its dynamic is described by
\[
e(k+1) = (A(k) + L_eC)e(k) + w(k) + L_ev(k). \quad (5)
\]
The error dynamic varies within a polytopic set, which is designed to guarantee stability by proper choosing the observer gain $L_e$ (here the stability is defined for the disturbance-free part of (5)). Another possibility is parameter-dependant observer gain as
\[
L_e(k) = \sum_{j=1}^J L_{e,j}\lambda_j(k). \quad (6)
\]
The observer gains $L_{e,j}, 1 \le j \le J$ are to satisfy LMI conditions in the following lemma.

Lemma 1. If there exists $P > 0$ and $Y_j = PL_{e,j}$ satisfying the following LMI constraints
\[
\begin{bmatrix}
P & P \\
PA_j + Y_jC & P
\end{bmatrix} \ge 0, \quad 1 \le j \le J,
\]
the observer gain $L_e = \sum_{j=1}^J \lambda_j(k)L_{e,j} = \sum_{j=1}^J \lambda_j(k)P^{-1}Y_j$ and the observer dynamic (5) is stable.

Proof. The observer dynamic with parameter-dependant observer gains is
\[
e(k+1) = \sum_{j=1}^J \lambda_j(k)\Phi_{o,j}e(k) + \tilde{w}(k), \quad (8)
\]
where $\tilde{w}(k) = w(k) + \sum_{j=1}^J L_{e,j}\lambda_j(k)v(k)$ and $\Phi_{o,j} = (A_j + L_{e,j}C)$. Since dynamic (8) is linear, the stability is analyzed on the nominal system without disturbances:
\[
c(k+1) = \Phi_{o}e(k) = \sum_{j=1}^J \lambda_j(k)\Phi_{o,j}e(k). \quad (9)
\]
The quadratic stability condition for the error dynamic is equivalent to
\[
P - \Phi_{o}^*P\Phi_{o} > 0. \quad (10)
\]
By using the Schur complement, it can be rewritten as
\[
\begin{bmatrix}
P & * \\
* & P
\end{bmatrix} > 0
\]
\[
\begin{bmatrix}
P & * \\
* & P\Phi_{o,j}^*P
\end{bmatrix} > 0, \quad 1 \le j \le J. \quad (12)
\]
With the introduction of new variables $Y_j = PL_{e,j}$, (7) is obtained. The corresponding observer gains $L_{e,j} = P^{-1}Y_j, 1 \le j \le J$.

For simplicity, the constant gain formulation is adopted in following sections. Since the estimation error dynamic is stable and driven by bounded external disturbances, it is possible to characterize its bound as in Mayne [2005].

Definition 1 (Mayne [2000]): A set $Z$ is robustly positive invariant for the discrete time system $x^{+} = f(x,w)$, where $w \in W$, if $f(x,w) \in Z$ for all $x \in Z, w \in W$.

The minimum robustly positive invariant set $E$ or its outer approximations can be calculated by approaches proposed in Kouramas [2005], Sui [2006] or Yu [2010]. Let us assume
that the initial estimation error $e(0) \in E$, which implies that

$$e(k) \in E, k \geq 0.$$  
(13)

Thus $E$ is considered as a bound of state estimation error, which will be used for control design in following sections.

**Remark 1:** To calculate $E$, the LPV system (5) is treated as polytopic system, which seems to be the only thing to do, but very conservative. The adoption of parameter dependant observer gain (6) is advantageous as compared to a constant gain $L_e$. It is helpful to reduce the uncertainty of observer dynamic. In some cases, it is possible to stabilize the observer dynamic, which may be not possible by using a simple constant gain. A less uncertain dynamic is also helpful for these algorithms (Kouramas [2005], Yu [2010]) to obtain a tighter approximation of $E_z$.

4. DISTURBANCE INVARIANT TUBE

Instead of controlling the state directly, output feedback MPC controls the estimated state. The real state will fall in the neighborhood of estimated state, $x(k) \in \hat{x}(k) \oplus E$, as analyzed in section 3. The prediction dynamic is

$$\dot{x}(k + 1) = A(k)\hat{x}(k) + B(k)u(k) + d(k),$$  
(14)

where $d(k)$ is the equivalent disturbance as $d(k) = L_eCe(k) + L_vv(k)$ and $d(k) \in D = L_eCE \oplus Lu_e$. We define the response contributed by $d(k)$ as $e_{xz}(k) = \hat{x}(k) - z(k)$, where $z(k)$ is the nominal estimated state trajectory without disturbance excitation. Control Input is parameterized by

$$u(k) = u_z(k) + u_e(k) = u_z(k) + Ke_{xz}(k).$$  
(15)

It is easy to derive that

$$\dot{z}(k) = z(k) + e_{xz}(k),$$  
(16)

$$z(k + 1) = A(k)z(k) + B(k)u_z(k),$$  
(17)

$$e_{xz}(k + 1) = \Psi_{xz}(k)e_{xz}(k) + d(k),$$  
(18)

where $\Psi_{xz}(k) = A(k) + B(k)K$. We assume that the controller gain $K$ is properly chosen such that the dynamic of $e_{xz}(k)$ is stable. It can be checked by the following lemma.

**Lemma 2.** If there exists a matrix $G > 0$ and $Y$, such that

$$\begin{bmatrix} G & * \\ A_j G + B_j Y & G \end{bmatrix} > 0, 1 \leq j \leq J,$$  
(19)

and controller gain $K = YG^{-1}$, the dynamic (18) of $e_{xz}$ is stable.

**Proof.** The proof is straightforward and similar to it for Lemma 1 above, thus omitted here.

Assumed such a controller gain $K$ exists, then the dynamic of $e_{xz}$ is governed by the polytopic model:

$$e_{xz}(k + 1) = \Psi_{xz}(k)e_{xz}(k) + d(k)$$
$$= \sum_{j=1}^{J} \lambda_j(k) \Psi_{xz}(k) + d(k),$$  
(20)

where $\Psi_{xz}(k) = A_j + B_jK$. It is desirable to characterize the bound of $e_{xz}$ excited by the equivalent disturbance $d(k)$.

Similar to the previous section, the minimum robustly positive invariant set $E_{xz}$ of dynamic (20) is calculated, such that if $e_{xz}(k) \in E_{xz}$, then $e_{xz}(k + 1) \in E_{xz}$ for any possible realization of $\Psi_{xz}(k)$ and $d(k) \in \mathcal{D}$. To enforce the input constraints (3), $u_z(k)$ has to satisfy a tightened constraints as $u_z(k) \in U_z = U \cap KE_{xz}$. An inner approximation of $U_z$ is given as $\hat{U}_z$, comprising component-wise constraints as:

$$|u_{z,j}(k)| \leq u_{z,j,\text{max}}, 1 \leq j \leq m, k \geq 0,$$  
(21)

such as $\hat{U}_z \subset U_z$.

**Remark 2:** As in section 3, it is possible to choose $K$ as parameter dependant controller gain, too. The difference of the resulting dynamic is that the system matrix is not linear, but quadratic function of the time varying variable $\lambda$ as

$$e_{xz}(k + 1) = \sum_{j=1}^{J} \lambda_j(k) \Psi_{xz}(k) + d(k).$$  
(22)

Although it still can be treated as polytopic system, new technique of less conservativeness for characterization of the robust positive invariant set for such systems is desirable.

5. QUASI-MIN-MAX MPC

As in Lu [2000], Quasi-min-max MPC utilizes the infinite horizon objective function

$$J_0^\infty(k) = x^T(k)Lx(k) + u^T(k)Ru(k)$$  
(23)

$$+ \sum_{i=1}^{\infty} x^T(k + i)Lx(k + i) + u^T(k + i)Ru(k + i).$$

The online optimization minimizes the upper bound of $J_0^\infty(k)$. For the output feedback case, it is not directly applicable. One intuitive solution is to replace state $x(k)$ by the estimated state $\hat{x}(k)$ for the optimization problem. However, since separation principle does not hold for nonlinear system generally, the simple modification does not provide stability or performance guarantee. The difficulty is introduced by the equivalent disturbance $d(k)$. Thus a remedy to the problem is to deal with the nominal state evolution $z(k)$ and the response caused by non-vanishing disturbances separately. As in section 4, the nominal prediction is given by $z(k)$ as

$$z(k + 1) = A(k)z(k) + B(k)u_z(k),$$  
(24)

$$u_z(k) \in U_z, z(k),$$

the nominal state, is the center trajectory of the state evolution tube $z(k) + e_{xz}(k)$. The optimization problem is formulated to minimize the upper bound of the nominal cost on $z(k)$ as

$$\min_{z(k), u_z(k), F(k)} \max_{A(k+i), B(k+i)} \gamma(k)$$  
(25)

subject to constraints:

$$z(k + 1) = A(k)z(k) + B(k)u_z(k),$$  
(26)

$$u_z(k) \in U_z, \hat{x}(k) \in \hat{x}(k) \oplus E_{xz},$$  
(27)

$$z(k + i + 1) = A(k+i)z(k + i) + B(k+i)u_z(k + i),$$  
(28)

$$F(k)z(k + i) \in \hat{U}_z, i \geq 1,$$  
(29)
\( z^T(k)Lz(k) + u^T_z(k)Ru_z(k) + z^T(k + 1)P(k)z(k + 1) \leq \gamma(k) \), (30)
\( \geq z^T(k + i)(P(k) - L - F^T(k)RF(k))z(k + i) \geq \max_{i \geq 1} |(YQ^{-1}z(k + i))j|^2 \) (31)

**Remark 3:** It is noted that the nominal trajectory, the center of the tube, is not a single one as in Mayne [2006] for LTI systems. The center trajectory itself is a tube, comprising of infinite possible trajectories, each corresponding to a particular realization of \([A(k + i)|B(k + i)], i \geq 1\). An illustration of the state evolution is depicted in Fig. 1.

![Illustration of state evolution](image)

Fig. 1. Illustration of state evolution (solid line: a trajectory of tube center corresponding to a particular realization \([A(k)|B(k)], k \geq 0\); grey area: the disturbance invariant tube along a particular tube center; area enclosed by dashed line: the bound or tube of the tube center.)

The next theorem shows how to convert the optimization problem (25-31) into a mathematically tractable formulation.

**Theorem 3.** The mathematical formulation of (25-31) is
\( \min_{z(k), u_z(k), Q(k), Y(k), X(k)} \gamma(k) \)
subject to
\( u_z(k) \in U_z, \hat{x}(k) = z(k) \oplus E_{xz}, \) (33)
\[
\begin{bmatrix}
1 & A(k)z(k) + B(k)u_z(k) & Q(k) & \ast & \ast \\
L^{1/2}z(k) & 0 & \gamma(k)I & \ast & \ast \\
R^{1/2}u_z(k) & 0 & 0 & \gamma(k)I
\end{bmatrix}
\]
\( > 0, \) (34)
\[
\begin{bmatrix}
Q(k) & A_jQ(k) + B_jY(k)Q(k) & \ast & \ast & \ast \\
L^{1/2}Q(k) & 0 & \gamma(k)I & \ast & \ast \\
R^{1/2}Y(k) & 0 & 0 & \gamma(k)I
\end{bmatrix}
\]
\( > 0, j \leq J(35) \)
\[
\begin{bmatrix}
X(k) & \ast \\
Y(k)^T & Q(k)
\end{bmatrix}
\]
\( > 0, \) with \( X_{jj} < u^2_{z,j_{max}}, j \leq m. \) (36)

**Proof.** It is obvious that (33) is identical to (27). Then divide Eqn. (30) by \( \gamma(k) \) and define \( Q^{-1}(k) = \gamma^{-1}(k)P(k) \), it becomes
\[
1 - \gamma^{-1}(k)z^T(k)Lz(k) - \gamma^{-1}(k)u^T_z(k)Ru_z(k) - z^T(k + 1)Q^{-1}(k)z(k + 1) \geq 0, \] (37)

By applying Schur complements, it is equivalent to
\[
\begin{bmatrix}
1 - (z^T(k)Lz(k) - u^T_z(k)Ru_z(k)) / \gamma(k) & * \\
* & \gamma(k)
\end{bmatrix}
\]
\( \geq 0 \) (38)
is positive semidefinite. This condition can be rewritten as
\[
\begin{bmatrix}
1 & * \\
* & -T^T(k)\gamma^{-1}(k) I
\end{bmatrix}
\]
\( \geq 0, \) (39)
where
\[
T(k) = \left[ \begin{array}{cc}
L^{1/2}z(k) & 0 \\
R^{1/2}u_z(k) & 0
\end{array} \right]. \] (40)
Repeat to apply the schur complement, the above inequality is expressed in LMI form as Eqn. (34). Next we prove that Eqn. (35) implies Eqn. (31). Define
\[
M(k) = P(k) - L - F^T(k)RF(k), \]
\[
N(k + i) = A(k + i) + B(k + i)F(k). \] (42)
Eqn. (31) is satisfied if
\[
M(k) - N^T(k + i)P(k)N(k + i) \geq 0. \] (43)
Multiplying by \( \gamma(k) \) and left- and right- multiplying by \( P^{-1}(k) \), it becomes
\[
H(k) = \gamma(k)P^{-1}(k)N(k + i) \gamma^{-1}(k)P(k) \times N^T(k + i) \gamma(k)P^{-1}(k) \geq 0. \] (44)
where \( H(k) = \gamma(k)P^{-1}(k)M(k)P^{-1}(k) \). Using schur complements, it is converted as
\[
\begin{bmatrix}
H(k) & * \\
N(k + i)Q(k) & Q(k)
\end{bmatrix}
\]
\( \geq 0, \) (45)
Define \( Y(k) = F(k)Q(k) \), it follows that
\[
\begin{bmatrix}
Q(k) & A(k + i)Q(k) + B(k + i)Y(k)Q(k) \\
0 & \gamma(k)I
\end{bmatrix}
\]
\( \leq 0, \) (46)
where
\[
T(k) = \left[ \begin{array}{cc}
L^{1/2}Q(k) & 0 \\
R^{1/2}Y(k) & 0
\end{array} \right]. \] (47)

By applying the Schur complement again, it require
\[
\begin{bmatrix}
Q(k) & A(k + i)Q(k) + B(k + i)Y(k)Q(k) & \ast & \ast & \ast \\
L^{1/2}Q(k) & 0 & \gamma(k)I & \ast & \ast \\
R^{1/2}Y(k) & 0 & 0 & \gamma(k)I
\end{bmatrix}
\]
is positive definite. This condition is valid for any \([A|B] \in \Omega\), if (35) is satisfied. Next the constraints on control input is considered. Since \( z^T(k + i)P(k)z(k + i) \) decreases with \( i \) and \( z^T(k + i)P(k)z(k + i) \leq \gamma(k) \) (due to (34,35)), \( z(k + i) \) will stay in the invariant set \( z(k + i)T^iPz(k + i) \leq \gamma(k) \).

The peak bound of input is
\[
\max_{i \geq 1} u_{z,j}(k + i)^2 = \max_{i \geq 1} |(YQ^{-1}z(k + i))j|^2 \]
\[
\leq z^T(k + i + 1)Q^{-1}z(k + i + 1) \]
\[
\leq |(YQ^{-1/2}z)|^2 = (YQ^{-1/2})jj. \] (51)
Then if there exists a symmetrical matrix $X$ such that (36) is satisfied, the input constraints satisfaction (29) is guaranteed. These complete the proof.

The online implementation of the proposed controller is as follows:

1. $k = 1$, $\tilde{x}(1)$ is given;
2. solve optimization problem (32-36) to obtain the optimal solution $z^*(k)$, $u^*_k$;
3. apply $u(k) = u^*_k(k) + K(\tilde{x}(k) - z^*(k))$ to the system (1);
4. Obtain the new estimated state $\hat{x}(k+1)$ via (4), $k = k + 1$ and return to step 2.

Remark 4: The differences between the optimization problem (32-36) and the one for state feedback MPC (Lu [2000]) are the additions of one decision variable $z(k)$ and one linear inequality constraint (33). Thus the Quasi-min-max approach is extended from state feedback case to output feedback case with modest increase on computational complexity.

Remark 5: It is noted that the proposed approach could also be considered as an extension of Tube MPC for LTI system (Mayne [2006]) to model uncertain case. It may be less conservative, as compared to the approach in Rawlings [2009], which treats the discrepancy between nominal model prediction and real system response as external disturbances.

**Theorem 4.** Assume that the initial state estimation error $e(k) \in E$. If a feasible solution can be determined at step $t = k$, the optimization problem (32-36) is feasible for all future steps $t > k$.

**Proof.** As in most MPC approaches, it will be shown that the optimal solution obtained from (32) at $t = k$, will be a feasible solution to (32) at $t = k + 1$. Assume at $t = k$, the optimal solution to optimization (32) is $z^*(k)$, $u^*_k(k), Q^*(k), Y^*(k), X^*(k), \gamma^*(k)$. We choose $z(k+1) = A(k)z^*(k) + B(k)u^*_k(k)$, $u^*_k(k + 1) = Y^*(k)Q^*(k)z^*(k) + (k + 1) = Y^*(k), \gamma^*(k)$ is asymptotically stable. As shown in section 4, $\hat{x}(k), x(k)$ are within a neighborhood of $z^*(k)$ as $\hat{x}(k) \in z^*(k) \cup U_z$ and $x(k) \in x(k) \cup U_x$. Thus (33) is satisfied at $t = k + 1$. Secondly, the satisfaction of (34.35) at $t = k + 1$ requires $z^*(k + 1)Lz(k + 1) + u^*_k(k + 1)Ru^*_k(k + 1) + z^*(k + 2)Pz(k + 2) \leq \gamma(k)$ and $z^*(k + 2 + i)Pz(k + 2 + i) \leq z^*(k + 1 + i)P(k) - L - FT^*(k)RF^*(k)(k + 1 + i), i \in N$, which are guaranteed to be fulfilled by (30, 31) at $t = k$. Lastly, (36) at $t = k + 1$ is satisfied since it is the same as the one at $t = k$. Thus the candidate is a feasible solution to optimization problem (32) at $t = k + 1$. This completes the proof.

The system property under the proposed controller (15) is explained in the following theorem.

**Theorem 5.** Assume that the initial state estimation error $e(k) \in E$. If the optimization problem is feasible initially, the closed system is stable and the state will evolve into a neighborhood of the origin eventually.

**Proof.** We first investigate the stability of $z^*(k)$, whose dynamic is represented conceptually by

$$ z(k + 1) = f(z^*(k), \hat{x}(k), d(k)). $$

Choose Lyapunov function as $V(z^*(k)) = z^T(k)Q^*(k)z^*(k) + u^*_k(k)Ru^*_k(k) + (A(k)z^*(k) + B(k)u^*_k(k))P(k)(A(k)z^*(k) + B(k)u^*_k(k)).$ Assume that $L > 0$, it follows that $V(\cdot) \geq 0$. Due to the recursive feasibility property, $Z^*(k + 1)$ is upper bounded as:

$$ V(z^*(k + 1)) = z^T(k)Q^*(k)z^*(k) + u^*_k(k)Ru^*_k(k) + (A(k)z^*(k) + B(k)u^*_k(k))P(k)(A(k)z^*(k) + B(k)u^*_k(k)). $$

To save space, the inequality (33) for $A(k), B(k), \gamma^*(k)$ is dropped in (53) and $\bar{u}(k + 1) = A(k)z^*(k) + B(k)u^*_k(k), \bar{u}(k + 1) = F^*(k)z^*(k + 1)$. The Lyapunov function decreases monotonically as

$$ \Delta V = V(z^*(k + 1)) - V(z^*(k)) \leq -z^T(k)Q^*(k)z^*(k) - u^*_k(k)Ru^*_k(k) < 0, $$

due to (30, 31). Thus $z^*(k)$ is asymptotic stable. As shown in section 4, $\hat{x}(k), x(k)$ are within a neighborhood of $z^*(k)$ as $\hat{x}(k) \in z^*(k) \cup U_z$ and $x(k) \in x(k) \cup U_x$. Thus (33) converges to the origin, $\hat{x}(k)$ and $x(k)$ will asymptotically converge to $E_z \cup U_z$, respectively.

### 6. Simulations

The following second order system model (Park [2011]) is used for simulation:

$$ x(k + 1) = A(\alpha(k))x(k) + B(\beta(k))u(k) + w(k), $$

$$ y(k) = Cx(k) + v(k), $$

where

$$ A(\alpha(k)) = \begin{bmatrix} 0.872 & -0.0623 \alpha(k) \\ 0.9935 & 0.997 \end{bmatrix}, $$

$$ B(\beta(k)) = \begin{bmatrix} 0.0935 \\ 0.00478 \end{bmatrix}, C = [0.333 -1] $$

We assume that the scheduling parameters $\alpha(k)$ and $\beta(k)$ belong to the following regions:

$$ \alpha(k) \in [1, 5], \quad \beta(k) \in [0.1, 1]. $$

The small disturbance and sensor noises are bounded as $|w|_\infty \leq 0.001$ and $|v| \leq 0.001$. The input constraint is assumed to be bounded as

$$ |u(k)| \leq 1, \quad k \geq 0, $$

The initial states of the system and the observer are assumed as $x(0) = [-1.5 -0.2]^T$ and $\hat{x}(0) = [-1.3 0]^T$, respectively. The observer gain $L_e = [-1.2060 -1.7289]^T$ and controller gain $K = [-2.2379 -1.3429]$ are adopted. The system response under the controller (15) is shown in Fig. 2. The real state $x(k)$ and estimated state $\hat{x}(k)$ are plotted by the blue and red line, respectively. In simulation, no non-vanishing disturbance was introduced, thus the state converges to the origin asymptotically. The control input at each step is shown in Fig. 3. It is clear that the constraints on input signal is enforced by the proposed MPC. Fig. 4 shows the worst case upper bound cost $\gamma(k)$. It is obvious that $\gamma(k)$ decreases monotonically.
The simple simulation demonstrates that the proposed output MPC stabilizes the LPV systems against various bounded uncertainties.

7. CONCLUSION

The tube based Quasi-min-max MPC is presented for control of input constrained LPV systems. The recursive feasibility and robust stability are guaranteed. However, some conservativeness exists. Two directions may be exploited to improve the performance. The first one is to utilize a parameter dependant offline controller $K$ in order to characterize a tighter positive invariant set; the second one is the conservativeness introduced by the ellipsoidal approximation of polyhedral set by the LMI condition. It is desirable to combine more advanced state feedback MPC approaches, such as those proposed in Besselmann [2010] recently, into output feedback scenario.

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